# Limit Laws for Coverage in Backbone-Sensor Networks

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**Abstract.** We study the coverage in sensor networks having two types of nodes, sensor and backbone nodes. Each sensor is capable of transmitting information over relatively small distances. The backbone nodes collect information from the sensors. This information is processed and communicated over an ad-hoc network formed by the backbone nodes, which are capable of transmitting over much larger distances. We consider two modes of deployment of sensors, one a Poisson-Poisson cluster model and the other a dependently-thinned Poisson point process. We deduce limit laws for functionals of vacancy in both models using properties of association for random measures.

**Keywords.** Poisson cluster process, ergodic, associated random measure, coverage, sensor networks

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# 1 Introduction

A sensor is a device that measures a physical quantity over a region and converts it into a signal which can be read by an instrument or an observer. The union of all such sensing regions in the sensor field is the coverage provided by the sensor network. Coverage of a sensor network provides a measure of the quality of surveillance that the network can provide. Some of the common applications of sensor network includes environmental monitoring, emergency rescue, ambient control and surveillance networks. Sensor nodes being deployed randomly, one typically models its location by a point process in an appropriate space. The sensing region across each sensor is described by a sequence of independent and identically distributed random sets. Hence sensor network coverage is generally analyzed by an equivalent "coverage process".

One of the main constraints in a sensor network is that the sensors have limited power and hence can transmit information only over short distances. They send the information sensed to some nearby base station or cluster head. The base stations form a backbone network that relays the information received from the sensors over larger distances. We refer to these special nodes as the backbone nodes. Another characteristic of sensor networks is the large number of nodes that are several orders of magnitude larger than those in ad-hoc networks. Thus it is natural to study limit laws for coverage in sensor networks.

We consider two deployments for the sensor and the backbone nodes. In the first model the backbone nodes are distributed according to a homogenous Poisson point process. Sensors are deployed independently around each backbone node according to another Poisson point process, thus giving rise to a Poisson-Poisson cluster model. In the second model the sensors and the backbones are deployed according to two independent Poisson processes. However, only the sensors that are within a certain distance from some backbone node can communicate the information sensed by them over the network. Such a scenario arises, for instance when the sensors have limited life and new sensors have to be deployed repeatedly.

For a detailed survey of the various issues relating to sensor networks, we refer the reader to [1]. For applications of such models to coverage and target tracking we refer the readers to [5], [6] and the references therein. Limit laws for coverage for Poisson point processes have been derived in [9] where these results are derived via a series of elaborate computations. We adopt the approach in [8] where these results are derived by first showing that the vacancy measure is an associated random measure and then using the properties of such random measures.

In order to describe our models precisely and state our results we need some notations.

## 1.1 Notations

Let $\mathbb{R}^d, d \geq 1$ be endowed with the usual Euclidean metric. Let $M(\mathbb{R}^d)$ be the space of Radon measures on $\mathbb{R}^d$ (all non-negative measures that are finite on bounded subsets) topologised by vague convergence, and let $\mathcal{B}(\mathbb{R}^d)$ be the corresponding Borel sigma field. $M_p(\mathbb{R}^d)$ denote the subspace of $M(\mathbb{R}^d)$ consisting of integer valued measures. For any measure $\mu \in M(\mathbb{R}^d)$ and $f \in L^1(\mu)$, we denote $\int_{\mathbb{R}^d} f d\mu$ by $\mu(f)$. $m(\cdot)$ will denote the Lebesgue measure on $\mathbb{R}^d$ and $\|\cdot\|_2$ is the $L_2$ norm with respect to $m$. For $c > 0$, let $B(x, c)$ denote the $d-$dimensional ball centered at $x \in \mathbb{R}^d$ of radius $c$.

*Definition* 1.1. A random measure is an $M(\mathbb{R}^d)$ valued random variable defined over some underlying probability space $(\Omega, \mathcal{F}, P)$.

For an extensive treatment of the theory of random measures we refer the reader to [10]. Define the partial order ($\leqq$) on $M(\mathbb{R}^d)$ by

$$\mu \leqq \nu \text{ if } \mu(B) \leq \nu(B), \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^d). \tag{1.1}$$

We say a that map $f : M(\mathbb{R}^d) \mapsto \mathbb{R}$ is non-decreasing (respectively non-increasing) if

$$\mu \leqq \nu \Rightarrow \mu(f) \leq \nu(f) \text{ (respectively } \mu(f) \geq \nu(f)). \tag{1.2}$$

*Definition* 1.2. An $M(\mathbb{R}^d)$ valued random measure $X$ is said to be associated if for each pair of bounded, Borel measurable, non-decreasing functions $f, g : M(\mathbb{R}^d) \mapsto \mathbb{R}$ we have,

$$Cov(X(f), X(g)) \geq 0. \tag{1.3}$$

Let $\mathcal{O}$ denote the space of all open subsets of $\mathbb{R}^d$. For any compact subset $K$ of $\mathbb{R}^d$ let $\mathcal{O}_K$ be the set of relatively open subsets of $K$. Let $\mathcal{G}$ be the sigma algebra on $\mathcal{O}$ (for details see [8]).

*Definition* 1.3. Given $T \in \mathbb{R}^d$, define the $T$-shift to be the map $\tau_T : M(\mathbb{R}^d) \mapsto M(\mathbb{R}^d)$ given by $(\tau_T\mu)(A) = \mu(T + A)$. A random measure $X$ on $\mathbb{R}^d$ is stationary if $\tau_T X$ has the same law as $X$, for all $T \in \mathbb{R}^d$. If $X$ is stationary, then the $T$-shift is *ergodic for* $X$, if for all $B \in \mathcal{B}(R^d)$ such that $\tau_T B = B$, we have $P(X^{-1}B)$ is either zero or one. $X$ is ergodic if for all $T \neq 0$, the $T$-shift is ergodic for $X$.

## 1.2 Model Definitions

We now define the two models precisely.

**Model I.**

Let $\mathcal{P} \equiv \{\xi_i, i \geq 1\}$ be a homogenous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda_1 > 0$. The points of $\mathcal{P}$ may be interpreted as the location of the backbone nodes. For each $i \geq 1$, let $N_i$ be independent Poisson random variables with mean $\lambda_2 > 0$, that are also independent of $\mathcal{P}$. Each backbone node $\xi_i \in \mathcal{P}$ acts as a cluster center around which a Poisson number $N_i$

of sensors are deployed uniformly in the region $\xi_i + D$, where $D$ is a bounded subset of $\mathbb{R}^d$. Let $\{\xi_{ij}\}_{i,j\geq 1}$, be a sequence of i.i.d. random variables distributed uniformly in $D$. Given $\xi_i$ and $N_i$, $\xi_i + \xi_{ij}$, $1 \leq j \leq N_i$, represents the location of the sensors that communicate with $\xi_i$. Let $\{D_{ij}\}_{i,j\geq 1}$ be a collection of i.i.d. random open sets (for an extensive treatment of random sets we refer [11]). $\xi_i + \xi_{ij} + D_{ij}$ represents the region sensed by the sensor located at $\xi_i + \xi_{ij}$. We will assume that the collection $\{D_{ij}\}_{i,j\geq 1}$ is independent of the sensor locations and is uniformly bounded, that is, $D_{ij} \subseteq B(0, c)$ almost surely, for some constant $c > 0$.

We define the $d$-dimensional volume of the region sensed by the sensors, called the coverage of the sensor network as

$$\begin{aligned}\Phi_1 &\equiv \bigcup_{i\geq 1} \left( \bigcup_{j=1}^{N_i} (\xi_i + \xi_{ij} + D_{ij}) \right) \\ &\equiv \bigcup_{i\geq 1} (\xi_i + C_i), \end{aligned} \tag{1.4}$$

where $C_i = \bigcup_{j=1}^{N_i} (\xi_{ij} + D_{ij}), i \geq 1$, is a sequence of random open sets distributed as $C$. Let $\sigma_1$ denote the law of $C$.

**Model II.**

As in the first model, let $\mathcal{P} \equiv \{\xi_i, i \geq 1\}$ be a homogenous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda_1 > 0$ representing the location of the backbone nodes. Let

$$\Phi_3 := \bigcup_{i\geq 1} (\xi_i + Y_i), \tag{1.5}$$

where $\{Y_i\}_{i\geq 1}$ is a sequence of i.i.d. random open sets independent of $\mathcal{P}$, such that $Y_i \stackrel{d}{=} Y$. Let $\mathcal{H} \equiv \{\eta_j, j \geq 1\}$ be another Poisson point process in $\mathbb{R}^d$ with intensity $\lambda_2 > 0$, independent of $\mathcal{P}$ and of the sequence $\{Y_i\}_{i\geq 1}$. $\mathcal{H}$ represents the location of the sensor nodes. Let $\{Z_j\}_{j\geq 1}$ be a sequence i.i.d. random open sets independent of everything else such that

$Z_j \stackrel{d}{=} Z$. Let $\sigma_2$ denote the law of $Z$. We assume that the shapes $Y_i, Z_j$ are uniformly bounded, that is, $Y_i \subseteq B(0, c_1)$, and $Z_j \subseteq B(0, c_2)$, almost surely for some positive constants $c_1, c_2$. The sensor located at $\eta_j$ covers the region $\eta_j + Z_j$ and successfully transmits the sensed information from this region to the backbone located at $\xi_i$ provided $\eta_j \in \xi_i + Y_i$. Let $I := \{j \geq 1 : \eta_j \in \Phi_3\}$, be the index set of the sensors that can successfully transmit information to some backbone node. Thus the total area covered in this model will be given by

$$\Phi_2 := \bigcup_{j \in I} (\eta_j + Z_j)\,. \tag{1.6}$$

# 2 Main Results

We now define the vacancy measure for the two models defined in Section 1.2. We first study some basic properties of this measure and use these to prove an almost sure convergence result and a central limit theorem for a sequence of scaled vacancy measures. All the results will hold under the assumptions made while defining these models without further explicit mention.

For $i = 1, 2, 3$, and any $x \in \mathbb{R}^d$ define the indicator random functions

$$\chi_i(x) = \begin{cases} 1 & \text{if } x \notin \Phi_i, \\ 0 & \text{otherwise.} \end{cases}$$

For any Borel measurable set $R \subseteq \mathbb{R}^d$, define $V^{(i)}(R)$ to be the vacancy in the region $R$, arising out of the coverage process $\Phi_i$, $i = 1, 2, 3$, respectively as

$$V^{(i)}(R) := \int_R \chi_i(x)\, dx, \;\; i = 1, 2, 3. \tag{2.1}$$

$V^{(i)}(R)$, $i = 1,2$, is the $d$-dimensional volume of the region within $R$ that is not sensed in the respective models. Thus the notion of vacancy and coverage are complementary to each other. Note that the random measure $V^{(i)}$, $i = 1,2$, satisfies $V^{(i)} \leq m$, and hence for any $f \in L^1(\mu)$, we can define

$$V^{(i)}(f) := \int f(x)\, dV^{(i)}(x). \tag{2.2}$$

**Proposition 2.1.** *The expectation and variance of the vacancy measure $V^{(i)}$ in the region $R \subseteq \mathbb{R}^d$, arising out of the coverage process $\Phi_i$, $i = 1,2$ are given by*

$$E[V^{(1)}(R)] = m(R)e^{-\lambda_1 E[m(C)]}, \tag{2.3}$$

$$E[V^{(2)}(R)] = m(R)e^{-\lambda_2 E[m(Z)]\left(1-e^{-\lambda_1 E[m(Y)]}\right)}, \tag{2.4}$$

$$Var[V^{(1)}(R)] = e^{-2\lambda_1 E[m(C)]} \int_R \int_R \left[e^{\lambda_1 E[m((x_1 - x_2 + C)\cap C)]} - 1\right]\, dx_1\, dx_2, \tag{2.5}$$

$$\begin{aligned} Var[V^{(2)}(R)] &= e^{-2\lambda_2 E[m(Z)](1-e^{-\lambda_1 E[m(Y)]})} \\ &\times \int_R \int_R \left[e^{\lambda_2 E(m[(x_1 - x_2 + Z)\cap Z])(1-e^{-\lambda_1 E[m(Y)]})} - 1\right]\, dx_1\, dx_2. \end{aligned} \tag{2.6}$$

The following Theorem is the key result that will allow us to conclude the limiting results for the scaled vacancy measure.

**Theorem 2.2.** *The vacancy measure $V^{(i)}$, $i = 1,2$, is an associated, ergodic random measure.*

We now define the scaled vacancy measures whose limiting behaviour is the main result of this paper. For $T > 0$, and $f \in L^1(m) \cap L^\infty(m)$, define for $i = 1,2$, the scaled vacancy measure as

$$V_T^{(i)}(f) := V^{(i)}(f(\cdot/T)), \tag{2.7}$$

**Theorem 2.3.** *For all $f \in L^1(m)$, almost surely*

*(i)*

$$\lim_{T\to\infty} T^{-d} V_T^{(1)}(f) = e^{-\lambda_1 E[m(C)]}\ m(f), \tag{2.8}$$

*and*

*(ii)*

$$\lim_{T\to\infty} T^{-d} V_T^{(2)}(f) = e^{-\lambda_2 E[m(Z)]\left(1-e^{-\lambda_1 E[m(Y)]}\right)}\ m(f), \tag{2.9}$$

Our next result is a Central Limit Theorem for the centered and scaled vacancy measure $Z_T^{(i)}$, $i = 1, 2$, defined as

$$Z_T^{(i)} := V_T^{(i)} - E\left[V_T^{(i)}(f)\right]. \tag{2.10}$$

**Theorem 2.4.** *For all $f \in L^1(m) \cap L^\infty(m)$ as $T \to \infty$,*

$$T^{-d/2} Z_T^{(i)}(f) \xrightarrow{d} N(0, \Gamma_i \|f\|_2^2), \qquad i = 1, 2, \tag{2.11}$$

*where*

$$\Gamma_1 = e^{-2\lambda_1 E[m(C)]} \int_{\mathbb{R}^d} \left(e^{\lambda_1 E[m((x+C)\cap C)]} - 1\right)\ dx < \infty,$$

*and*

$$\Gamma_2 = e^{-2\lambda_2 E[m(Z)]\left(1-e^{-\lambda_1 E[m(Y)]}\right)} \int_{\mathbb{R}^d} \left(e^{\lambda_2 (1-e^{-\lambda_1 E[m(Y)]}) E[m((y+Z)\cap Z))]} - 1\right)\ dy < \infty.$$

We deduce a "functional" corollary of Theorem 2.4. Let $S(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$ and let $S'(\mathbb{R}^d)$ denote the corresponding dual space of tempered distributions.

**Corollary 2.5.** *Under the conditions of Theorem 2.4, the map $f \longmapsto \left[V_T^{(i)}(f) - E[V_T^{(i)}(f)]\right]$, $i = 1, 2$, $f \in S(\mathbb{R}^d)$, defines an $S'(\mathbb{R}^d)$ valued random variable. For $i = 1, 2$, as $T \to$*

$\infty$, $T^{-d/2}Z_T^{(i)}$ *converges in distribution to a Gaussian* $S'(\mathbb{R}^d)$ *valued random variable* $W$ *with mean 0 and variance given by* $E[W(f)]^2 = \Gamma_i \, \|f\|_2^2$, $f \in S(\mathbb{R}^d)$, *where* $\Gamma_i$, *are as in Theorem 2.4.*

In Model II, a sensor that gathers information may not be able to transmit the same to a backbone. We call such sensor as "inactive sensor". Similarly the backbone node located at $\xi_i$ which does not have any sensor within the region $\xi_i + Y_i$ will be termed an 'idle backbone". The next result gives the expected proportion of "idle backbones" and "inactive sensors" in a fixed region $R$.

**Proposition 2.6.** *Let* $N$ *be the number of "idle backbones" and* $M$ *be the number of "inactive sensors" in region* $R$ *in Model II. Then*

$$\frac{E[N]}{m(R)} = \lambda_1 e^{-\lambda_2 E[m(Y)]}. \tag{2.12}$$

$$\frac{E[M]}{m(R)} = \lambda_2 e^{-\lambda_1 E[m(Y)]}. \tag{2.13}$$

Using the above proposition we derive the minimum value of $E[m(Y)]$ which will ensure that there will be no "idle backbones" and "inactive sensors" with a specified large probability.

**Theorem 2.7.** *Fix* $\alpha \in (0,1)$ *and let* $\theta$ *be the volume of the unit ball in* $\mathbb{R}^d$. *Then* $Pr[N = 0] \geq \alpha$ *and* $Pr[M = 0] \geq \alpha$, *provided*

$$E[m(Y)] \geq \max\left[\left[-\frac{\ln\left(\frac{-\ln\alpha}{\lambda_1 m(R)}\right)}{\lambda_2\theta}\right]^{1/d}, \left[-\frac{\ln\left(\frac{-\ln\alpha}{\lambda_2 m(R)}\right)}{\lambda_1\theta}\right]^{1/d}\right]. \tag{2.14}$$

# 3 Proof of Main Results

*Proof of Proposition 2.1.* The simple computations for the mean and vacancy are similar to those on pp. 128, 129 [9], to which we refer the readers for details. Recall the definition of $V^{(1)}(R)$ from (2.1). By Fubini's theorem, homogeneity, symmetry of the Poisson process we have,

$$\begin{aligned} E[V^{(1)}(R)] &= \int_R P[\ x \notin \xi_i + C_i, \quad \text{for all } i \geq 1]\ dx \\ &= m(R) P[\ \xi_i \notin -C_i, \quad \text{for all } i \geq 1] \\ &= m(R) e^{-\lambda_1 E[m(C)]}. \end{aligned}$$

$$\begin{aligned} E[\chi(x_1)\chi(x_2)] &= P[x_1 \notin \xi_i + C_i \text{ and } x_2 \notin \xi_i + C_i, \text{ for all } i \geq 1] \\ &= P[\xi_i \notin x_1 - x_2 + C_i \text{ and } \xi_i \notin C_i, \text{ for all } i \geq 1] \\ &= P[\xi_i \notin (x_1 - x_2 + C_i) \cup C_i, \text{ for all } i \geq 1] \\ &= e^{-\lambda_1 E[m((x_1 - x_2 + C) \cup C)]} \\ &= e^{-2\lambda_1 E[m(C)] + \lambda_1 E[m((x_1 - x_2 + C) \cap C)]}. \end{aligned}$$

Hence,

$$Cov[\chi(x_1), \chi(x_2)] = e^{-2\lambda_1 E[m(C)]} \times \left(e^{\lambda_1 E[m((x_1 - x_2 + C) \cap C)]} - 1\right),$$

and

$$\begin{aligned} Var[V^{(1)}(R)] &= \int_R \int_R Cov[\chi(x_1), \chi(x_2)]\ dx_1\ dx_2 \\ &= e^{-2\lambda_1 E[m(C)]} \times \int_R \int_R \left(e^{\lambda_1 E[m((x_1 - x_2 + C) \cap C)]} - 1\right)\ dx_1\ dx_2. \end{aligned}$$

We now compute the mean and variance of the vacancy for Model II. Let $\{\eta_j, j \geq 1\}$ and $\{\xi_i, i \geq 1\}$ be the location of the sensors and the backbones respectively. A point $x \in \mathbb{R}^d$ is sensed or covered if we can find $i, j \geq 1$, satisfying the following two conditions,

$$x \in \eta_j + Z_j, \qquad \eta_j \in \xi_i + Y_i.$$

Define the sets

$$A_j(x) = \bigcup_{i \geq 1} \left[ (\xi_i + Y_i) \bigcap (x - Z_j) \right],$$

$$A(x) = \bigcup_{i \geq 1} \left[ (\xi_i + Y_i) \bigcap (x - Z) \right].$$

A point $x \in \mathbb{R}^d$ is not covered if for all $j \geq 1,\ \eta_j \notin A_j(x)$. Hence,

$$E[V^{(2)}(R)] = \int_R P[\eta_j \notin A_j(x), \text{ for all } j \geq 1]\, dx. \tag{3.1}$$

Again by calculations similar to those on pp. 128 [9], we have

$$P[\ \eta_j \notin A_j(x), \text{ for all } j \geq 1\ ] = e^{-\lambda_2 E[m(A(x))]}. \tag{3.2}$$

$$\begin{aligned} E[m(A(x))] &= E\left[ m\left( \bigcup_{i \geq 1} \left[ (\xi_i + Y_i) \bigcap (x - Z) \right] \right) \right] \\ &= E\left[ m\left( x - Z \right) \right] - E[V^{(3)}(x - Z)] \\ &= E\left[ m\left( x - Z \right) - m\left( x - Z \right) e^{-\lambda_1 E[m(Y)]} \right] \\ &= E\left[ m\left( Z \right) \right] \left( 1 - e^{-\lambda_1 E[m(Y)]} \right). \end{aligned} \tag{3.3}$$

From (3.1), (3.2) and (3.3) we have,

$$E[V^{(2)}(R)] = m(R) e^{-\lambda_2 E[m(Z)] \left(1 - e^{-\lambda_1 E[m(Y)]}\right)}. \tag{3.4}$$

$$
\begin{aligned}
E[\chi(x_1)\ \chi(x_2)] &= P[\eta_j \notin A_j(x_1), \quad \text{for all } j \geq 1 \text{ and } \eta_j \notin A_j(x_2), \quad \text{for all } j \geq 1] \\
&= P[\eta_j \notin A_j(x_1) \cup A_j(x_2), \quad \text{for all } j \geq 1] \\
&= e^{-\lambda_2 E[m(A(x_1) \cup A(x_2))]} \\
&= e^{-2\lambda_2 E[m(Z)](1-e^{-\lambda_1 E[m(Y)]})} e^{\lambda_2 E[m(A(x_1) \cap A(x_2))]}. \qquad (3.5)
\end{aligned}
$$

Now by calculations similar to those in (3.3),

$$
E\left[m\left(A(x_1) \cap A(x_2)\right)\right] = E\left[m\left((x_1 - x_2 + Z) \cap Z\right)\right]\left(1 - e^{-\lambda_1 E[m(Y)]}\right).
$$

Hence,

$$
Var[V^{(2)}(R)] = \int_R \int_R \left(E[\chi(x_1)\ \chi(x_2)] - E[\chi(x_1)]E[\chi(x_2)]\right)\ dx_1\ dx_2
$$

$$
= e^{-2\lambda_2 E[m(Z)](1-e^{-\lambda_1 E[m(Y)]})} \int_R \int_R \left[e^{\lambda_2 E[m((x_1-x_2+Z)\cap Z)](1-e^{-\lambda_1 E[m(Y)]})} - 1\right]\ dx_1\ dx_2.
$$

□

*Proof of Theorem 2.2.* (i) We first show that $V^{(1)}$ is associated. Let $I = [-\frac{1}{2}, -\frac{1}{2})^d, d \geq 1$, and for $N \in \mathbb{N}$, define

$$
\pi_{NI}(\xi + C) := (\xi + C) \cap NI.
$$

Let $\mathcal{P}_1$ be the Poisson point process on $\mathbb{R}^d \times \mathcal{O}$ with intensity $\lambda_1(m \times \sigma_1)$, where $\lambda_1 > 0, m$ is the Lebesgue measure and $\sigma_1$ is the distribution of $C$. The coverage process $\Phi_1$ is the random open set

$$
\Phi_1 \equiv \bigcup_{(\xi,C)\in\mathcal{P}_1} (\xi + C),
$$

with vacancy corresponding to $\Phi_1$ given by the random measure

$$
V(dx) = 1_{\mathbb{R}^d \setminus \Phi_1}(x)\ dx. \qquad (3.6)
$$

Consider the Poisson process

$$\mathcal{R}_N^{(1)} = \{\pi_{NI}(\xi + C) : \quad (\xi, C) \in \mathcal{P}_1,\ \xi \in NI\}.$$

Let $R_N^{(1)} \in \mathbb{M}_p(\mathcal{O}_{NI})$ be the measure defined as

$$R_N^{(1)}(.) = \sum_{x_i \in \mathcal{R}_N^{(1)}} \delta_{x_i}(.).$$

Since the underlying process $\mathcal{P}_1$ is an independently marked Poisson point process, $R_N^{(1)}$ is an infinitely divisible $\mathbb{M}_p(\mathcal{O}_{NI})$ valued random variable. By Theorem 1.1 [7], $R_N^{(1)}$ is an associated random measure. For any $\gamma \in \mathbb{M}_p(\mathcal{O}_{NI})$, let

$$W(\gamma) := \bigcup_{S \in Supp(\gamma)} S,$$

where $Supp(\gamma)$ is the support of the measure $\gamma$. Define the map $H_N : \mathbb{M}_p(\mathcal{O}_{NI}) \mapsto M(\mathbb{R}^d)$ by,

$$\left(H_N(\gamma)\right)(dx) := 1_{\mathbb{R}^d \setminus W(\gamma)}(x)\ m(dx). \tag{3.7}$$

Let $\gamma_1, \gamma_2 \in \mathbb{M}_p(\mathcal{O}_{NI})$, and let $\leqq_p$ denote the partial ordering in $\mathbb{M}_p(\mathcal{O}_{NI})$ (Note that the partial ordering in $\mathbb{M}_p(\mathcal{O}_{NI})$ can be defined similar to that in (1.1) with obvious modifications). Observe that

$$\gamma_2 \leqq_p \gamma_1 \Rightarrow W(\gamma_2) \subseteq W(\gamma_1) \Rightarrow H_N(\gamma_1) \leqq H_N(\gamma_2), \tag{3.8}$$

where $\leqq$ is as in (1.1). Hence the map $H_N$ is non-increasing. Since $R_N^{(1)}$ is associated by Theorem 3.2 [2],

$$V_N^{(1)} = H_N(R_N^{(1)})$$

is an associated random measure on $\mathbb{R}^d$. Since $V_N^{(1)} \to V^{(1)}$, almost surely, as $N \to \infty$, it follows by Lemma 2.2 (ii) [7] that the random measure $V^{(1)}$ is associated.

(ii) To prove the result for $V^{(2)}$, we consider the Poisson point process $\mathcal{P}_2$ on $\mathbb{R}^d \times \mathcal{O}$, with intensity $\lambda_2(m \times \sigma_2)$, where $\lambda_2, \sigma_2$ are as in the definition of Model II. The coverage process $\Phi_2$ is the random open set

$$\Phi_2 \equiv \bigcup_{(\eta,Z)\in\mathcal{P}_2} (\eta + Z),$$

with vacancy defined in a similar way as in the previous case. Consider the process

$$\mathcal{R}_N^{(2)} := \{\pi_{NI}(\eta + Z) : \quad (\eta, Z) \in \mathcal{P}_2,\ \eta \in NI\}.$$

Let

$$S_N := \{\pi_{NI}(\eta + Z), \eta \in NI, Z \in \mathcal{O}, Z \subseteq B(0, c_2)\}.$$

For $\omega = E \times S_N, E \subseteq \mathcal{O}(\omega \in \mathcal{O} \times S_N)$, we define the function $F_N : \mathcal{O} \times S_N \mapsto S_N$, as

$$F_N(\omega) := \{\pi_{NI}(\eta + Z), \eta \in NI \cap E, Z \in \mathcal{O}, Z \subseteq B(0, c_2)\}.$$

Let

$$\begin{aligned} T_N &= F_N(\Phi_3 \times \mathcal{R}_N^{(2)}) \\ &= \{\pi_{NI}(\eta + Z), \eta \in NI \cap \Phi_3, Z \in \mathcal{O}, Z \subseteq B(0, c_2)\}. \end{aligned}$$

Let $R_N^{(2)} \in \mathbb{M}_p(\mathcal{O}_{NI})$, be the measure defined as

$$R_N^{(2)}(.) = \sum_{x_i \in T_N} \delta_{x_i}(.).$$

Since $F_N$ is a non-decreasing function on $\mathcal{O} \times S_N$, $R_N^{(2)}$ is an associated random measure.

The rest of the proof is similar to the first case with appropriate modifications.

(iii) We now show that the vacancy measure $V^{(i)}$ is ergodic for $i = 1, 2$. The stationarity of $V^{(i)}, \ i = 1, 2$, follows since the underlying point process is a homogeneous Poisson point process in both cases. By Theorem 3.3 [7], it suffices to show that for any compact set $K \in \mathbb{R}^d$ and any unit vector $u \in \mathbb{R}^d$,

$$\lim_{T\to\infty} Cov[V^{(i)}(K), V^{(i)}(Tu + K)] = 0, \quad i = 1, 2. \tag{3.9}$$

$$\begin{aligned} E[V^{(i)}(K)\, V^{(i)}(Tu + K)] &= E\left[\left(\int_K \chi_i(x_1)\, dx_1\right)\left(\int_{Tu+K} \chi_i(x)\, dx\right)\right] \\ &= E\left[\left(\int_K \chi_i(x_1)\, dx_1\right)\left(\int_K \chi_i(x_2 + Tu)\, dx_2\right)\right] \\ &= E\left[\int_K \int_K \chi_i(x_1)\, \chi_i(x_2 + Tu)\, dx_1\, dx_2\right] \\ &= \int_K \int_K E\left[\chi_i(x_1)\, \chi_i(x_2 + Tu)\right]\, dx_1\, dx_2. \end{aligned} \tag{3.10}$$

By calculations leading to (2.5) and (2.6) we have respectively,

$$Cov[V^{(1)}(K), V^{(1)}(Tu + K)] = e^{-2\lambda_1 E[m(C)]} \int_K \int_K \left[e^{\lambda_1 E[m((x_1 - x_2 - Tu + C)\cap C)]} - 1\right]\, dx_1\, dx_2, \tag{3.11}$$

$$\begin{aligned} Cov[V^{(2)}(K), V^{(2)}(Tu + K)] &= e^{-\lambda_2 E[m(Z)]\left(1 - e^{-\lambda_1 E[m(Y)]}\right)} \\ &\quad \times \int_K \int_K \left(e^{\lambda_2 E[m((x_1 - x_2 - Tu + Z)\cap Z)] m\left(1 - e^{-\lambda_1 E[m(Y)]}\right)} - 1\right)\, dx_1\, dx_2. \end{aligned} \tag{3.12}$$

Since $e^x - 1 \leq xe^x$, for all $x \geq 0$, we obtain respectively from (3.11) and (3.12),

$$\begin{aligned}
&Cov[V^{(1)}(K), V^{(1)}(Tu+K)] \\
&\leq \lambda_1 \int_K \int_K E\left[m\left((x_1 - x_2 - Tu + C) \cap C\right)\right] \, dx_1 \, dx_2 \\
&\leq \lambda_1 \, (m(K))^2 \, m(B(0,c)),
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&Cov[V^{(2)}(K), V^{(2)}(Tu+K)] \\
&\leq \lambda_2 (1 - e^{-\lambda_1 E[m(Y)]}) \int_K \int_K E\left[m\left((x_1 - x_2 - Tu + Z) \cap Z\right)\right] \, dx_1 \, dx_2 \\
&\leq \lambda_2 (1 - e^{-\lambda_1 E[m(Y)]}) \, (m(K))^2 \, m(B(0,c_2)).
\end{aligned} \tag{3.14}$$

Since the shapes $C, Z$ are uniformly bounded, the integrands in (3.11) and (3.12) tends to zero, as $T \to \infty$. Hence by Bounded Convergence Theorem we have (3.9), thereby completing the proof. $\square$

*Proof of Theorem 2.3.* For $n \in \mathbb{N}$ and $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{N}^d$, set $I(n; \mathbf{k}) = \prod_{i=1}^d [0, k_i 2^{-n})$. Note that for $N \in \mathbb{N}$,

$$V_N^{(i)}(I(d; \mathbf{k})) = \sum_{r_1=0}^{N-1} ... \sum_{r_d=0}^{N-1} V\left((r_1 k_1, ..., r_d k_d) 2^{-n} + I(n; \mathbf{k})\right), \; i = 1, 2. \tag{3.15}$$

As $V^{(i)}(x + I(n; \mathbf{k})) \leq m(I(n; \bar{k}))$, for any $x \in \mathbb{R}^d$, and $i = 1, 2$. By Theorem 2.2, $V^{(i)}$'s are ergodic for $i = 1, 2$. Hence by Theorem 9, p.679 [4]

$$N^{-d} V_N^{(i)}(I(n; \mathbf{k})) \to E\left[V^{(i)}(I(n; \mathbf{k})\right], \tag{3.16}$$

almost surely, as $N \to \infty$ in $\mathbb{N}$, $i = 1, 2$. We further have

$$T^{-d} V_T^{(1)}(I(n; \mathbf{k})) \to E\left[V^{(i)}(I(n; \mathbf{k})\right], \tag{3.17}$$

almost surely, as $T \to \infty$, in $\mathbb{R}$. For $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{Z}^d$, define

$$J(n; \mathbf{k}) := \prod_{i=1}^{d} [k_i 2^{-n}, (k_i + 1) 2^{-n}).$$

Elementary inclusion exclusion principle shows that

$$T^{-d} V_T^{(1)}(J(n; \mathbf{k})) \to E\left[V^{(i)}(J(n; \mathbf{k})\right], \tag{3.18}$$

almost surely, as $T \to \infty$, when $k_1, .., k_d \geq 0$. By (2.3) and (2.4) we have respectively,

$$T^{-d} V_T^{(1)}(J(n; \mathbf{k})) \to m(J(n; \mathbf{k})) e^{-\lambda_1 E[m(C)]}, \tag{3.19}$$

and

$$T^{-d} V_T^{(2)}(J(n; \mathbf{k})) \to m(J(n; \mathbf{k})) e^{-\lambda_2 E[m(Z)]} \left(1 - e^{-\lambda_1 E[m(Y)]}\right), \tag{3.20}$$

almost surely, as $T \to \infty$, when $k_1, .., k_d \geq 0$. We observe that (3.19) and (3.20) holds similarly for other orthants, for all $\mathbf{k}$). Let $\mathcal{C}$ denote the countable class of functions of the form $\sum_{l=1}^{L} c_l 1_{J(n; k^{(l)})}$ for some $L, n \in \mathbb{N}, c_1, ..., c_L \in \mathbb{Q}$ and $\mathbf{k}^{(l)} = (k_1^l, ..., k_d^l) \in \mathbb{Z}^d$. By linearity it follows that, for all $f \in \mathcal{C}$ we have respectively,

$$T^{-d} V_T^{(1)}(f) \to e^{-\lambda_1 E[m(S)]}\ m(f), \tag{3.21}$$

and

$$T^{-d} V_T^{(2)} \to e^{-\lambda_2 E[m(Z)]} \left(1 - e^{-\lambda_1 E[m(Y)]}\right)\ m(f), \tag{3.22}$$

almost surely, as $T \to \infty$. $\mathcal{C}$ is dense in $L^1(m)$, and $T^{-d} V_T^{(1)} \leq m$, for all $T > 0$, and $i = 1, 2$. Hence (2.8) and (2.9) follows.

$\square$

*Proof of Theorem 2.4.* Let $I = [-\frac{1}{2}, -\frac{1}{2})^d$. The result follows from Theoerm 4.4 [7], and Theorem 2.2, once we show that

$$\sup_{T>0} Cov[V^{(i)}(I), V^{(i)}(TI)] = \Gamma_i < \infty, \tag{3.23}$$

where $\Gamma_i,\ i = 1, 2$ are as defined in Theorem 2.4. By calculations leading to (3.11) and (3.12) we have respectively,

$$Cov[V^{(1)}(I), V^{(1)}(TI)] = e^{-2\lambda_1 E[m(C)]} \int_I \int_{TI} \left[ e^{\lambda_1 E[m((x_1 - x_2 + C) \cap C)]} - 1 \right] \ dx_1 \ dx_2, \tag{3.24}$$

$$\begin{aligned} Cov[V^{(2)}(I), V^{(2)}(TI)] &= e^{-\lambda_2 E[m(Z)] \left(1 - e^{-\lambda_1 E[m(Y)]}\right)} \\ &\times \int_I \int_{TI} \left( e^{\lambda_2 E[m((x_1 - x_2 + Z) \cap Z)] m(1 - e^{-\lambda_1 E[m(Y)]})} - 1 \right) \ dx_1 \ dx_2. \end{aligned} \tag{3.25}$$

The quantity on the right hand side of (3.24) and (3.25) increases to $\Gamma_1$ and $\Gamma_2$ respectively, as $T \to \infty$. The random shapes $C, Y, Z$ being uniformly bounded, we have $\Gamma_i < \infty,\ i = 1, 2$, thereby completing the proof. $\square$

*Proof of Corollary 2.5.* For both the models the result follows from Theorem 3.1 [3] and Theorem 5.3 (i) [3] . $\square$

*Proof of Lemma 2.6.* Let $N$ be the number of "idle backbones" in region $R$, and let

$$Q := \bigcup_{i\geq 1}\left(\left(\eta_i + Y_i\right)\right)\bigcap R\right).$$

Let $A_m$ denote the event that there are $m$ backbones in $R$.

$$\begin{aligned}
P[N=k] &= \sum_{m=0}^{\infty} P[N=k|A_m]\times P[A_m] \\
&= \sum_{m=k}^{\infty} P[N=k|A_m]\times P[A_m] \\
&= \sum_{m=k}^{\infty}\left(e^{-\lambda_1 m(R)}\times\frac{(\lambda_1 m(R))^m}{m!}\right)\times P[k \text{ out of } m \text{ backbones } \notin Q] \\
&= \sum_{m=k}^{\infty}\binom{m}{k}\left(\frac{E[m(Q)]}{m(R)}\right)^{m-k}\left(1-\frac{E[m(Q)]}{m(R)}\right)^{k}\times\left(e^{-\lambda_1 m(R)}\times\frac{(\lambda_1 m(R))^m}{m!}\right) \\
&= \left[e^{-\lambda_1 m(R)}\times\frac{(\lambda_1 m(R))^k}{k!}\left(1-\frac{E[m(Q)]}{m(R)}\right)^{k}\right]\times\sum_{m=k}^{\infty}\frac{(\lambda_1 m(R))^{m-k}}{(m-k)!}\left(\frac{E[m(Q)]}{m(R)}\right)^{m-k} \\
&= e^{-\lambda_1 m(R)\left(1-\frac{E[m(Q)]}{m(R)}\right)}\times\frac{\left[\lambda_1 m(R)\left(1-\frac{E[m(Q)]}{m(R)}\right)\right]^k}{k!}, \qquad k=0,1,2,... \qquad (3.26)
\end{aligned}$$

Hence by Proposition 2.1, we have

$$\begin{aligned}
E[N] &= \lambda_1 m(R)\left(1-\tfrac{E[m(Q)]}{m(R)}\right) \\
&= \lambda_1 m(R) e^{\lambda_2 m(Y)}.
\end{aligned}$$

This gives us (2.12).

Similarly if we define $M$ to be the number of "inactive sensors" in region $R$, and $T := \bigcup_{i\geq 1}\left(\left(\xi_i + Y_i\right)\bigcap R\right)$ by exactly similar calculations one can show

$$P[M=k] = e^{-\lambda_2 m(R)\left(1-\frac{E[m(T)]}{m(R)}\right)}\times\frac{\left[\lambda_2 m(R)\left(1-\frac{E[m(T)]}{m(R)}\right)\right]^k}{k!}, \qquad k=0,1,2,... \qquad (3.27)$$

Hence we have,

$$E[M] = \lambda_2 m(R) e^{\lambda_1 E[m(Y)]}, \tag{3.28}$$

thereby proving (2.13). $\square$

*Proof of Theorem 2.7.* For $\alpha \in (0,1)$ we calculate the minimum value $E[m(Y)]_\alpha$ for which $P[N=0] \geq \alpha$, and $P[M=0] \geq \alpha$. Let $E^{(b)}[m(Y)]_\alpha$ and $E^{(s)}[m(Y)]_\alpha$ be the minimum value for which $P[N=0] \geq \alpha$, and $P[M=0] \geq \alpha$ respectively. Then $E[m(Y)]_\alpha = \max\{E^{(b)}[m(Y)]_\alpha, E^{(s)}[m(Y)]_\alpha\}$. From (3.26), we obtain

$$P[N=0] \geq \alpha \Rightarrow \left(E^{(b)}[m(Y)]_\alpha\right)^d \geq \left[ -\frac{\ln\left(\frac{-\ln\alpha}{\lambda_1 m(R)}\right)}{\lambda_2 \theta} \right]. \tag{3.29}$$

On simplification we have,

$$E^{(b)}[m(Y)]_\alpha = \begin{cases} \left[ -\frac{\ln\left(\frac{-\ln\alpha}{\lambda_1 m(R)}\right)}{\lambda_2 \theta} \right]^{1/d} & \text{if } \alpha \in (e^{-\lambda_1 m(R)}, 1) \\ \\ 0 & \text{if } \alpha \in (0, e^{-\lambda_1 m(R)}], \end{cases}$$

$E^{(b)}[m(Y)]_\alpha$ being the least value required to ensure $P[N=0] \geq \alpha$, for any $\alpha \in (0,1)$. A similar calculation for "inactive sensors" shows that ,

$$E^{(s)}[m(Y)]_\alpha = \begin{cases} \left[ -\frac{\ln\left(\frac{-\ln\alpha}{\lambda_2 m(R)}\right)}{\lambda_1 \theta} \right]^{1/d} & \text{if } \alpha \in (e^{-\lambda_2 m(R)}, 1) \\ \\ 0 & \text{if } \alpha \in (0, e^{-\lambda_2 m(R)}]. \end{cases}$$

Hence (2.14) follows by choosing the maximum of $E^{(b)}[m(Y)]_\alpha$ and $E^{(s)}[m(Y)]_\alpha$. $\square$

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