Lossy Distributed Source Coding with Side Information

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Abstract— **In a typical sensor network scenario a goal is to monitor a spatio-temporal process through a number of inexpensive sensing nodes, the key parameter being the fidelity at which the process has to be estimated at distant locations. We study such a scenario in which multiple encoders transmit their correlated data at finite rates to a distant and common decoder. In particular, we derive inner and outer bounds on the rate region for the random field to be estimated with a given mean distortion.**

Keywords: Sensor networks, distributed source coding

I. Introduction

A commonly used model of sensor networks assumes multiple encoders transmitting at finite rates to a distant, common decoder. Since the observations of these sensors are inherently correlated, it is possible to achieve lower rates using astute techniques than would have been required if sensors compressed their respective observations disregarding others'. Moreover, the availability of side information at the encoders or/and at the decoder can also considerably enlarge the achievable rate region. This is true irrespective of the required fidelity at the decoder.

For the case when observations are to be decoded losslessly with arbitrarily low error probability, Slepian and Wolf [1] proved the coding theorem for two sensors. Cover [2] extended their results to an arbitrary number of discrete sources with ergodic memory using a important technique now known as "random binning". Inspired by Slepian-Wolf results, Wyner and Ziv [3] obtained the rate distortion region for a source when the decoder has access to side information. Their result requires the encoder to communicate at higher rate than it will if it too had access to the same side information. The latter result (when the encoder and the decoder have side information) was first obtained by Gray (See, [4]) and is generally known as conditional rate distortion theorem. The difference in the rates stems from the Markov Chain condition present in the Wyner-Ziv result. Gray's result ([4]) is generalized to the case where the encoder and the decoder have common information and the decoder may possibly have extra information in [5]. The most important contributions to the lossy Slepian-Wolf problem are those of Berger and Tung [4] in the form of an inner and an outer bound on the rate distortion region. Despite numerous attempts (e.g. [6], [7], [8]), exact rate distortion region is still unknown. Recently, Gastpar [9] studied the multiterminal source coding problem when the decoder has access to side information. He derived an inner and an outer bound on the rate region and proved the tightness of his bounds for the case when the sources are conditionally independent given side information. In this paper, we study the situation depicted in Fig 1. We consider discrete memoryless dependent finite alphabet sources X_1, X_2 and side information random variables Y_1, Y_2 and Y with known joint distribution $p(x_1, x_2, y_1, y_2, y)$, i.e, $(X_{1i}, X_{2i}, Y_{1i}, Y_{2i}, Y_i), i \geq 1$ form an i.i.d. sequence with distribution $p(x_1, x_2, y_1, y_2, y)$. Our interest lies in the rate region that allows the decoder to satisfy the distortion constraints

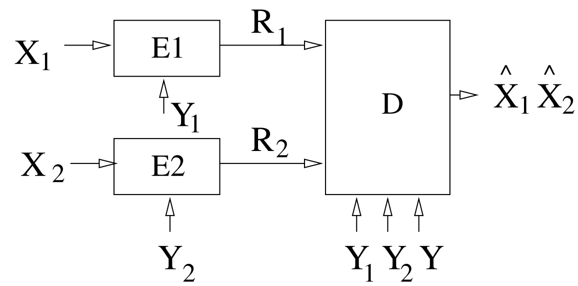

Fig. 1. The Problem Studied: lossy encoders and decoder with side information.

$$Ed_1(X_1, \hat{X}_1) \leq D_1, \quad Ed_2(X_2, \hat{X}_2) \leq D_2, \tag{1}$$

where $\hat{X}_i$ is the estimate of X_i produced by the receiver and d_i are arbitrary, bounded distortion measures. We will generalize these results to multiple sources also. Though the proofs here are limited to finite alphabets, we believe that these results can be extended without modifications to the case of continuous alphabets. Our results generalize the results in [4], [5], [9], [1], [3].

A related problem is the CEO problem ([10], [11]). Our setup reduces to it if we take $X_i = S + Z_i$, $i = 1, 2$ where Z_i are independent observation noises. Our proof techniques will provide these results which will be reported elsewhere.

Our results can be used in the following sensor network scenario. Multiple sensor nodes are sensing a random field. The overall random field is divided into disjoint sub-regions. Each sub-region has a clusterhead (sensor node) to which all the sensor nodes in the subregion transmit their observations directly ([12]). The clusterheads are connected (via a wireline network, or they have powerful batteries) to the central node. Due to correlations in the random field, sensor nodes can significantly reduce their transmission rates to the cluster head. Furthermore, the clusterhead itself can be a sensor node which senses data that is correlated to the observations sent by other sensor nodes. One can further reduce the data rate if the neighbouring sensor nodes communicate directly with each other (because of multiple access channel in the subregion this will be sensed by the decoder also) before transmitting to the clusterhead, or if they relay observations of some other sensors. The reduction in transmission rate due to the correlation in the sensor field could be obtained using the results in [4] and due to the sensor data at the clusterhead from the results in [9]. Our results provide these reductions and also due to prior communication between sensor nodes, or relaying via the side informations Y_1 and Y_2 in Fig 1. This result was not available before.

If in the above sensor network scenario we also add that the sensor nodes communicate their observations to the clusterhead via a common multiple access wireless channel, this becomes a system in which the source-channel separation theorem does not hold (See, [13]). Thus for optimal transmission joint source-channel coding is required. Under these conditions, the problem of minimal rates for a given mean distortion is addressed in [14].

The rest of the paper is organized as follows. In Section II, we obtain an achievable rate region for two sensors followed by an outer bound in section III. The special cases mentioned above are provided in Section IV. We generalize these results to the case of $N \geq 2$ sources in Section V. We summarize and conclude the paper in Section VI.

II. An inner bound

The essence of this section is the following inner bound on the rate distortion region of the problem at hand. In the following for random variable X, X^n denotes an i.i.d. sequence of length n with the distribution of X.

The rate vector $\mathbf{R} \triangleq (R_k, k \in \mathcal{S})$, where $\mathcal{S}$ is the set of sources, is said to be admissible with distortions $\mathbf{D} \triangleq (D_k, k \in \mathcal{S})$ if for any $\varepsilon > 0$, there exists n_0 such that for all $n > n_0$ there exist encoders $f_{E,k}^n : \mathcal{X}_k^n \times \mathcal{Y}_k^n \to \mathcal{C}_k^{(n)}$ with $\log|\mathcal{C}_k^{(n)}| \leq n(R_k + \varepsilon)$, $k \in \mathcal{S}$, and, decoders $f_{D,j}^n : \prod_{i \in \mathcal{S}} \mathcal{C}_i^{(n)} \to \mathcal{X}_j$, $j \in \mathcal{S}$ such that $Ed(X_j^n, \hat{X}_j^n) \triangleq \frac{1}{n}E[\sum_{i=1}^n d(X_{ji}, \hat{X}_{ji})] \leq D_j + \varepsilon$, $j \in \mathcal{S}$, where $\hat{X}_j^n = f_{D,j}^{(n)}(Y^n, Y_k^n, W_k^n, k \in \mathcal{S})$.

Let $\mathcal{R}(D) \triangleq \{\mathbf{R} : \mathbf{R} \text{ is } \mathbf{D} \text{ - admissible}\}$.

As in the special cases mentioned above, we will not be able to find $\mathcal{R}(D)$. This Section provides a set $\mathcal{R}_{in}(D) \subset \mathcal{R}(D)$, and Section III will provide $\mathcal{R}_{out}(D) \supset \mathcal{R}(D)$.

Theorem 2.1: Given the joint distribution $p(x_1, x_2, y_1, y_2, y)$, define $\mathcal{W}_{in}(D)$ as the set of random vectors $\mathbf{W} \triangleq (W_1, W_2)$ satisfying the following conditions ($X - Y - Z$ indicates that (X, Y, Z) forms a Markov sequence):

1) $W_1 - X_1Y_1 - W_2X_2Y_2Y$, $W_2 - X_2Y_2 - W_1X_1Y_1Y$ and
2) There exist decoder functions $f_{D,1}(.)$ and $f_{D,2}(.)$ such that $\frac{1}{n}Ed(X_i^n, \hat{X}_i^n) \leq D_i + \varepsilon_1$, where $\hat{X}_i^n = f_{D,i}^{(n)}(W_1^n, W_2^n, Y_1^n, Y_2^n, Y^n)$, $i \in \{1, 2\}$, for a given $\varepsilon_1 > 0$.

Let $R(\mathbf{W})$ be the set of all rate vectors (R_1, R_2) such that

$$\begin{aligned} R_1 &> I(X_1; W_1|W_2Y_1Y_2Y) \\ R_2 &> I(X_2; W_2|W_1Y_1Y_2Y) \\ R_1 + R_2 &> I(X_1X_2; W_1W_2|Y_1Y_2Y) \end{aligned} \quad (2)$$

then $\mathcal{R}_{in}(\mathbf{D}) \triangleq \overline{conv}\{\bigcup_{\mathbf{W} \in \mathcal{W}_{in}(\mathbf{D})} R(\mathbf{W})\} \subset \mathcal{R}(D)$, where $\overline{conv}\{.\}$ is the convex hull of its argument.

We now show the achievability of all points in the rate region $R_{in}(D)$. Let $T_\varepsilon^n(X)$ be the set of strongly δ-typical sequences of length n generated according to the distribution $p_X(x)$ ([15, p. 288,358], [4]). The following lemma, introduced by Berger ([4], [15]), is handy while proving many multiterminal source coding results. It ensures the transitivity of joint typicality.

Lemma 2.2 (Markov Lemma): Suppose $X - Y - Z$. If for a given $(x^n, y^n) \in T_\varepsilon^n(X, Y)$, Z^n is drawn $\sim \prod_{i=1}^n p(z_i|y_i)$, then with high probability $(x^n, y^n, Z^n) \in T_\varepsilon^n(X, Y, Z)$ for n sufficiently large.

The proofs of following Lemmas are simple and omitted for the sake of brevity. The following extension of the Markov lemma is required for proving our results.

Lemma 2.3 (extended Markov Lemma 1): Suppose $W_1 - X_1Y_1 - X_2W_2Y_2Y$ and $W_2 - X_2Y_2 - X_1W_1Y_1Y$. If for a given $(x_1^n, x_2^n, y_1^n, y_2^n, y^n) \in T_\varepsilon^n(X_1, X_2, Y_1, Y_2, Y)$, W_1^n and W_2^n are drawn respectively $\sim \prod_{i=1}^n p(w_{1i}|x_{1i}y_{1i})$ and $\prod_{i=1}^n p(w_{2i}|x_{2i}y_{2i})$, then with high probability $(x_1^n, x_2^n, y_1^n, y_2^n, y^n, W_1^n, W_2^n) \in T_\varepsilon^n(X_1X_2Y_1Y_2YW_1W_2)$ for n sufficiently large.

■

The following lemma is a generalization of Lemma 2.3 to more than two encoders. This will be used in Section V.

Lemma 2.4 (extended Markov Lemma 2): Suppose $W_i - X_iY_i - X_{\{i\}^c}W_{\{i\}^c}Y_{\{i\}^c}Y$ for all $i \in \mathcal{E}$, where $\mathcal{E}$ is the set of encoders, and $X_{\{i\}^c} \triangleq (X_k, k \in \mathcal{E} \setminus \{i\})$. If for a given $(x_i^n, y_i^n, y^n, i \in \mathcal{E}) \in T_\varepsilon^n(X_i, Y_i, Y, i \in \mathcal{E})$, W_i^n is drawn $\sim \prod_{j=1}^n p(w_{ij}|x_{ij}y_{ij})$, then with high probability $(x_i^n, y_i^n, y^n, W_i^n, i \in \mathcal{E}) \in T_\varepsilon^n(X_iY_iYW_i, i \in \mathcal{E})$ for n sufficiently large.

We further need the following lemmas during the course of the proof of Theorem 2.1.

Lemma 2.5: If $(\hat{W}_1^n, W_2^n, Y_1^n, Y_2^n, Y^n) \sim p(w_2y_1y_2y)p(w_1)$, then $\Pr\Big((\hat{W}_1^n, W_2^n, Y_1^n, Y_2^n, Y^n) \in T_\varepsilon^n\Big) \leq 2^{-n\{I(W_1;W_2Y_1Y_2Y) - 3\varepsilon\}}$.

Lemma 2.6: If $(\hat{W}_1^n, \hat{W}_2^n, Y_1^n, Y_2^n, Y^n) \sim p(y_1y_2y)p(w_1)p(w_2)$, then $\Pr\Big((\hat{W}_1^n, \hat{W}_2^n, Y_1^n, Y_2^nY^n) \in T_\varepsilon^n\Big) \leq 2^{-n\{I(W_1;W_2Y_1Y_2Y) + I(W_2;W_1Y_1Y_2Y) - I(W_1;W_2|Y_1Y_2Y) - 4\varepsilon\}}$.

Proof of Theorem 2.1 : Fix $p(w_1|x_1y_1)$ and $p(w_2|x_2y_2)$ as well as $f_{D,1}(.)$, $f_{D,2}(.)$ satisfying the distortion constraints.

Codebook Generation : Let $R_k' = I(X_kY_k; W_k) + \varepsilon_1$ for $k = 1, 2$, and some $\varepsilon_1 > 0$. Generate $2^{nR_1'}$ codewords of length n, sampled iid from the marginal distribution $p(w_1)$. Label them as $w_1(i)$, with $i \in \{1, 2, \ldots, 2^{nR_1'}\}$. Similary, generate $2^{nR_2'}$ codewords of length n, sampled iid from the marginal distribution $p(w_2)$. Label them as $w_2(j)$, with $j \in \{1, 2, \ldots, 2^{nR_2'}\}$.

For $k \in \{1, 2\}$, take 2^{nR_k} random bins with indices b_k. Randomly assign to each codeword $w_k(.)$ a bin index $b_k \in \{1, 2, \ldots, 2^{nR_k}\}$. Let $B_k(b_k)$ be the set of all codewords in the bin indexed by b_k. This information is sent to the decoder.

Encoding: For $k \in \{1, 2\}$, given source sequences X_k^n and Y_k^n, the k^{th} encoder looks for a codeword $W_k^n(i)$ such that $(X_k^n, Y_k^n, W_k^n(i)) \in T_\varepsilon^n(X_k, Y_k, W_k)$. It sends the bin index b_k to which $W_k^n(i)$ belongs.

Decoding: The decoder searches for a jointly typical $(Y_1^n, Y_2^n, Y^n, W_1^n(i), W_2^n(j))$ such that $w_1^n(i) \in B_1(b_1)$ and $w_2^n(j) \in B_2(b_2)$. If there exists a unique such choice calculate $\hat{X}_i^n = f_{D,i}(W_1^n(i), W_2^n(j), Y_1^n, Y_2^n, Y^n)$, $i \in \{1, 2\}$ otherwise declare an error and incur the maximum distortion.

In the following we first show that as $n \to \infty$ the probability of the event when the decoder declares an error tends to zero under our conditions. The error can occur under the following four disjoint events $\mathbf{E1} - \mathbf{E4}$.

E1 The encoders do not find codewords.

From standard rate distortion theory $\lim_{n\to\infty} P(\mathbf{E1}) = 0$ if $R_1' > I(X_1Y_1; W_1)$ and $R_2' > I(X_2Y_2; W_2)$.

E2 The codewords are not jointly typical with (y_1^n, y_2^n, y^n).

Probability of this event goes to zero from Lemma 2.3.

E3 There exists another codeword $w_1^n(\hat{i}) \in B_1(b_1)$ which is jointly typical with $(W_2^n(j), Y_1^n, Y_2^n, Y^n)$.

From Lemma 2.5,

$$\begin{aligned} P(\mathbf{E3}) &\leq |B(b_1)|2^{-n\{I(W_1;W_2Y_1Y_2Y)-\delta\}} && (3)\\ &\leq 2^{n(R_1'-R_1)}2^{-n\{I(W_1;W_2Y_1Y_2Y)-\delta\}} && (4)\end{aligned}$$

which goes to zero if

$$\begin{aligned} R_1 > \;& R_1' - I(W_1; W_2Y_1Y_2Y) && (5)\\ >\;& I(X_1Y_1; W_1) - I(W_1; W_2Y_1Y_2Y) && (6)\\ =\;& H(W_1|W_2Y_1Y_2Y) - H(W_1|X_1Y_1) && (7)\\ =\;& H(W_1|W_2Y_1Y_2Y)\\ & -H(W_1|X_1Y_1Y_2YW_2) && (8)\\ =\;& I(X_1; W_1|W_2Y_1Y_2Y) && (9)\end{aligned}$$

where (8) follows from the Markov chain condition $W_1 - X_1Y_1 - X_2W_2Y_2Y$.

Similarly, by symmetry of the problem we require $R_2 > I(X_2; W_2|W_1Y_1Y_2Y)$.

E4 There exist codewords $w_1^n(\hat{i}) \in B_1(b_1)$ and $w_2^n(\hat{j}) \in B_2(b_2)$ which are jointly typical with (Y_1^n, Y_2^n, Y^n). From Lemma 2.6,

$$\begin{aligned} P(\mathbf{E4}) &\leq |B(b_1)|.|B(b_2)|2^{-n\{I(W_1;W_2Y_1Y_2Y)\}} && (10)\\ &\quad 2^{-n\{I(W_2;W_1Y_1Y_2Y)-I(W_1;W_2|Y_1Y_2Y)-6\varepsilon\}}\\ &\leq 2^{n(R_1'-R_1+R_2'-R_2)}2^{-n\{I(W_1;W_2Y_1Y_2Y)\}}\\ &\quad 2^{-n\{I(W_2;W_1Y_1Y_2Y)-I(W_1;W_2|Y_1Y_2Y)-6\varepsilon\}}\end{aligned}$$

which goes to zero if

$$\begin{aligned} R_1 + R_2 &> R_1' + R_2' - I(W_1; W_2Y_1Y_2Y)\\ &\quad -I(W_2; W_1Y_1Y_2Y) + I(W_1; W_2|Y_1Y_2Y)\\ &> I(X_1Y_1; W_1) + I(X_2Y_2; W_2) -\\ &\quad I(W_1; W_2Y_1Y_2Y) - I(W_2; W_1Y_1Y_2Y) +\\ &\quad I(W_1; W_2|Y_1Y_2Y) && (11)\\ &= I(X_1; W_1|Y_1Y_2YW_2) + I(X_2; W_2|Y_1Y_2YW_1)\\ &\quad -I(W_1; W_2|Y_1Y_2Y) && (12)\\ &= H(W_1|Y_1Y_2YW_2) - H(W_1|X_1Y_1Y_2YW_2)\\ &\quad +H(W_2|Y_1Y_2YW_1) - H(W_2|X_2Y_1Y_2YW_1)\\ &\quad +H(W_2|Y_1Y_2Y) - H(W_2|Y_1Y_2YW_1)\\ &= H(W_1W_2|Y_1Y_2Y) - H(W_1|X_1Y_1Y_2YW_2)\\ &\quad -H(W_2|X_2Y_1Y_2YW_1) && (13)\\ &= H(W_1W_2|Y_1Y_2Y) - H(W_1|X_1X_2Y_1Y_2Y)\\ &\quad -H(W_2|X_2Y_1Y_2YW_1X_1) && (14)\\ &= H(W_1W_2|Y_1Y_2Y) - H(W_1W_2|X_1X_2Y_1Y_2Y)\\ &= I(X_1X_2; W_1W_2|Y_1Y_2Y) && (15)\end{aligned}$$

where (12) follows from (9) and (14) from the Markov chain conditions $W_1 - X_1Y_1 - X_2Y_2W_2Y$, $W_2 - X_2Y_2 - X_1W_1Y_1Y$.

Thus as $n \to \infty$, with probability tending to 1, the decoder finds the correct sequence $(W_1^n(i), W_2^n(j))$ which are is jointly strongly typical with (Y_1^n, Y_2^n, Y^n). Since for large n, the empirical distribution of $(W_1^n(i), W_2^n(j), Y_1^n, Y_2^n, Y^n)$ is close to its joint distribution which by assumption has distortion (D_1, D_2), our claim follows. ■

III. An Outer Bound

In this section, we obtain the following outer bound on the rate distortion region of our problem.

Theorem 3.1: $\mathcal{R}(\mathbf{D}) \subseteq \mathcal{R}'(\mathbf{D})$ where $\mathcal{R}'(\mathbf{D})$ is the set of all rate pairs (R_1, R_2) such that there exists a pair (W_1, W_2) of random variables satisfying $W_1 - X_1Y_1 - X_2Y_2Y$ and $W_2 - X_2Y_2 - X_1Y_1Y$ for which the following conditions are satisfied:

$$\begin{aligned} R_1 &\geq I(X_1X_2; W_1|W_2Y_1Y_2Y),\\ R_2 &\geq I(X_1X_2; W_2|W_1Y_1Y_2Y),\\ R_1 + R_2 &\geq I(X_1X_2; W_1W_2|Y_1Y_2Y)\end{aligned}$$

and for which there exist functions $f_{D,1}(.)$ and $f_{D,2}(.)$ such that

$$\begin{aligned} &\frac{1}{n}Ed_1(X_1^n, f_{D,1}(W_1^n, W_2^n, Y_1^n, Y_2^n, Y^n)) \leq D_1 + \varepsilon,\\ &\frac{1}{n}Ed_2(X_2^n, f_{D,2}(W_1^n, W_2^n, Y_1^n, Y_2^n, Y^n)) \leq D_2 + \varepsilon. \end{aligned} \quad (16)$$

Proof: We follow the usual steps for proving the converse. Let $E1$ be one of the 2^{nR_1} indices sent by the first encoder and $E2$ be that of second. Then,

$$\begin{aligned} nR_1 &\geq H(E1)\\ &= H(E1|E2) + I(E1; E2)\\ &\geq H(E1|E2)\\ &= I(E1; X_1^nX_2^nY_1^nY_2^nY^n|E2)\\ &= I(E1; Y_1^nY_2^nY^n|E2) + I(E1; X_1^nX_2^n|E2Y_1^nY_2^nY^n)\\ &\geq I(E1; X_1^nX_2^n|E2Y_1^nY_2^nY^n)\\ &= \sum_{i=1}^n I(X_{1i}X_{2i}; E1|E2Y_1^nY_2^nY^nX_1^{i-1}X_2^{i-1})\\ &= \sum_{i=1}^n \{H(X_{1i}X_{2i}|E2Y_1^nY_2^nY^nX_1^{i-1}X_2^{i-1})\\ &\quad -H(X_{1i}X_{2i}|E1E2Y_1^nY_2^nY^nX_1^{i-1}X_2^{i-1})\}\\ &= \sum_{i=1}^n \{H(X_{1i}X_{2i}|W_{2i}Y_{1i}Y_{2i}Y_i)\\ &\quad -H(X_{1i}X_{2i}|W_{1i}W_{2i}Y_{1i}Y_{2i}Y_i)\}\\ &= \sum_{i=1}^n I(X_{1i}X_{2i}; W_{1i}|W_{2i}Y_{1i}Y_{2i}Y_i)\end{aligned}$$

where we have defined $W_{1i} = (E_1X_1^{i-1}X_2^{i-1}Y_1^{-i}Y_2^{-i}Y^{-i})$ and $W_{2i} = (E_2X_1^{i-1}X_2^{i-1}Y_1^{-i}Y_2^{-i}Y^{-i})$. Here, $Y^{-i} \triangleq (Y^n \setminus Y_i)$ and, the validity of the Markov Chains can be confirmed using simple information theoretic inequalities. A similar sequence of inequalities hold for a bound on nR_2. For the bound on the sum rate, we have :

$$\begin{aligned} n(R_1 + R_2) &\geq H(E1E2)\\ &= I(E1E2; X_1^nX_2^nY_1^nY_2^nY^n)\end{aligned}$$

$$\begin{aligned}
&\geq I(E1E2; X_1^n X_2^n | Y_1^n Y_2^n Y^n) \\
&= \sum_{i=1}^{n} I(X_{1i}X_{2i}; E1E2 | Y_1^n Y_2^n Y^n X_1^{i-1} X_2^{i-1}) \\
&= \sum_{i=1}^{n} \{H(X_{1i}X_{2i} | Y_1^n Y_2^n Y^n X_1^{i-1} X_2^{i-1}) \\
&\quad - H(X_{1i}X_{2i} | E1E2 Y_1^n Y_2^n Y^n X_1^{i-1} X_2^{i-1})\} \\
&= \sum_{i=1}^{n} \{H(X_{1i}X_{2i} | Y_{1i}Y_{2i}Y_i) \\
&\quad - H(X_{1i}X_{2i} | W_{1i}W_{2i}Y_{1i}Y_{2i}Y_i)\} \\
&= \sum_{i=1}^{n} I(X_{1i}X_{2i}; W_{1i}W_{2i} | Y_{1i}Y_{2i}Y_i)
\end{aligned}$$

and the result follows from the standard convexity arguments (For details see, [4],[15]). ■

We conclude this section by noting that analogous to the original multiterminal source coding problem, the achievability proof of the inner bound reckons on the stronger Markov chain conditions $W_1 - X_1Y_1 - X_2Y_2W_2Y$ and $W_2 - X_2Y_2 - X_1Y_1W_1Y$ which are required to make sure the joint typicality of the codewords (Lemma 2.3), while the outer bound is obtained using weaker conditions $W_1 - X_1Y_1 - X_2Y_2Y$ and $W_2 - X_2Y_2 - X_1Y_1Y$. Currently, we are unaware of any weaker assumption (than the Markov chain requirement) which yields the transitivity of joint typicality. Moreover, it is desirable to know the conditions under which the two bounds coincide. This happens when the probability distributions in the outer bound give no more freedom than those in inner bound. These results may prove instrumental in setting the probability space so as to make sure that the two probability distributions coincide.

IV. Some Special Cases

In this section, we consider several special cases of the problem. The first case requires the reconstruction of one source with arbitrarily low mean distortion for the second source. We determine the exact rate distortion region for this problem. In second case, we consider the situation when the two sources are conditionally independent given the side information.

A. Rate Distortion Region for $D_2 = 0$

Theorem 4.1: For a given distortion $D_1 = D$, the rate pair (R_1, R_2) is achievable if and only if

$$\begin{aligned}
&R_1 \geq I(X_1; W_1 | Y_1 Y_2 Y X_2), R_2 \geq H(X_2 | W_1 Y_1 Y_2 Y), \\
&R_1 + R_2 \geq H(X_2 | Y_1 Y_2 Y) + I(X_1; W_1 | X_2 Y_1 Y_2 Y),
\end{aligned} \tag{17}$$

where W_1 is an auxiliary random variable satisfying $W_1 - X_1Y_1 - X_2Y_2Y$ and for which there exist functions $f_{D,1}(.)$ and $f_{D,2}(.)$ such that

$$\begin{aligned}
&\frac{1}{n} E d_1(X_1^n, f_{D,1}(W_1^n, W_2^n, Y_1^n, Y_2^n, Y^n)) \leq D + \varepsilon_1, \\
&\frac{1}{n} E d_2(X_2^n, f_{D,2}(W_1^n, W_2^n, Y_1^n, Y_2^n, Y^n)) \leq \varepsilon_1.
\end{aligned} \tag{18}$$

Proof: Take $W_2 = (X_2Y_2)$. Then the constraints $W_2 - X_2Y_2 - W_1X_1Y_1Y$ and $W_2 - X_2Y_2 - X_1Y_1Y$ of the inner and the outer bounds respectively, vanish and that the remaining constraints coincide. The result, finally, follows from trivial simplification. ■

We observe that this result coincides with the previously known special cases including the results of Slepian and Wolf [1], Wyner and Ziv [3] and Berger [7].

To provide a feel for the reduction in rates that can be obtained via the correlations between X_1, X_2 and the side information Y_1, Y_2, Y we illustrate the rate regions with an example. In this example $D_1 = D_2 = 0$ and hence from Theorem 4.1 we obtain an exact rate region $\mathcal{R}(D)$.

Example: Let X_1, X_2, Y_1, Y_2, Y be binary random variables with the following distributions: $p(x_1, x_2) = \frac{1}{3}$ if $x_1 = x_2$ otherwise $\frac{1}{6}$, $Y = (X_1 + X_2)$ *modulo* 2, $Y_1 = (X_2 + Z_2)$ *modulo* 2, and $Y_2 = (X_1 + Z_1)$ *modulo* 2, where Z_1, Z_2 are equiprobable binary random variables independent of each other X_1, X_2, and Y.

If Y_1, Y_2, Y are absent then the rate region is $R_1 \geq 0.9184$, $R_2 \geq 0.9184$, and $R_1 + R_2 \geq 1.9184$. This provides the compression due to correlation, without which the minimum required rates are $R_1 = R_2 = 1$. If side information Y only is provided then we get $R_1 \geq 0$, $R_2 \geq 0$, and $R_1 + R_2 \geq 1.0$. If Y and Y_1 only are provided then we need $R_1 + R_2 \geq 0.811$, and if Y_1, Y_2, and Y are provided then $R_1 + R_2 \geq 0.719$.

B. Rate Distortion Region when $X_1 \perp X_2 \,|(Y_1, Y_2, Y)$

Theorem 4.2: If X_1 and X_2 are conditionally independent given (Y_1, Y_2, Y), then the rate pair (R_1, R_2) is achievable if and only if

$$\begin{aligned}
&R_1 \geq I(X_1; W_1 | Y_1) - I(W_1; Y_2 Y | Y_1), \\
&R_2 \geq I(X_2; W_2 | Y_2) - I(W_2; Y_1 Y | Y_2),
\end{aligned} \tag{19}$$

where W_1 and W_2 are auxiliary random variables satisfying $W_1 - X_1Y_1 - X_2Y_2Y$, $W_2 - X_2Y_2 - X_1Y_1Y$ and for which there exist functions $f_{D,1}(.)$ and $f_{D,2}(.)$ such that

$$\begin{aligned}
&\frac{1}{n} E d_1(X_1^n, f_{D,1}(W_1^n, W_2^n, Y_1^n, Y_2^n, Y^n)) \leq D_1 + \varepsilon_1, \\
&\frac{1}{n} E d_2(X_2^n, f_{D,2}(W_1^n, W_2^n, Y_1^n, Y_2^n, Y^n)) \leq D_2 + \varepsilon_1.
\end{aligned} \tag{20}$$

Proof: The theorem can be established using the observations that under the given hypothesis, $I(X_1; W_1 | W_2 Y_1 Y_2 Y) = I(X_1; W_1 | Y_1 Y_2 Y) = I(X_1; W_1 | Y_1) - I(W_1; Y_2 Y | Y_1)$ and that the term $I(W_1; W_2 | Y_1 Y_2 Y)$ in the sum rate is zero. Therefore, the sum rate bound becomes just the sum of the two side bounds, and hence can be omitted. Moreover, in this case the additional degrees of freedom in the larger Markov chain (i.e., in achievable rate region) do not lower the involved rates. ■

This result can be intuitively justified by noting that given the side information at the decoder, the two sources are independent and hence the situation becomes as good as two independent sources transmitting to a common receiver. This is further confirmed by observing that the rate distortion region in this special case is a square. Note that, this result generalizes the results of Gastpar ([9]).

V. Generalization to Multiple Sources

Let $(Y, Y_k, X_k, k \in \mathcal{S})$ be a discrete memoryless source with joint probability distribution $p(y, y_k, x_k, k \in \mathcal{S})$ on $\mathcal{Y} \times \prod_{i \in \mathcal{S}} (\mathcal{X}_i \times \mathcal{Y}_i)$ which is observed by $|\mathcal{S}| \geq 2$ sensors in a distributed fashion. Sensor k observes (X_k, Y_k) and communicates at rate R_k with a distant decoder which has access to $(Y, Y_k, k \in \mathcal{S})$. The goal is to obtain the minimal rates $(R_k, k \in \mathcal{S})$ needed so that $(X_k, k \in \mathcal{S})$ can be recovered with the given expected distortion.

Theorem 5.1 (Inner Bound): Given the joint distribution $p(y, y_k, x_k, k \in \mathcal{S})$, define $\mathcal{W}_{in}(D)$ as the set of random vectors $\mathbf{W} \triangleq (W_k, k \in \mathcal{S})$ satisfying the following conditions :

1) $W_k - X_kY_k - (W_{(\mathcal{S}\setminus k)}X_{(\mathcal{S}\setminus k)}Y_{(\mathcal{S}\setminus k)}Y)$, and
2) There exist functions $f_{D,k}(.)$, $k \in \mathcal{S}$ such that $\frac{1}{n}Ed(X_k^n, \hat{X}_k^n) \leq D_k + \varepsilon_1$,
where $\hat{X}_k^n = f_{D,k}(Y^n, Y_j^n, W_j^n, j \in \mathcal{S})$, $k \in \mathcal{S}$.

Let $R(\mathbf{W})$ be the set of all rate vectors $(R_k, k \in \mathcal{S})$ such that

$$\sum_{i \in \mathcal{A}} R_i \geq I(X_{\mathcal{A}}; W_{\mathcal{A}}|W_{\mathcal{A}^c}Y_SY), \qquad \text{for all } \mathcal{A} \subset \mathcal{S} \tag{21}$$

then $\mathcal{R}_{in}(\mathbf{D}) \triangleq \overline{conv}\{\bigcup_{\mathbf{W}\in\mathcal{W}_{in}(\mathbf{D})} R(\mathbf{W})\} \subset \mathcal{R}(D)$.

The proof of this theorem is similar to that of Theorem 2.1. In particular, after fixing the auxiliary random variables and decoding functions satisfying the distortion constraints, encoder k generates $2^{nR_k'}$ codewords of length n, sampled iid from the marginal distribution $p(w_k)$ and randomly assigns them to 2^{nR_k} bins. This information is sent to the decoder. After observing (x_k^n, y_k^n), it looks for a jointly typical triplet (x_k^n, y_k^n, w_k^n) and sends the index of the bin to which w_k^n belongs. The decoder looks for a $|\mathcal{S}|$ tuple in the bins indexed by the received indices which is jointly typical with $(Y, Y_k^n, k \in \mathcal{S})$ and calculates the estimates from the decoder functions $f_{D,k}$ if there exists a unique choice. Otherwise, an error is declared incurring the maximum distortion. The probability of error can be analyzed exactly as in Theorem 2.1 resulting in $(2^{|\mathcal{S}|} - 1)$ rate constraints, one for each possible error.

Finally, let us note that this inner bound can be specialized to the results of Gastpar [9], and Berger and Tung [4]. Gastpar's results can be obtained by taking $(Y_k, k \in \mathcal{S}) \perp (Y, X_k, k \in \mathcal{S})$, while those of Berger and Tung by putting $(Y, Y_k, k \in \mathcal{S}) \perp (X_k, k \in \mathcal{S})$ and $\mathcal{S} = \{1, 2\}$. Since, we were not able to find an inner bound on the multiterminal source coding problem (Berger and Tung's results) for the case of $|\mathcal{S}| > 2$ in the existing literature, we write it as a corollary for further reference.

Corollary 5.2 (Multiterminal Source Coding inner bound): Given the joint distribution $p(x_k, k \in \mathcal{S})$, define $\mathcal{W}_{in}(D)$ as the set of random vectors $\mathbf{W} \triangleq (W_k, k \in \mathcal{S})$ satisfying the following conditions :

1) $W_k - X_k - (W_{(\mathcal{S}\setminus k)}X_{(\mathcal{S}\setminus k)})$, and
2) There exist functions $f_{D,k}(.)$, $k \in \mathcal{S}$ such that $\frac{1}{n}Ed(X_k^n, \hat{X}_k^n) \leq D_k + \varepsilon$,
where $\hat{X}_k^n = f_{D,k}(W_j^n, j \in \mathcal{S})$, $k \in \mathcal{S}$.

Let $R(\mathbf{W})$ be the set of all rate vectors $(R_k, k \in \mathcal{S})$ such that

$$\sum_{i \in \mathcal{A}} R_i \geq I(X_{\mathcal{A}}; W_{\mathcal{A}}|W_{\mathcal{A}^c}), \qquad \text{for all } \mathcal{A} \subset \mathcal{S} \tag{22}$$

then $\mathcal{R}_{in}(\mathbf{D}) \triangleq \overline{conv}\{\bigcup_{\mathbf{W}\in\mathcal{W}_{in}(\mathbf{D})} R(\mathbf{W})\} \subset \mathcal{R}(D)$. ■

The outer bound for the region $\mathcal{R}(D)$ is given in

Theorem 5.3 (Outer Bound): Given the joint distribution $p(y, y_k, x_k, k \in \mathcal{S})$, define $\mathcal{W}_{out}(D)$ as the set of random vectors $\mathbf{W} \triangleq (W_k, k \in \mathcal{S})$ satisfying the following conditions :

1) $W_k - X_kY_k - (X_{(\mathcal{S}\setminus k)}Y_{(\mathcal{S}\setminus k)}Y)$, and
2) There exist functions $f_{D,k}(.)$, $k \in \mathcal{S}$ such that $\frac{1}{n}Ed(X_k^n, \hat{X}_k^n) \leq D_k + \varepsilon$,
where $\hat{X}_k^n = f_{D,k}(Y^n, Y_j^n, W_j^n, j \in \mathcal{S})$, $k \in \mathcal{S}$.

Let $R(\mathbf{W})$ be the set of all rate vectors $(R_k, k \in \mathcal{S})$ such that

$$\sum_{i \in \mathcal{A}} R_i \geq I(X_{\mathcal{S}}; W_{\mathcal{A}}|W_{\mathcal{A}^c}Y_SY), \qquad \text{for all } \mathcal{A} \subset \mathcal{S} \tag{23}$$

then $\mathcal{R}_{out}(\mathbf{D}) \triangleq \{\bigcup_{\mathbf{W}\in\mathcal{W}_{out}(\mathbf{D})} R(\mathbf{W})\} \supset \mathcal{R}(D)$.

Proof: Similar to the two-source case. ■

VI. CONCLUSIONS

In this paper, we investigated the distributed lossy compression with various assumptions on the side information at the encoders and the decoder. The side information at the encoders can be interpreted as the information exchanged among the encoders (sensor nodes), or the information relayed by them to the decoder. We derived inner and outer bounds on the rate distortion region of our problem and further obtained exact rate distortion region in two important special cases. Our results generalize various results known in distributed source coding. The results are of particular interest to distributed compression and signal processing in low power sensor networks.

REFERENCES

[1] D. Slepian and J. K. Wolf, "Noiseless coding of correlated information sources," *IEEE Trans. Inform. Theory, IT-19:471-480*, 1973.

[2] T. M. Cover, "A proof of the data compression theorem of slepian and wolf for ergodic sources," *IEEE Trans. Inform. Theory, IT-22:226-228*, 1975.

[3] A. Wyner and J. Ziv, "The rate distortion function for source coding with side information at the receiver," *IEEE Trans. Inform. Theory, IT-22:1-11*, 1976.

[4] T. Berger, *Multiterminal Source Coding in The Information Theory Approach to Communications*, G. Longo, Ed. Springer-Verlag, N.Y., 1977.

[5] M. Fleming and M. Effros, "Rate-distortion with mixed types of side information," *ISIT*, June 2003.

[6] A. H. Kaspi and T. Berger, "Rate-distortion for correlated source with partially separated encoders," *IEEE Trans. Inform. Theory, IT-28:828-840*, 1982.

[7] T. Berger and R. W. Yeung, "Multiterminal source coding with one distortion criterion," *IEEE Trans. Inform. Theory, IT-35:228-236*, 1989.

[8] Y. Oohama, "Gaussian multiterminal source coding," *IEEE Trans. Inform. Theory, IT-43:1912-1923*, 1997.

[9] M. Gastpar, "The wyner-ziv problem with multiple sources," *IEEE Trans. Inform. Theory, IT-50:2762-2768, Nov. 2004*, November 2004.

[10] T. Berger, Z. Zhang, and H. Viswanathan, "The ceo problem," *IEEE Trans. Inform. Theory, IT-42:887-902*, May 1996.

[11] Y. Oohama, "The rate distortion function for quadratic gaussian ceo problem," *IEEE Trans. Inform. Theory, IT-44:1057-1070*, May 1998.

[12] S. J. Baek, G. Veciana, and X. Su, "Minimizing energy consumption in large-scale sensor networks through distributed data compression and hierarchical aggregation," *IEEE JSAC, Vol 22, No 6*, Aug 2004, pp. 1130-1140.

[13] T. M. Cover, A. E. Gamal, and M. Salehi, "Multiple access channels with arbitrarily correlated sources," *IEEE Trans. Inform. Theory, IT-26:648-657*, 1980.

[14] V. K. Varshneya and V. Sharma, "Lossy coding for multiple access communication with various side information," *Accepted for publication in IEEE WCNC 2006*.

[15] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. John Wiley, N.Y., 1991.