Some Generalizations of Linear Cyclic Pursuit

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Abstract—In this paper, the behaviour of a group of autonomous mobile agents under cyclic pursuit is studied. Cyclic pursuit is a simple distributed control law, in which the agent i pursues agent $i + 1$ modulo n . The agents are assumed to be linear and identical, with no kinematic constraints on motion. It is shown here that the point of convergence of the system of agents is a function of the controller gain. The bounds on the controller gain for convergence to occur is also studied. The results generalize available results in the literature.

I. INTRODUCTION

This paper deals with the problem of linear cyclic pursuit in a multi-vehicle system. Multi-vehicle systems are groups of autonomous mobile agents used in military surveillance, rescue missions, space and oceanic explorations, and other automated collaborative operations. Linear cyclic pursuit uses simple local interaction between these vehicles to obtain desired global behaviour.

The pursuit strategies are designed to mimic the behaviour of biological organisms like dogs, birds, ants or beetles. They are commonly referred to as the 'bugs' problem. Bruckstein et al. [1] modelled the behavior of ants, crickets and frogs with continuous and discrete pursuit laws and examined the possible evolution of global behaviour such as the convergence to a point, collision, limit points or periodic motion. Convergence to a point in linear pursuit is the starting point to the analysis of achievable global formation among a group of autonomous mobile agents as discussed in [2],[3]. The paper [2] also deals with the evolution of the formation of these agents with respect to the possibility of collision. The stability aspect of both the linear and nonlinear pursuit has been studied in [4]. Pursuit strategies are one of the *behavior-based* techniques. Other techniques include *agreement problem* as studied in [5],[6].

This work has been inspired by the problem addressed in Marshall et al. [4]. They considered n identical autonomous mobile agents in linear cyclic pursuit. This cyclic pursuit is called linear since there is no non-holonomic constraint on the agents. The proof of stability of such a system available in [4] is based on the assumption of equal controller gain for all the agents. In this paper, a more general case is considered where the controller gain need not be the same for all the agents. This is more practical since all the n agents are unlikely to be identical. Yet, a stable behavior can be obtained.

The paper is organized as follows: Section 2 deals with the mathematical modelling of the n -agent system. Analysis of the expected behavior is discussed in Section 3. In Section 4

simulated result of the behavior of the system is shown and Section 5 concludes the paper.

II. EQUATION OF LINEAR PURSUIT

Linear pursuit is formulated considering the classical n -bugs problem. Each bug is modelled as an autonomous mobile agent. The agents are ordered from 1 to n , and the agent i pursues agent $i + 1$ modulo n .

The agents start from any arbitrary positions on a plane. The position of the agents at time $t > 0$ is given by

$$z_i(t) = [x_i(t), y_i(t)]^T \in R^2, i = 1, 2, \dots, n. \quad (1)$$

The kinematic that agent i follows is

$$\dot{z}_i = u_i \quad (2)$$

where u_i is the control input given by

$$u_i = k_i(z_{i+1} - z_i), \quad \forall i \quad (3)$$

Thus, the velocity of the i^{th} agent is proportional to the distance between the i^{th} and the $i + 1^{th}$ agent. The control variable for the agent i is the gain k_i .

A special case of the above pursuit law is considered in [4]. The gains of the agents are taken as $k_i = k, \forall i$, where k is a constant. Here, the analysis of a more general case where k_i 's can have different values, is carried out.

III. ANALYSIS FOR DIFFERENT CONTROLLER GAINS

The x_i and y_i coordinates of the i^{th} agent evolve independently for all i . Therefore, they can be decoupled into two identical linear systems of equations as

$$\dot{X} = AX \quad (4)$$

where

$$A = \begin{bmatrix} -k_1 & k_1 & 0 & \dots & \dots & 0 \\ 0 & -k_2 & k_2 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -k_{n-1} & k_{n-1} \\ k_n & 0 & 0 & \dots & 0 & -k_n \end{bmatrix}$$

The stable formation of the n -agents system can be analyzed through a single equation (4) since the equation of motions in the two directions are identical. The stability of the Equation (4) depends on the eigenvalues of A . If it is stable, then the equilibrium point is given by $\dot{x} = 0$. This implies that $x_1 = x_2 = \dots = x_n = \bar{x}$, that is, all the agents converge to a point $\bar{x}$.

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The stability of the system is proved using Gershgorin's theorem [7].

Greshgorin's Theorem [7]. Let $A = [a_{ij}] \in M_n$, and let

$$R_i(A) \equiv \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|, \quad 1 \leq i \leq n$$

denote the *deleted absolute row sums* of A . Then, all the eigenvalues of A are located in the union of n discs

$$\bigcup_{j=1}^n \left\{ z \in \mathbb{C} : |z - a_{ij}| \leq R_i(A) \right\} \equiv G(A)$$

Therefore if there is a $n \times n$ square matrix A , then n circles can be drawn with centers at the diagonal elements of A , i.e., a_{ii} , $i = 1, 2, \dots, n$ and radius equal to the norms of the sum of the other elements on the same row $\sum_{j \neq i} |a_{ij}|$. Such circles are called Gershgorin's discs. All the eigenvalues of A lie in the region formed by the union of all these n discs.

Theorem 1. Consider n planar agents with kinematics given by (4). For all initial conditions and for $k_i > 0$, $\forall i$, all the agents converges to a point.

Proof. Gershgorin disc theorem is applied to (4). Figure 1 shows the Gershgorin's discs in the complex plane. All the discs lie on the left half of the s -plane for $k_i > 0 \forall i$. Therefore, all the eigenvalues of A must lie on the left half plane or at the origin.

Fig. 1. Greshgorin disc for system (4) with all the gains positive

Now, the characteristic equation of A is

$$\prod_{i=1}^n (\lambda + k_i) - \prod_{i=1}^n (k_i) = 0 \quad (5)$$

From this equation, it is evident that only one eigenvalue lies at the origin provided all gains are positive. However, the stability

of the linear system does not dependent on this root as is proved below.

Let a similarity transformation be performed on A . The non-singular transformation matrix P is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/k_1 & 1/k_2 & \dots & \dots & 1/k_n \end{bmatrix} \quad (6)$$

Therefore, the matrix $B (= PAP^{-1})$ has the same eigenvalues as that of A from the properties of similarity transformation. In the new coordinate system, $\hat{X} = PX$. Hence, (4) now becomes

$$\dot{\hat{X}} = \begin{bmatrix} -k_1 & k_1 & \dots & \dots & 0 \\ 0 & -k_2 & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & k_{n-2} & 0 \\ \frac{-k_n k_{n-1}}{k_1} & \frac{-k_n k_{n-1}}{k_2} & \dots & -(k_n + k_{n-1}) & k_n k_{n-1} \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \hat{X} \quad (7)$$

Here, $\dot{\hat{x}}_n = 0$. Therefore, $\hat{x}_n$ does not change with time. Let $\hat{x}_n = a$. Then the above equation can be re-written as

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \vdots \\ \hat{x}_{n-1} \end{bmatrix} = C \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_{n-2} \\ \hat{x}_{n-1} \end{bmatrix} + D \quad (8)$$

where,

$$C = \begin{bmatrix} -k_1 & k_1 & \dots & 0 \\ 0 & -k_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_{n-2} \\ -k_n k_{n-1}/k_1 & \dots & \dots & -(k_n + k_{n-1}) \end{bmatrix} \quad (9)$$

$$D = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k_n k_{n-1} a \end{bmatrix} \quad (10)$$

Therefore, the dynamics of the system (8) depends on the $(n-1)$ eigenvalues of C . Hence, the zero eigenvalue does not

play any role in the dynamics of the system given by (7) and (8) and so also for system given by (4).

Thus, the linear system (4) is stable and for any initial condition, equilibrium will be reached in finite time. This implies that the n agents will finally meet at a point. $\square$

Theorem 2. Consider n planar agents with kinematics (4). The agents will converge to a point if one of the agent p has negative controller gain $\bar{k} < k_p < 0$ and all other agents have $k_i > 0$, $\forall i, i \neq p$ where $\bar{k}$ is given by

$$\bar{k} = -\frac{\prod_{j=1, j \neq p}^n k_j}{\sum_{i=1, i \neq p}^n \prod_{j=1, j \neq i, j \neq p}^n k_j}$$

Proof. Following the proof of Theorem 1, one eigenvalue of the linear system (4) will be at the origin. However, it will not effect the dynamic stability of the system. With one gain negative, the Gershgorin's disc is shown in Figure 2. There is one disc on the right half side of the s -plane. However, so long any of the $(n-1)$ roots of A do not lie in the right hand side disc, the system is stable. It is possible to find out the value of the gains till which the roots are all on the left half plane.

Fig. 2. Greshgorin disc for system (4) with one of the gains negative

Consider one agent say, p^{th} , whose gain is decreased from positive to negative. At some negative value, the system will become unstable, that is, one of the eigenvalue will cross the imaginary axis. Since this root always has to remain within the Gershgorin's disc, it must pass through the origin to move from the left to the right side of the plane, as can be seen from Figure 2. Moreover, at the point the root reaches the origin, the system becomes unstable. This point gives the maximum negative gain that the system can have, maintaining its stability.

Now, the point at which the root crosses the origin is found from the characteristic equation of A . Expanding (5)

$$\lambda(\lambda^{n-1} + f_1(k)\lambda^{n-2} + f_2(k)\lambda^{n-3} + \dots + f_n(k)) = 0 \quad (11)$$

where each $f_i(k)$ is a function of k_i 's. For a second root to be at the origin, $f_n(k) = 0$. Now

$$f_n(k) = \sum_{i=1}^n \prod_{\substack{j=1, \\ j \neq i}}^n k_j = 0 \quad (12)$$

One can express one of the gains (say k_p) as a function of the other gains as follows

$$k_p = -\frac{\prod_{j=1, j \neq p}^n k_j}{\sum_{i=1, i \neq p}^n \prod_{j=1, j \neq i, j \neq p}^n k_j} = \bar{k} \quad (13)$$

It implies that the minimum value of k_p that ensures stability is bounded by (13) and hence so long as $k_p > \bar{k}$ the vehicles will converge. $\square$

IV. SIMULATION RESULTS

A system of 5 agents are considered. The initial positions of the agents are $Z = [(10, -1), (7, 2), (0, 10), (-7, 5), (4, -8)]$. The gains are taken as $k = [4, 6, 8, 10, 12]$. The trajectories of each agent are plotted and is shown in Figure 3. The agents converge to $\bar{z} = [4.55, 1.6]^T$

Fig. 3. The trajectories of 5 vehicles, all gains positive

Assuming the gains of agents 2 to 5 to remain as given, we can compute the bound $\bar{k}$ for k_1 as -2.86 from (13). Consider that the gain of agent 1 is $k_1 = -1$, while the other gains remain the same. The simulated trajectories are shown in Figure 4. The agents converges at $\bar{z} = [17.5, -4.6]^T$

Assigning a lower value of $k_1 = -3$, the trajectories of the agents are shown in Figure 5. The agents do not converge to any point. They continue to move in a straight line.

V. CONCLUSION

Pursuit strategies has recently been of much interest among researchers. In this paper, the convergence properties of linear cyclic pursuit is studied. Since the point of convergence of

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Fig. 5. The trajectories of 5 vehicles, one gain less than the minimum value required for stability. The vehicles do not converge to a point.

the agents is a function of the agents' gain, this point can be easily controlled. In fact, in some applications the final point at which all the UAVs should reach is fixed. It can be a target at which all the agents should arrive simultaneously. This problem has been addressed in [8]-[10] under different framework. Using pursuit law the vehicles can also be made to reach the target point starting from any arbitrary position by controlling the gains and using only local information. The task of designing control strategies to achieve this objective is a part of future work in the direction.

VI. ACKNOWLEDGEMENT

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