A novel stochastic locally transversal linearization (LTL) technique for engineering dynamical systems: Strong solutions

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Abstract

Most available integration techniques for stochastically driven engineering dynamical systems are based on stochastic Taylor expansions of the response variables and thus require numerical modelling of multiple stochastic integrals (MSI-s). Since the latter is an extremely involved numerical task and becomes inaccurate for higher level MSI-s, these methods fail to achieve an accuracy beyond a limited order. Recently, the first author has proposed a locally transversal linearization (LTL) technique that completely avoids the use of Taylor-like expansions in the construction of the integration map [Proc. Roy. Soc. Ser. A, 457 (2001) 539; Int. J. Numer. Methods Eng. 61 (2004) 764]. A crucial step in the implementation of the LTL method is to arrive at a conditionally linearized solution which is then made to transversally intersect the non-linear solution manifold in the associated phase space. The present paper is the first part of an investigation consisting of two parts to considerably simplify and numerically expedite the generation of the conditionally linearized solution without affecting the local and global error orders of the original LTL method. In particular, the derivation of the conditionally linear form of the stochastic differential equations is done in such a way that the corresponding fundamental solution matrix (and consequently its inverse) remains unchanged during the entire integration process. In this part of the work, only strong (path wise) stochastic solutions are constructed. Through formal error estimates, it is verified that the present version of the LTL method has the same error orders as its older counterpart. Further, a host of numerical examples on stochastically driven non-linear oscillators are presented to illustrate its superior computational speed and ease

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of implementation. In a companion paper related to this work, a weak form of the alternative LTL approach will be derived.

1. Introduction

Sample path-wise numerical integration of noise-driven engineering dynamical systems cannot generally be performed beyond a limited level of accuracy, especially when the noise processes are modelled using (filtered) white noises [1]. A white noise process does not have a valid graph because the associated Wiener process is only continuous and not differentiable. In addition to non-differentiability, sample increments of a Wiener process change by $O(h^{\frac{1}{2}})$ in a time interval of h. One way to generate consistently higher orders of accuracy while developing a numerical algorithm for strong (pathwise) solutions of SDEs is to use the stochastic Taylor (either Ito–Taylor or Stratonovich–Taylor) expansion [2]. The stochastic Euler method, which may be thought of as a stochastic Taylor expansion with $O(h)$ terms, has $O(h)$ and $O(h^{\frac{1}{2}})$ local and global errors of convergence respectively [3–5]. Milstein [6,7] has also proposed a strong Taylor scheme of local error $O(h)$. By adding more terms to the Milstein scheme, strong Taylor schemes of $O(h^{\frac{3}{2}})$ and $O(h^2)$ have been proposed, among others, by Wagner and Platen [8], Milstein [7] and Kloeden and Platen [9]. A comprehensive review of all these methods may be found in Kloeden and Platen [2]. However, a major obstacle in using higher order Taylor approximations is the difficulty in evaluations of first and higher order derivatives of drift and diffusion terms. This is compounded with the daunting task of numerically evaluating the multiple stochastic integrals (MSI-s). The stochastic Runge–Kutta scheme provides an alternative integration strategy. Rumelin [10] systematically investigated stochastic Runge–Kutta schemes of strong order 1.0. For SDE-s driven by a 1-dimensional Wiener process, the stochastic Heun scheme (SHS) is, by far, one of the most general as well as accurate integration scheme of strong (local) order 1.5 (see [1]). *However, for SDE-s with a higher dimensional Wiener process vector, SHS generally only yields a local error order 1.0 unless certain conditions on the gradients of the diffusion terms are met with.* Recently, several other approaches for strong and weak solutions of SDE-s have appeared in the literature. They include methods based on polynomial chaos expansions [11], stochastic Runge–Kutta using B-series [12], and a Lie algebraic approach [13,14]. A Lie algebraic integrator for a stochastically driven system may be thought of as a geometric integrator in the sense that certain intrinsic or special structures of the solution are preserved during numerical integration. This imparts a far greater reliability to this class of methods than is achievable by methods based on implicit or explicit forms of the Ito–Taylor expansion.

An implicit, semi-analytical integration method, called the locally transversal linearization (LTL), has been proposed by Roy [15] for non-linear, stochastically driven dynamical systems. The essence of the method is to locally construct a set of conditionally linear and easily integrable (non-unique) system of SDE-s such that, given a discretization of the time axis, solution vector of the linearized SDE-s transversally intersects that of the original SDE-s at the points of discretization (grid points). The discretized solution vector may then be found by locating the points of transversal intersections. A host of extensions of the method along with detailed theoretical error analyses of the linearization approach, especially as applied to problems in non-linear stochastic engineering dynamics, has been studied by the same author [16]. In its present form, the LTL-based methods involve computations of the fundamental solution matrix (FSM) and its inverse in each time step, crucial for the construction of a linearized solution. Numerical evaluation of the FSM over any time step requires the exponentiation of certain (possibly quite large) system matrices. Thus the computational effort for evaluating these functions at each time step is quite high. In order to further reduce the computational cost and to impart a greater level of flexibility in the implementation of the procedure, one must also explore ways of minimizing the evaluations of the transcendental terms, such as sine, cosine, exponential and hyperbolic functions. However, a reduction in the computational effort should not come at the cost of a loss of accuracy. Finally, the semi-analytical form of the locally linearized solution of the LTL method and the associated exponential form of the solution are pointers to the possibility of derivation of a geometric form of the method. These observations naturally provide an impetus for the development of alternative tools based on the basic philosophy of the transversal linearization approach.

This paper constitutes the first part of an investigation that considers an alternative version of the stochastic LTL method avoiding repeated and laborious evaluations of the FSM over each time step. The basic idea behind this new approach is to treat the non-linear part of the drift vector as a conditionally known constant or time dependent force vector and the state dependent (multiplicative) part of the diffusion vectors as conditionally additive diffusion vectors. In the process, the FSM of the linearized SDE-s becomes state and time-invariant. In this part of the work, only strong solutions are constructed. A rigorous error estimate for the displacement and velocity vectors suggest that the local and global error orders in this new implementation remain the same as in the ones in the previous versions of the stochastic LTL method. A limited numerical illustration of the method for a few low-dimensional non-linear oscillators demonstrates its ready applicability and computational advantages. A weak form of this alternative LTL approach will be provided in a companion paper.

2. The methodology

Consider a very general n-DOF non-linear stochastic engineering dynamical system in the following canonical form

$$\ddot{X} = A(X,\dot{X},t) + \sum_{r=1}^{q} G_r(X,\dot{X},t)\dot{W}_r(t), \tag{1}$$

where $X, \dot{X} \in \Re^{2n}, A(X,\dot{X},t) : \Re^n \times \Re^n \times \Re \to \Re^n$ is an n-dimensional drift vector function (non-linear in X and $\dot{X}$ and with or without explicit time dependence). The diffusion vector $\sum_{r=1}^{q} G_r(X,\dot{X},t)\dot{W}_r(t)$ may be split into a constant or time dependent vector $\sum_{r=1}^{q}\sigma_r(t)\dot{W}_r(t)$ and a state and (possibly) time dependent vector $\sum_{r=1}^{q} B_r(X,\dot{X},t)\dot{W}_r(t)$. Thus, the diffusion terms may be written as

$$\sum_{r=1}^{q} G_r(X,\dot{X},t)\dot{W}_r(t) = \sum_{r=1}^{q}\sigma_r(t)\dot{W}_r(t) + \sum_{r=1}^{q} B_r(X,\dot{X},t)\dot{W}_r(t). \tag{1a}$$

In what follows the pth scalar element of the n-dimensional diffusion vectors $B_r(X, \dot{X}, t)$ and $\sigma_r(X, \dot{X}, t)$ for each r would be denoted as $B_r^{(p)}(X, \dot{X}, t)$ and $\sigma_r^{(p)}(X, \dot{X}, t)$ respectively. Here, $\{W_r(t)|r = 1, 2, \ldots, q\}$ denotes a set of independently evolving Wiener processes with $W_r(0) = 0$ and $E[|W_r(t) - W_r(s)|^2] = (t - s)$, $t > s$. It may be noted that the over-dot (meaning differentiation w.r.t. time, t) over $W_r(t)$ needs to be construed in a formal sense since Wiener processes are only continuous and not differentiable in t. From this point of view, the acceleration vector $\ddot{X}$ also does not make much of mathematical sense and thus it would be preferable to cast the system of second order Eq. (1) via the following incremental form in the state space:

$$
\begin{aligned}
&\mathrm{d}X_1 = X_2\,\mathrm{d}t \\
&\mathrm{d}X_2 = A(X_1, X_2, t)\,\mathrm{d}t + \sum_{r=1}^{q} \sigma_r(t)\,\mathrm{d}W_r(t) + \sum_{r=1}^{q} B_r(X_1, X_2, t)\dot{W}_r(t),
\end{aligned}
\tag{2a}
$$

where $X_1 = \{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)}\}^T$ and $X_2 = \{x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(n)}\}^T$ are respectively the displacement and velocity vector components of the $2n$ dimensional response vector $\widehat{X} = \{X_1^T, X_2^T\}^T$. Similarly, the vectors A and σ_r may be written in terms of their scalar components as $A = \{A^{(p)}|p = 1, 2, \ldots, n\}$ and $\sigma_r = \{\sigma_r^{(p)}|p = 1, 2, \ldots, n\}$. It is assumed that these vector drift and diffusion functions are measurable with respect to all the arguments, Lipschitz continuous and have appropriate growth bounds (not necessarily linear). Thus the sample continuity of any realization of the (separable) non-linear flow $\phi_t(\omega, X_1(0), X_2(0))$ for any $\omega \in \Omega$ (Ω being the event space) is assured provided that the (norms of) initial displacement and velocity vectors, $X_1(0), X_2(0) \in R^n$ (taken to be deterministic without the loss of any generality), are bounded. In all the discussion to follow, no differentiability requirements are imposed on the drift and diffusion vectors, unless specifically mentioned on the contrary. Finally, it is assumed that the diffusion vector may be decomposed into its linear, time-invariant and non-linear, parametric constituent parts as follows:

$$
A(X_1, X_2, t) = A_l(X_1, X_2) + f_e(t) + A_n(X_1, X_2, t),
\tag{2b}
$$

where the vector $A_l^{(p)} = \{A_l^{(p)}|p = 1, \ldots, n\}$ is time invariant and linear in X_1, X_2; $f_e(t) = \{f_e^{(p)}(t)|p = 1, \ldots, n\}$ is the external excitation vector and $A_n(X_1, X_2, t) = \{A_n^{(p)}(X_1, X_2, t)|p = 1, \ldots, n\}$ is non-linear in X_1 and X_2. Moreover, A_n also contains any (non-stochastic time varying) parametric excitations that may be non-conservative in nature.

Let the subset of the time axis over $[0, T]$ be ordered such that $0 = t_0 < t_1 < t_2 < \; < t_i < \ldots < t_N = T$ and $h_i = t_i - t_{i-1}$ where $i \in Z^+$. The purpose of the stochastic LTL method is to replace the non-linear system of SDEs (2) by a suitably chosen set of L linear systems of SDE-s, wherein the ith linear system should, in a sense, be a representative of the non-linear flow over the ith time interval $T_i = (t_{i-1}, t_i]$. Such a replacement is non-unique and using the basic steps for constructing the LTL equations [15], the following system of conditionally linearized SDEs constitutes a valid LTL system corresponding to Eq. (2) over the interval T_i:

$$
\begin{aligned}
&\mathrm{d}\overline{X}_1 = \overline{X}_2\,\mathrm{d}t \\
&\mathrm{d}\overline{X}_2 = (A_l(\overline{X}_1, \overline{X}_2) + \bar{\psi}_v(t))\,\mathrm{d}t + \sum_{r=1}^{q} \sigma_r(t_i)\,\mathrm{d}W_r(t) + \sum_{r=1}^{q} B_r(X_{1,i}, X_{2,i}, t_i)\,\mathrm{d}W_r(t).
\end{aligned}
\tag{3}
$$

The conditionally known time-dependent function $\bar{\psi}_v(t)$ is defined as

$$\bar{\psi}_v(t) = A_n(X_{1,i}, X_{2,i}, t_i) + f_e(t) \text{and } X_{1,i} \stackrel{\Delta}{=} X_1(t_i), X_{2,i} \stackrel{\Delta}{=} X_2(t_i). \tag{4}$$

In Eq. (3), $\overline{X}_1, \overline{X}_2$ denote the LTL-based approximations to X_1 and X_2 respectively. It may be noted that the linear, time-invariant part of the drift vector should admit the following representation:

$$A_l(X_1, X_2) = -[K]X_1 - [D]X_2, \tag{5}$$

where $[K]$ and $[D]$ are respectively the time-invariant stiffness and damping matrices. Denoting by $u_i = (X_{1,i}, X_{2,i}, t_i)$ and $\bar{u}_i = (\overline{X}_{1,i}, \overline{X}_{2,i}, t_i)$, the state-space solutions (discrete points) to the non-linear and linearized SDE-s at $t = t_i$, one may readily observe (see [15] for details) that the corresponding tangent spaces T_{ui} and $T_{\bar{u}i}$ (constructed respectively at u_i and $\bar{u}_i$) are transversal (non-tangential). It is known that the local evolutions of solutions $\Phi_t(\omega, X_{1,i-1}, X_{2,i-1})$, $\overline{\Phi}_t(\omega, X_{1,i-1}, X_{2,i-1})$ for $t \in (t_{i-1}, t_i]$ and $\omega \in \Omega$ fixed, of the non-linear and linearized SDE-s respectively are governed by their respective tangent spaces. One may thus argue that Φ_t and $\overline{\Phi}_t$ are transversal to each other on and around $t = t_i$. It is moreover observed that the vector fields of the non-linear and linearized SDE-s (Eqs. (2a) and (3) respectively) are instantaneously identical at $t = t_i$. If it is possible to construct the solution $(\overline{X}_1, \overline{X}_2, t)$ of the linearized SDE-s (Eq. (3)) explicitly around $t = t_i$, then the unknown discrete solution $(X_{1,i}, X_{2,i}, t_i)$ may be determined as the point of intersection of the linearized and non-linear solutions at $t = t_i$. Before attempting to construct the linearized solution, it is convenient to introduce a $2n$-dimensional (conditional) forcing vector $\bar{\psi}(t)$ as

$$\bar{\psi}(t) = \{\bar{\psi}_h^T, \bar{\psi}_v^T\}^T, \tag{6}$$

where

$$\bar{\psi}_h(t) = \{0\}_{n\times 1}, \ \bar{\psi}_v(t) \text{is as defined in Eq.(4)}. \tag{7}$$

Eq. (3) has to be solved subject to the known initial condition vector $(X_{1,i-1}^T, X_{2,i-1}^T)^T \in \Re^{2n}$. Closed-form solution, $\overline{X}(t) = \{\overline{X}_1^T(t), \overline{X}_2^T(t)\}^T$, for Eq. (3) may now be constructed in terms of the unknown state vector, $\widehat{X}_i = \{X_{1,i}^T, X_{2,i}^T\}^T$, as

$$\overline{X}(t_i) = [\Phi(t_i, t_{i-1})]\left\{\widehat{X}_{i-1} + \int_{t_{i-1}}^{t_i} [\Phi^{-1}(s, t_{i-1})]\bar{\psi}(s)\,\mathrm{d}s + \int_{t_{i-1}}^{t_i} [\Phi^{-1}(s, t_{i-1})] \times \sum_{r=1}^{q} (\bar{\sigma}_r(s) + \overline{B}_{r,i})\,\mathrm{d}W_r(s)\right\}. \tag{8}$$

In the above expression, $\bar{\sigma}_r(s) = \{\{0\}, \{\sigma_r(s)\}\}^T; \overline{B}_{r,i} = \{\{0\}^T, \{B_r(X_{1,i}, X_{2,i}, t_i)\}^T\}^T$ are $2n$-dimensional diffusion vectors, obtained by pre-augmenting $\sigma_r(s)$ and $B_{r,i}$ with an n-dimensional zero vector, $\{0\}$. The superscript 'T ' stands for vector transposition. Moreover, $\Phi(t, t_{i-1})$ is the fundamental solution matrix (FSM) associated with the transversally linearized vector field of SDE-s (Eq. (3)) and has the following simple form:

$$\Phi(t, t_{i-1}) = \exp\{[A](t - t_{i-1})\}, \tag{9}$$

where the $2n \times 2n$ constant coefficient matrix, A, has the form

$$[A] = \begin{bmatrix} [0] & [I] \\ -[K] & -[D] \end{bmatrix}. \tag{10}$$

Details of evaluating the matrix exponent, as needed to obtain the RHS of Eq. (8), will be considered shortly. Presently, the desired vector, $\widehat{X}_i = \{X_{1,i}, X_{2,i}\}^T$, may be obtained by enforcing the $2n$ constraint conditions:

$$\overline{X}_{1,i} = X_{1,i} \text{ and } \overline{X}_{2,i} = X_{2,i}. \tag{11}$$

It may be noted that the imposition of equalities (Eq. (11)) implies a transversal intersection of the linearized and non-linear solution manifolds at $t = t_i$ [15]. These equalities constitute $2n$ (non-linear) algebraic equations in as many unknowns. Thus the vector $\widehat{X}_i$ as found by solving Eq. (11) is a (not necessarily unique because of a possible multiplicity of solutions owing to non-linearity) desired solution of Eq. (3) at $t = t_i$.

3. Error estimates

The sample path, $(X_1(t), X_2(t))$ traced by Eq. (2a) is, in general, different from the approximated LTL solution, $(\overline{X}_1(t), \overline{X}_2(t))$ corresponding to Eq. (3), *unless the trajectory is in a phase-independent regime* (see [15] for details). Thus, in the more common case of phase-dependent solutions (these include transient solutions), the signed instantaneous error at $t = t_i$ may be defined as the following vector in R^{2n}:

$$E_i = \{\{E^j_{1,i}\}^T, \{E^{(j)}_{2,i}\}^T\}^T = \{\{(x^{(j)}_{1,i} - \bar{x}^{(j)}_{1,i})\}^T, \{(x^{(j)}_{2,i} - \bar{x}^{(j)}_{2,i}\}^T\}^T,$$

where $j = 1, \ldots, n$, and the instantaneous Euclidean error norm is denoted as $e_i = \|(\overline{X}^T_1, \overline{X}^T_2)^T_i - (X^T_1, X^T_2)^T_i\|$. This (signed error) vector may be treated as a vector of conditional random variables such that the local initial condition vector, $(X^T_{1,i-1}, X^T_{2,i-1})^T$ is deterministic and $(X^T_{1,i-1}, X^T_{2,i-1})^T = (\overline{X}^T_{1,i-1}, \overline{X}^T_{2,i-1})^T$. For a simplicity in further presentation, the superscripts $'T'$ on $2n$-dimensional vectors are omitted in cases where there is no scope of confusion.

Definition 3.1. Let r_m and r_s respectively denote the orders of the mean and mean square of the conditional (local) error with respect to the uniformly chosen time step size, $h = t_i - t_{i-1}$. The chosen uniformity of the time step size is only for convenience of discussion to follow and may be readily extended for cases with non-uniform or adoptive step sizes. Then, one can define the following local error bounds:

$$\|E\{(X_{1,i}, X_{2,i}) - (\overline{X}_{1,i}, \overline{X}_{2,i})\}\| \leqslant Q(1 + \|(X_{1,i-1}, X_{2,i-1})\|^2)h^{r_m}, \tag{12}$$

$$[E\|(X_{1,i}, X_{2,i}) - (\overline{X}_{1,i}, \overline{X}_{2,i})\|^2]^{\frac{1}{2}} \leqslant Q(1 + \|(X_{1,i-1}, X_{2,i-1})\|^2)h^{r_s} \tag{13}$$

provided that the positive real constant $Q \in R^+$ does not depend on r_m, r_s and h.

Proposition 3.2. *Let $r_s \geqslant 1/2$ and $r_m \geqslant r_s + 1/2$. Then, one has the following bound on the global error*:

$$[E\|(X_{1,i},X_{2,i})-(\overline{X}_{1,i},\overline{X}_{2,i})|^2]^{\frac{1}{2}} \leqslant Q(1+|(X_{1,0},X_{2,0})|^2)h^{r_s-\frac{1}{2}}. \tag{14}$$

That is, the global order of accuracy of a local (i.e., single-step) method, constructed using a one-step approximation, is $r_g = r_s - 1/2$.

Proof. For proof, one may refer to the monograph by Milstein [7, pp. 12–17]). □

The essence of the error analysis to follow is based on stochastic Ito–Taylor expansions of the non-linear and conditionally linear vector fields. These expansions are in turn derived based on a repeated application of Ito's formula, which, as adapted specifically for Eq. (2), is stated below:

$$\begin{aligned} f(X_1(t),X_2(t),t) &= f(X_{1,i-1},X_{2,i-1},t_{i-1}) + \sum_{r=1}^{q}\int_{t_{i-1}}^{s} \Lambda_r f(X_1(s),X_2(s),s)\,\mathrm{d}W_r(s) \\ &\quad + \int_{t_{i-1}}^{s} Lf(X_1(s),X_2(s),s)\,\mathrm{d}s, \end{aligned} \tag{15a}$$

where f is any sufficiently smooth (scalar or vector) function of its arguments, $t \geqslant t_{i-1}$. The operators Λ_r and L as applied to the deterministic function $f(X_1, X_2, t)$ of the response vectors are given by:

$$\begin{aligned} \Lambda_r f &= \sum_{j=1}^{n}\{\sigma_r^{(j)}(t) + B_r^{(j)}(X_1,X_2,t)\}\frac{\partial f(X_1,X_2,t)}{\partial x_2^{(j)}} \\ Lf &= \frac{\partial f}{\partial t} + \sum_{j=1}^{n} x_2^{(j)}\frac{\partial f}{\partial x_1^{(j)}} + \sum_{j=1}^{n}\{A_l^{(j)} + A_n^{(j)}\}\frac{\partial f}{\partial x_2^{(j)}} \\ &\quad + 0.5\sum_{r=1}^{q}\sum_{k=1}^{n}\sum_{l=1}^{n}\left\{(\sigma_r^{(k)} + B_r^{(k)})(\sigma_r^{(l)} + B_r^{(l)})\frac{\partial^2 f}{\partial x_2^{(k)}\partial x_2^{(l)}}\right\} \end{aligned} \tag{15b,c}$$

For convenience of further discussion, it is also necessary to define a multiple stochastic integral.

Definition 3.3. A multiple stochastic integral (MSI) of the kth level is defined as

$$I_{j_1,j_2,\ldots,j_k} = \int_{t_{i-1}}^{t_i} \mathrm{d}W_{j_k}(s)\int_{t_{i-1}}^{s} \mathrm{d}W_{j_{k-1}}(s_1)\int_{t_{i-1}}^{s_1}\cdots\int_{t_{i-1}}^{s_{k-2}} \mathrm{d}W_{j_1}(s_{k-1}), \tag{16}$$

where the integers $j_1, j_2, \ldots, j_k$ take values in the set $\{0,1,2,\ldots,q\}$ and $I_{j_1,j_2,\ldots,j_k}$ is called the kth Ito multiple integral. Moreover, $\mathrm{d}W_0(s)$ is taken to indicate $\mathrm{d}s$. Moreover, the following proposition is essential.

Proposition 3.4. *One has* $E(I_{j_1,j_2,\ldots,j_k}) = 0$ *if there exists at least one* $j_l \neq 0$, $l = 1,2,\ldots,k$. *On the other hand,* $E(I_{j_1,j_2,\ldots,j_k}) = \mathrm{O}(h^k)$ if $j_l = 0 \forall l \in [0,k]$. *Moreover,*

$$\begin{aligned} &[E(I_{j_1,j_2,\ldots,j_k})^2]^{\frac{1}{2}} = \mathrm{O}(h^w), \\ &\text{where } w = \sum_{l=1}^{k}(2-\bar{j}_l)/2, \bar{j}_l = 1 \quad \text{if} \quad j_l \neq 0, \ \text{else } \bar{j}_l = 0. \end{aligned} \tag{17}$$

Proof. The first part of the above proposition regarding the mean is quite straightforward. For the second part, involving Eq. (17), reference is made to the monographs by Milstein [7] or Kloeden and Platen [2]. □

3.1. The case of only additive noises ($B_r = 0$)

To begin with, the case of purely additive stochastic excitations (possibly with either constant or time dependent coefficients), i.e., $\sigma_r(t) = \{\sigma_r^{(p)}(t)|p = 1, \ldots, n\}$, is considered.

Proposition 3.5. *Consider the LTL-based replacement given by* Eq. (3) *corresponding to the system of SDE-s in* Eq. (2a) *with* $B_r = 0$, $r = 1, \ldots q$. *Let the drift vector in* Eq. (2a) *be at least once differentiable with respect to t and at least twice differentiable with respect to all the displacement and velocity components. Moreover let the diffusion vector be at least once differentiable with respect to t. Then the exact solution of the LTL-based SDE-s of* Eq. (3) *has local error orders of* $O(h)^{2.5}$ *and* $O(h)^{1.5}$ *for displacement and velocity components respectively. The global orders of accuracy are less by order* 0.5 *than their corresponding local counterparts.*

Proof. For simplicity of the discussion to follow, the linear, time-invariant and external forcing vectors, earlier denoted via $A_l\ (X_1, X_2)$ and $f_e(t)$ respectively, are added together to form a new time-variant vector $A_f(X_1, X_2, t)$, i.e., $A_f(X_1, X_2, t) = A_l(X_1, X_2)(X_1, X_2) + f_e(t)$. Now, using the original vector field as in Eq. (2a), the displacement components, $x_1^{(p)}, p = 1, \ldots, n$, may be expanded in an Ito–Taylor series as

$$x_{1,i}^{(p)} = x_{1,i-1}^{(p)} + x_{2,i-1}^{(p)}h + \sum_{r=1}^{q} \sigma_r^{(p)}(t_{i-1})I_{r0} + \{A_l^{(p)}(X_{1,i-1}, X_{2,i-1}) + A_n^{(p)}(X_{1,i-1}, X_{2,i-1}, t_{i-1})\}$$
$$\times \frac{h^2}{2} + T_1^{(p)}, \tag{18}$$

where

$$T_1^{(p)} = \sum_{r=1}^{q} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \int_{t_{i-1}}^{s_1} \Lambda_r\{A_l^{(p)}(X_1(s_2), X_2(s_2))$$
$$+ A_n^{(p)}(X_1(s_2), X_2(s_2), s_2)\}\,\mathrm{d}W_r(s_2)\,\mathrm{d}s_1\,\mathrm{d}s + \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \int_{t_{i-1}}^{s_1} L\{A_l^{(p)}(X_1(s_2), X_2(s_2))$$
$$+ A_n^{(p)}(X_1(s_2), X_2(s_2), s_2)\}\,\mathrm{d}s_2\,\mathrm{d}s_1 ds - \sum_{r=1}^{q} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \int_{t_{i-1}}^{s_1} \dot{\sigma}_r^{(p)}(s_2)\,\mathrm{d}s_2\,\mathrm{d}W_r(s_1)\,\mathrm{d}s. \tag{19}$$

An implicit form of Eq. (18) may be written by noting that

$$A_n^{(p)}(X_{1,i-1}, X_{2,i-1}, t_{i-1}) = A_n^{(p)}(X_{1,i}, X_{2,i}, t_i) - \sum_{r=1}^{q} \int_{t_{i-1}}^{t_i} \Lambda_r A_n^{(p)}(X_1(s), X_2(s), s)\,\mathrm{d}W_r(s)$$
$$- \int_{t_{i-1}}^{t_i} LA_n^{(p)}(X_1(s), X_2(s), s)\,\mathrm{d}s \tag{20}$$

so that Eq. (18) takes the form

$$x_{1,i}^{(p)} = x_{1,i-1}^{(p)} + x_{2,i-1}^{(p)} h + \sum_{r=1}^{q} \sigma_{r,i-1}^{(p)} I_{r0} + \{A_l^{(p)}(X_{1,i-1}, X_{2,i-1}) + A_n^{(p)}(X_{1,i}, X_{2,i}, t_i)\} h^2/2 + R_1^{(p)}. \quad (21)$$

The displacement remainder is finally given as

$$R_1^{(p)} = T_1^{(p)} - \left(\sum_{r=1}^{q} \int_{t_{i-1}}^{t_i} \Lambda_r \{A_n^{(p)}(X_1(s), X_2(s), s)\} \, \mathrm{d}W_r(s) + \int_{t_{i-1}}^{t_i} L\{A_n^{(p)}(X_1(s), X_2(s), s)\} \, \mathrm{d}s \right) h^2/2. \quad (22)$$

A similar expansion for the LTL-based displacement component, $\bar{x}_{1,i}^{(p)}$ ($p = 1, 2, \ldots, n$), subject to same initial conditions, $(X_{1,i-1}, X_{2,i-1})$, leads to

$$\bar{x}_{1,i}^{(p)} = x_{1,i-1}^{(p)} + x_{2,i-1}^{(p)} h + \sum_{r=1}^{q} \sigma_{r,i-1}^{(p)} I_{r0} + [A_l^{(p)}(X_{1,i-1}, X_{2,i-1}) + A_n^{(p)}(X_{1,i}, X_{2,i}, t_i)] h^2/2 + \overline{R}_1^{(p)}, \quad (23)$$

where the displacement remainder corresponding to the LTL equation is

$$\begin{aligned}\overline{R}_1^{(p)} = &\sum_{r=1}^{q} \sum_{m=1}^{q} \int_0^t \int_0^s \int_0^{s_1} \Lambda_r \sigma_r^{(p)}(s_2) \, \mathrm{d}W_m(s_2) \, \mathrm{d}W_r(s_1) \, \mathrm{d}s \\ &+ \sum_{r=1}^{q} \int_0^t \int_0^s \int_0^{s_1} L(\sigma_r^p(s_2) \, \mathrm{d}s_2 \, \mathrm{d}W_r(s_1)) \, \mathrm{d}s \\ &+ \sum_{r=1}^{q} \int_0^t \int_0^s \int_0^{s_1} \Lambda_r A_l^{(p)}(X_1(s_2), X_2(s_2)) \, \mathrm{d}W_r(s_2) \, \mathrm{d}s_1 \, \mathrm{d}s \\ &+ \int_0^t \int_0^s \int_0^{s_1} L A_l^{(p)}(X_1(s_2), X_2(s_2)) \, \mathrm{d}s_2 \, \mathrm{d}s_1 \, \mathrm{d}s. \end{aligned} \quad (24)$$

If the constraint conditions $x_{1,i}^{(p)} = \bar{x}_{1,i}^{(p)}, x_{2,i}^{(p)} = \bar{x}_{2,i}^{(p)}$ are satisfied, then it is observed that the expansions for $x_{1,i}^{(p)}$ and $\bar{x}_{1,i}^{(p)}$ match except for the remainder terms, viz. $R_1^{(p)}$ and $\overline{R}_1^{(p)}$. Thus the error order for $\bar{x}_1^{(p)}$ is determined by the order of signed scalar measure $E_{1,i}^{(p)} = R_1^{(p)} - \overline{R}_1^{(p)}$. Taking expectation of this term followed by the use of inequality (Eqs. (12) and (17)) leads to the following estimate (for some real constant vector $\overline{Q} = \{\overline{Q}^{(p)} | p = 1, \ldots, n\}$):

$$E(|E_{1,i}^{(p)}|) \leqslant \overline{Q}^{(p)}(1 + \|X_{1,i-1}, X_{2,i-1}\|^2) h^3 \quad (25)$$

so that $r_m = 3.0$. In deriving the above, use has been made of the identity $\Lambda_r \sigma_r^{(p)}(s) = 0$. Similarly, using inequality (13) and Eq. (17) one has the following estimate of the mean square error:

$$|E(E_{1,i}^{(p)})^2|^{\frac{1}{2}} \leqslant \overline{Q}^{(p)}(1 + \|X_{1,i-1}, X_{2,i-1}\|^2) h^{2.5} \quad (26)$$

so that $r_s = 2.5$. Now, a direct use of proposition (1) (noting that $r_m = 3.0 = r_s + \frac{1}{2}$ and $r_s > 1/2$) leads to the global error order $r_g = 2.0$ for the displacement components obtained via the LTL scheme of this section. In a similar manner, one can expand the velocity components, $x_{2,i}^{(p)}$ and $\bar{x}_{2,i}^{(p)}$, respectively based on the original and LTL-based vector fields as

$$x_{2,i}^{(p)} = x_{2,i-1}^{(p)} + \sum_{r=1}^{q} \sigma_{r,i-1}^{(p)} I_r + [A_l^{(p)}(X_{1,i-1}, X_{2,i-1}) + A_n^{(p)}(X_{1,i}, X_{2,i}, t_i)]h + R_2^{(p)}, \tag{27}$$

where

$$\begin{aligned} R_2^{(p)} = &-\left[\sum_{r=1}^{q} \int_0^t \Lambda_r A_n^{(p)}(X_1(s), X_2(s), s)\,\mathrm{d}W_r(s) + \int_0^t LA_n^{(p)}(X_1(s), X_2(s), s)\,\mathrm{d}s\right] h \\ &+ \sum_{r=1}^{q}\sum_{m=1}^{q} \int_0^t\int_0^s \Lambda_m \sigma_r^{(p)}(s_1)\,\mathrm{d}W_m(s_1)\,\mathrm{d}W_r(s) + \sum_{r=1}^{q} \int_0^t\int_0^s L\sigma_r^{(p)}(s_1)\,\mathrm{d}s_1\,\mathrm{d}W_r(s) \\ &+ \sum_{r=1}^{q} \int_0^t\int_0^s \Lambda_r A_l^{(p)}(X_1(s_1), X_2(s_1), s_1)\,\mathrm{d}W_r(s_1)\,\mathrm{d}s + \sum_{r=1}^{q} \int_0^t\int_0^s \Lambda_r A_n^{(p)}\,\mathrm{d}W_r(s_1)\,\mathrm{d}s \\ &+ \int_0^t\int_0^s LA_n^{(p)}(X_1(s_1), X_2(s_1), s_1)\,\mathrm{d}s_1\,\mathrm{d}s \end{aligned} \tag{28}$$

and

$$\bar{x}_{2,i}^{(p)} = x_{2,i-1}^{(p)} + \sum_{r=1}^{q} \sigma_{r,i-1}^{(p)} I_r + [A_l^{(p)}(X_{1,i-1}, X_{2,i-1}) + A_n^{(p)}(X_{1,i}, X_{2,i}, t_i)]h + \overline{R}_2^{(l)}, \tag{29}$$

where

$$\begin{aligned} \overline{R}_2^{(p)} = &\sum_{r=1}^{q}\sum_{m=1}^{q} \int_0^t \Lambda_m \sigma_r^{(p)}\,\mathrm{d}W_m(s_1)\,\mathrm{d}W_r(s) + \sum_{r=1}^{q} \int_0^t L\sigma_r^{(p)}(s_1)\,\mathrm{d}s_1\,\mathrm{d}W_r(s) \\ &+ \sum_{r=1}^{q} \int_0^t\int_0^s \Lambda_r A_l^{(p)}(X_1(s_1), X_2(s_1))\,\mathrm{d}W_r(s_1)\,\mathrm{d}s \\ &+ \int_0^t\int_0^s LA_l^{(p)}(X_1(s_1), X_2(s_1))\,\mathrm{d}s_1\,\mathrm{d}s. \end{aligned} \tag{30}$$

It may be noted that the third term of Eq. (28) and first term of Eq. (30) are zero as diffusion vector σ is either constant or time dependent.

As before, one observes that Eqs. (27) and (29) match except for the velocity remainders, $R_2^{(p)}$ and $\overline{R}_2^{(p)}$. Denoting $E_{2,i}^{(p)} = x_{2,i}^{(p)} - \bar{x}_{2,i}^{(p)}$, the following estimates are easily derived:

$$E(|E_{2,i}^{(p)}|) \leqslant \overline{Q}^{(p)}(1 + \|X_{1,i-1}, X_{2,i-1}\|^2)h^2, \tag{31}$$

$$|E(E_{2,i}^{(l)})^2|^{\frac{1}{2}} \leqslant \overline{Q}^{(l)}(1 + \|X_{1,i-1}, X_{2,i-1}\|^2)h^{\frac{3}{2}}. \tag{32}$$

Using Proposition 3.2, it is clear that the local and global error orders for the LTL-based velocity vector, $\overline{X}_2$, are respectively given by $r_s = 1.5$ and $r_g = 1.0$. Error orders for velocity components are therefore one order less than their corresponding values for the displacement components. □

3.2. The case of multiplicative noises

Proposition 3.6. *Consider the LTL-based replacement given by* Eq. (3) *corresponding to the system of SDE-s in* Eq. (2a) *with* $B_r(X_1, X_2, t) \neq 0$, $r = 1, \ldots q$. *Let the original drift and diffusion vectors be at least once differentiable with respect to t and at least twice differentiable with respect to all the displacement and velocity components. Then the exact solution of the LTL-based SDE-s of* Eq. (3) *has local error orders of* $O(h)^{2.0}$ *and* $O(h)^{1.0}$ *for displacement and velocity components respectively. In case the diffusion vector* $B_r(X_1, X_2, t) = B_r(X_1, t)$ *is not a function of the velocity vector, then the exact solution of the LTL-based SDE-s of* Eq. (3) *has local error orders of* $O(h)^{2.5}$ *and* $O(h)^{1.5}$ *for displacement and velocity components respectively* (*as in the case of purely additive noises*). *In each case, the global orders of accuracy are less by order* 0.5 *than their corresponding local counterparts.*

Proof. Consider Eq. (2) with $B_r(X_1, X_2, t) \neq 0$ for one or more $r \in [1,q]$ and following the same logic as for the case of additive noises, an implicit form of the stochastic Taylor expansion for the displacement variable $x_{1,i}^{(p)}$ may be written as

$$x_{1,i}^{(p)} = x_{1,i-1}^{(p)} + x_{2,i-1}^{(p)}h + \left(\sum_{r=1}^{q} \sigma_{r,i-1}^{(p)} + \sum_{r=1}^{q} B_{r,i}^{(p)}\right) I_{ro} + (A_{l,i-1}^{(p)} + A_{n,i}^{(p)})h^2/2 + \Delta_{1,i}^{(p)}, \tag{33}$$

where the local error $\Delta_{1,i}^{(p)}$ is given by:

$$\begin{aligned}
\Delta_{1,i}^{(p)} = & \sum_r \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} \Lambda_r A_l^{(p)}(X(s_2))\,\mathrm{d}W_r(s_2)\,\mathrm{d}s_1\,\mathrm{d}s + \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} LA_l^{(p)}(X(s_2))\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}s \\
& - \left(\sum_r \int_{t_{i-1}}^{t_i} \Lambda_r A_n^{(p)}(X(u),u)\,\mathrm{d}W_r(u)\right)h^2/2 - \left(\int_{t_{i-1}}^{t_i} LA_n^{(p)}(X(u),u)\,\mathrm{d}u\right)h^2/2 \\
& + \sum_r \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} \Lambda_r A_n^{(p)}(X(s_2),s_2)\,\mathrm{d}W_r(s_2)\,\mathrm{d}s_1\,\mathrm{d}s \\
& + \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} LA_n^{(p)}(X(s_2),s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}s + \sum_r \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} \dot{\sigma}_r^{(p)}(s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}s \\
& - \sum_r \left(\sum_m \int_{t_{i-1}}^{t_i} \Lambda_r B_r^{(p)}(X(u),u)\,\mathrm{d}W_r(u)\right) I_{ro} - \sum_r \left(\int_{t_{i-1}}^{t_i} LB_r^{(p)}(X(u),u)\,\mathrm{d}u\right) I_{ro} \\
& + \sum_r \sum_m \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} \Lambda_m B_r^{(p)}(X(s_2),s_2)\,\mathrm{d}W_m(s_2)\,\mathrm{d}W_r(s_1)\,\mathrm{d}s \\
& + \sum_r \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}\int_{t_{i-1}}^{s_1} LB_r^{(p)}(X(s_2),s_2)\,\mathrm{d}s_2\,\mathrm{d}W_r(s_1)\,\mathrm{d}s.
\end{aligned} \tag{34}$$

A corresponding expansion of the vector field for the LTL-based displacement component, $\bar{x}_{1,i}^{(l)}$ is:

$$\bar{x}_{1,i}^{(p)} = \bar{x}_{1,i-1}^{(p)} + \bar{x}_{2,i-1}^{(l)}h + \sum_r (\sigma_{r,i-1}^{(p)} + B_r^{(p)}(X_{1,i}, X_{2,i}, t_i)) I_{r0} + (A_{l,i-1}^{(p)} + A_{n,i}^{(p)})h^2/2 + \overline{\Delta}_{1,i}^{(p)}, \tag{35}$$

where the remainder term,

$$\overline{\Delta}_{1,i}^{(p)} = \sum_r \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \int_{t_{i-1}}^{s_1} \Lambda_r A_l^{(p)}(X(s_2))\,\mathrm{d}W_r(s_2)\,\mathrm{d}s_1\,\mathrm{d}s + \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \int_{t_{i-1}}^{s_1} LA_l^{(p)}(X(s_2))\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}s$$
$$+ \sum_r \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{s} \int_{t_{i-1}}^{s_1} \dot{\sigma}_r^{(p)}(s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}s. \tag{36a}$$

In deriving the above remainder term, use has been made of the identities:

$$\Lambda_m \sigma_r^{(p)} = 0, \forall m, r[1, q], \tag{36b}$$

$$A_{n,i-1}^{(p)} = A_{n,i}^{(p)} - \sum_r \int_{t_{i-1}}^{t_i} \Lambda_n^{(p)}(X(u), u)\,\mathrm{d}W_r(u) - \int_{t_{i-1}}^{t_i} LA_n^{(p)}(X(u), u)\,\mathrm{d}u, \tag{36c}$$

$$B_{r,i-1}^{(p)} = B_{r,i}^{(p)} - \sum_r \int_{t_{i-1}}^{t_i} \Lambda_r B_r^{(p)}(X(u), u)\,\mathrm{d}W_r(u) - \int_{t_{i-1}}^{t_i} LB_r^{(p)}(X(u), u)\,\mathrm{d}u. \tag{36d}$$

With the displacement error vector defined as $E_{1,i} = \{E_{1,i}^{(p)}\} = \{x_{1,i}^{(p)} - \bar{x}_{1,i}^{(p)}\}, p = 1, \ldots, n$, one may readily see, using the identities (3) and (11), that the stochastic Taylor expansions (33) and (35) precisely match except for their respective remainders, $\Delta_1^{(p)}$ and $\overline{\Delta}_1^{(p)}$, so that one has $\{E_{1,i}^{(p)}\} = \{\Delta_1^{(p)} - \overline{\Delta}_1^{(p)}\}$. Based on propositions 1 and 2, one has

$$E(|E_{1,i}^{(p)}|) \leqslant \overline{Q}^{(p)}(1 + |X_{i-1}|^2)h^3, \tag{37}$$

$$|E(E_{1,i}^{(p)})^2|^{\frac{1}{2}} \leqslant \overline{Q}^{(p)}(1 + |X_{i-1}|^2)h^2 \tag{38}$$

and hence the local and global displacement errors are respectively $r_s = 2.0$ and $r_g = 1.5$, i.e., 0.5 less than their corresponding values in the purely additive case (Section 3.1). Thus the accuracy of the present version of the LTL method is generally less under multiplicative noises. Expansions similar to (33) through (36) may also be readily derived for the velocity components, $x_{2,i}^{(p)}$ and $\bar{x}_{2,i}^{(p)}$, and their corresponding remainders, $\Delta_2^{(p)}$ and $\overline{\Delta}_2^{(p)}$. It may consequently be shown that the local and global error orders under a general set of multiplicative noise vectors are $r_s = 1.0$ and $r_g = 0.5$.

The Special Case of $B_r(X_1, X_2, t) = B_r(X_1, t)$:

It is however interesting to note that in the specific case of multiplicative coefficients being functions of displacements alone, i.e., $B_r(X_1, X_2, t) = B_r(X_1, t)$, the first and third terms of order $O(h^2)$ in Eq. (34) vanish (via Eq. 15(b)) and thus $r_s = 2.5$ and $r_g = 2.0$, which is the same as for the purely additive case. Similarly, it can be shown that the error order for the velocity vector is same as that of the purely additive case, i.e., $r_s = 1.5$ and $r_g = 1.0$. □

It is worth noting at this stage that, unlike the stochastic Heun method, no restrictions are imposed in this study on the number of independent white noise processes driving the system for the above error estimates to hold true. For a ready reference, the error orders for different cases as discussed in Sections 3.1 and 3.2 are shown in Table 1.

Table 1
Error orders in the new stochastic LTL method

State space vector	Local error			Global error		
	Additive noise $B_r = 0$	Multiplicative noise $B_r = B_r(X, \dot{X}, t)$	Multiplicative noise $B_r = B_r(X, t)$	Additive noise $B_r = 0$	Multiplicative noise $B_r = B_r(X, \dot{X}, t)$	Multiplicative noise $B_r = B_r(X, t)$
Displacement	2.5	2.0	2.5	2.0	1.5	2.0
Velocity	1.5	1.0	1.5	1.0	0.5	1.0

4. Certain computational issues

Computations of the fundamental solution matrix and its inverse, crucial for the construction of a linearized solution, require exponentiation of certain (possibly quite large) system matrices. For $t_{i-1} < t < t_i$, let it be required to numerically obtain $[\Phi(t, t_{i-1}) = \exp\{[A](t - t_{i-1})\}$. An 'exact' evaluation of $[\Phi(t, t_{i-1})]$ needs the determination of eigenvalues and eigenvectors of $[A]$. Since this is generally a computationally expensive task, especially for higher dimensional systems, a more expedient way to evaluate $[\Phi\ (t, t_{i-1})]$ would be via a suitable truncation of the deterministic Taylor expansion followed by the retention of the first few terms (may be four or even fewer) as

$$[\Phi(t, t_{i-1})] = [U] + [A](t - t_{i-1}) + \frac{1}{2!}[A]^2(t - t_{i-1})^2 + \frac{1}{3!}[A]^3(t - t_{i-1})^3 + \cdots \tag{39}$$

where $[U]$ is an $2n \times 2n$ identity matrix. A similar scheme may also be utilized to obtain the inverse $\Phi^{-1}(t, t_{i-1}) = \exp\{-[A](t - t_{i-1})\}$. In this case,

$$[\Phi^{-1}(t, t_{i-1})] = [U] - [A](t - t_{i-1}) + \frac{1}{2!}[A]^2(t - t_{i-1})^2 - \frac{1}{3!}[A]^3(t - t_{i-1})^3 + \cdots \tag{40}$$

The minimum number of terms in the Taylor expansion of the FSM should be dictated by the associated error orders of displacement and velocity components. It needs to be mentioned here that a truncated Taylor expansion for computing matrix exponentials is certainly not the best way to do so. Indeed, 19 different ways of computing matrix exponentials have been reviewed in a recent article [17]. While the expansion scheme in Eqs. (39) and (40) does not interfere with the formal orders of accuracy of the LTL method, it may indeed reduce the stochastic numerical stability of the method. Moreover, it needs to be noted that the concept of transversal linearization has the potential of being modified into a geometrical integration method, wherein the exponential form of the solution could be forced to evolve on a Lie manifold. In such a case, to be considered in a future article, the scheme for computing matrix exponentials must be modified.

Implementation of the linearization methods requires an appropriate modelling of the vector Wiener process $\{W_r(t)|r = 1, \ldots, q\}$. This is numerically done by independently generating $N(0,1)$ a set of Gaussian random vectors $\{g_i^{(r)}|r = 1, \ldots, q\}$ for every positive integer i (including zero) corresponding to the time interval t_i. One thus has

$$\Delta W_{r,i} = W_r(t_i) - W_r(t_{i-1}) = g_i^{(r)}\sqrt{h}, \tag{41}$$

where $h = t_i - t_{i-1}$ is the uniform time step size.

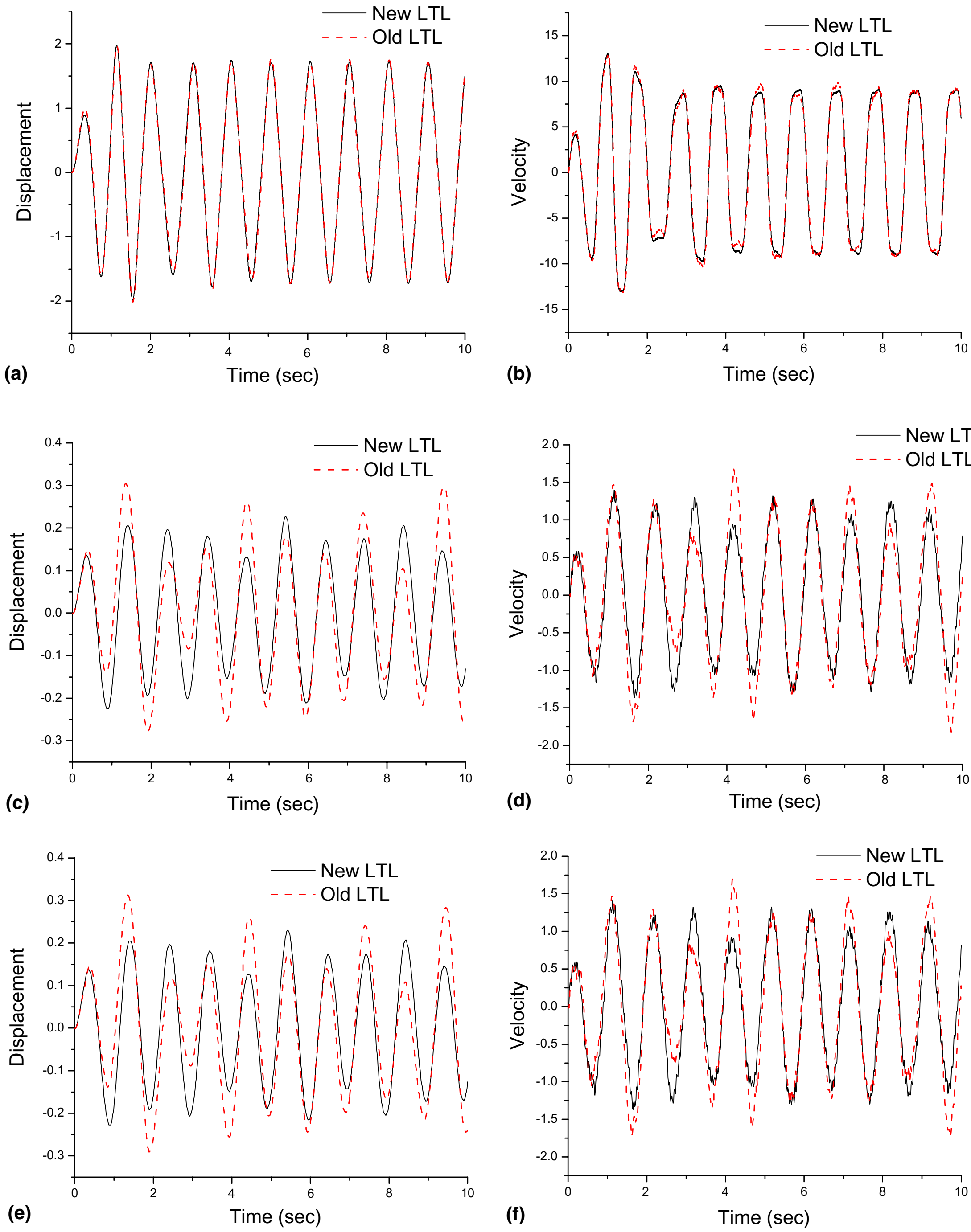

Fig. 1. Displacement and velocity plots via old and new LTL methods; $\varepsilon_2 = 0.5$, $\varepsilon_3 = 0.1$, $h = 0.01$ s. (a,b): $\varepsilon_4 = 0.1$, $\varepsilon_5 = 0.0$; (c,d): $\varepsilon_4 = 0.5$, $\varepsilon_5 = 0.0$; (e,f): $\varepsilon_4 = 0.5$, $\varepsilon_5 = 0.5$.

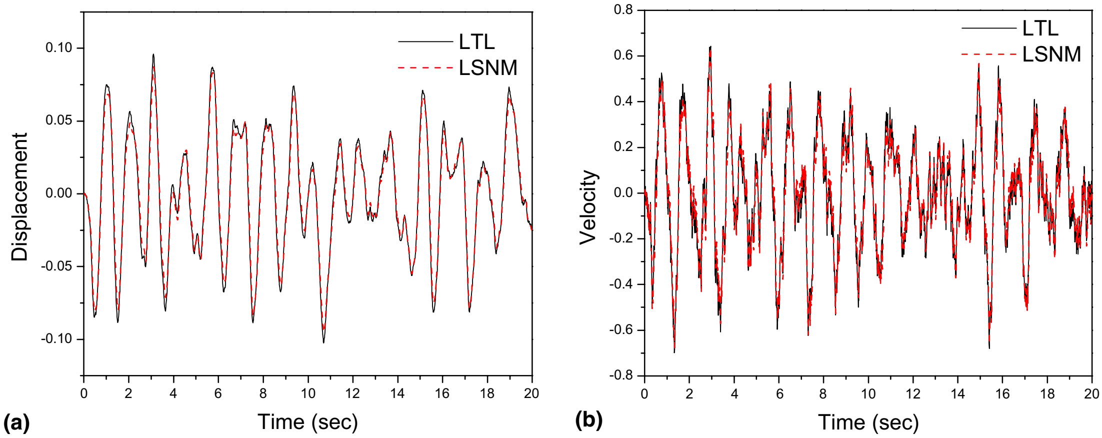

Fig. 2. Duffing oscillator under weak additive noise: (a) displacement histories and (b) velocity histories ($\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 0.0$, $\varepsilon_4 = 0.5$, $\varepsilon_5 = 0.0$).

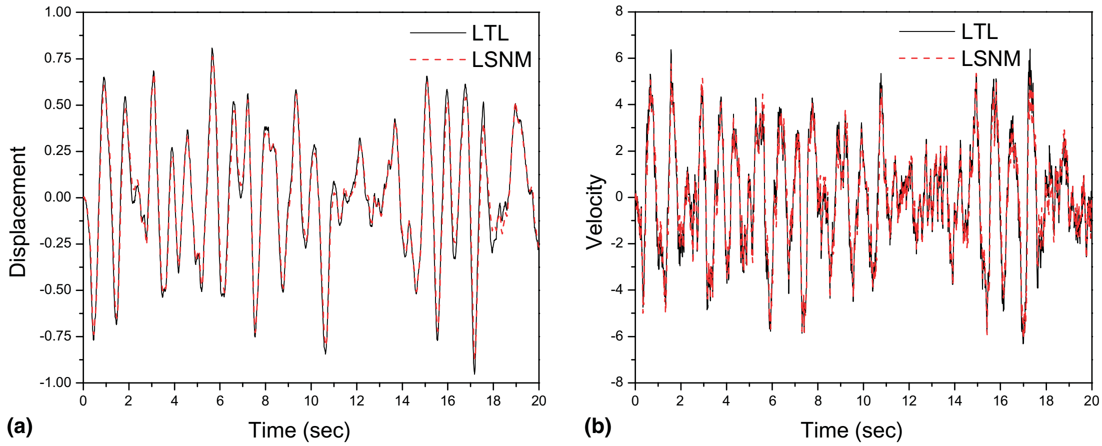

Fig. 3. Duffing oscillator under medium additive noise: (a) displacement histories and (b) velocity histories ($\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 0.0$, $\varepsilon_4 = 5.0$, $\varepsilon_5 = 0.0$).

It may be noted from Eq. (8) that the construction of particular solutions for the linearized equations involve determining the Gaussian stochastic integrals of the forms $\int_{t_{i-1}}^{t_i} \psi(s)\,\mathrm{d}W_r(s)$ for $u,r = 1,2,\ldots,q$. Interpreted according to Ito, expectations of both these integrals are zero. Moreover, one has for the standard deviation of the integral

$$\sigma_1 = \left\{ E\left(\left| \int_{t_{i-1}}^{t_i} \psi(s)\,\mathrm{d}W_r(s) \right|^2 \right) \right\}^{\frac{1}{2}} = \left\{ \int_{t_{i-1}}^{t_i} \psi^2(s)\,\mathrm{d}s \right\}^{\frac{1}{2}}. \tag{42}$$

In this work, the RHS of the above equation is generated using a 3-point Gauss quadrature and finally the stochastic integral is obtained as $\int_{t_{i-1}}^{t_i} \psi(s)\,\mathrm{d}W_r(s) = g_i^{(r)}\sigma_1$.

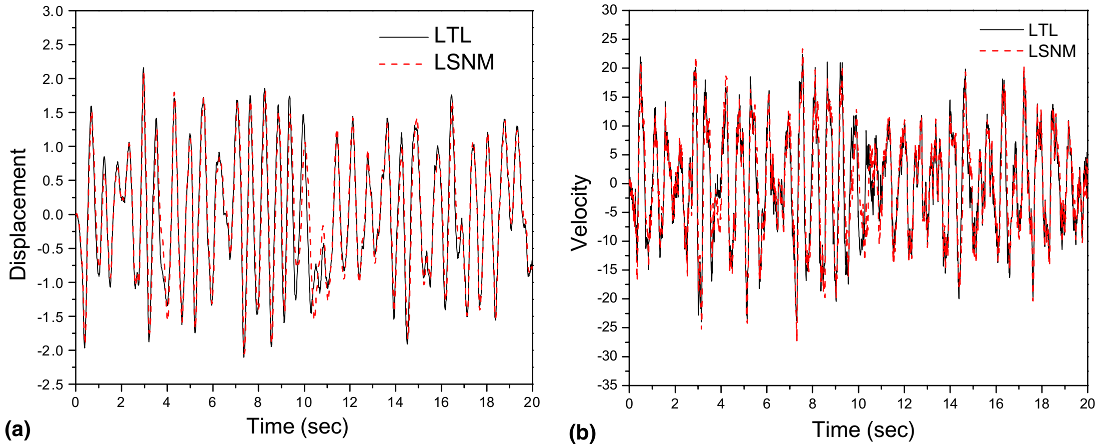

Fig. 4. Duffing oscillator under strong additive noise: (a) displacement histories and (b) velocity histories ($\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 0.0$, $\varepsilon_4 = 20.0$, $\varepsilon_5 = 0.0$).

5. The basic or old LTL method

In order to appreciate the development of the present LTL methodology, a passing reference may be made to the basic or the old LTL method developed by the first author. The Eq. (1) may be cast via the following incremental form in the state space:

$$\begin{aligned} \mathrm{d}X_1 &= X_2\,\mathrm{d}t \\ \mathrm{d}X_2 &= \{A_l(X_1, X_2) + f(t) + A_n(X_1, X_2, t)\}\,\mathrm{d}t + \sum_{r=1}^{q}\{\sigma_r(t) + B_r(X_1, X_2, t)\}\,\mathrm{d}W_r(t), \end{aligned} \tag{43}$$

where the suffices l and n stand for linearized and non-linear vector fields. First, it is assumed that the non-linear vector function $A_n(X_1, X_2, t)$ may be decomposed as

$$\{A_n(X_1, X_2, t)\} = -[C_1(X_1, X_2, t)]\{X_2\} - [K_1(X_1, X_2, t)]\{X_1\}, \tag{44}$$

where C_1 and K_1 stand respectively for state and time dependent damping and stiffness matrices. Moreover, the linear part of the drift vector, $A_l(X_1, X_2)$, may be written as

$$\{A_l(X_1, X_2)\} = -[C]\{X_2\} - [K]\{X_1\}, \tag{45}$$

where $[C]$ and $[K]$ are respectively the damping and stiffness matrices with constant coefficients. The linearized SDE-s at $t = t_i$ according to the old LTL scheme take the form

$$\begin{aligned} \mathrm{d}\overline{X}_1 &= \overline{X}_2\,\mathrm{d}t \\ \mathrm{d}\overline{X}_2 &= -[[C] + [C_1(X_{1,i}, X_{2,i}, t_i)]]\{\overline{X}_2\}\,\mathrm{d}t - [[K] + [K_1(X_{1,i}, X_{2,i}, t_i)]]\{\overline{X}_1\}\,\mathrm{d}t \\ &\quad + \{f(t)\}\,\mathrm{d}t + \sum_{r=1}^{q}\{\sigma_r(t) + B_r(X_{1,i}, X_{2,i}, t)\}\,\mathrm{d}W_r(t). \end{aligned} \tag{46}$$

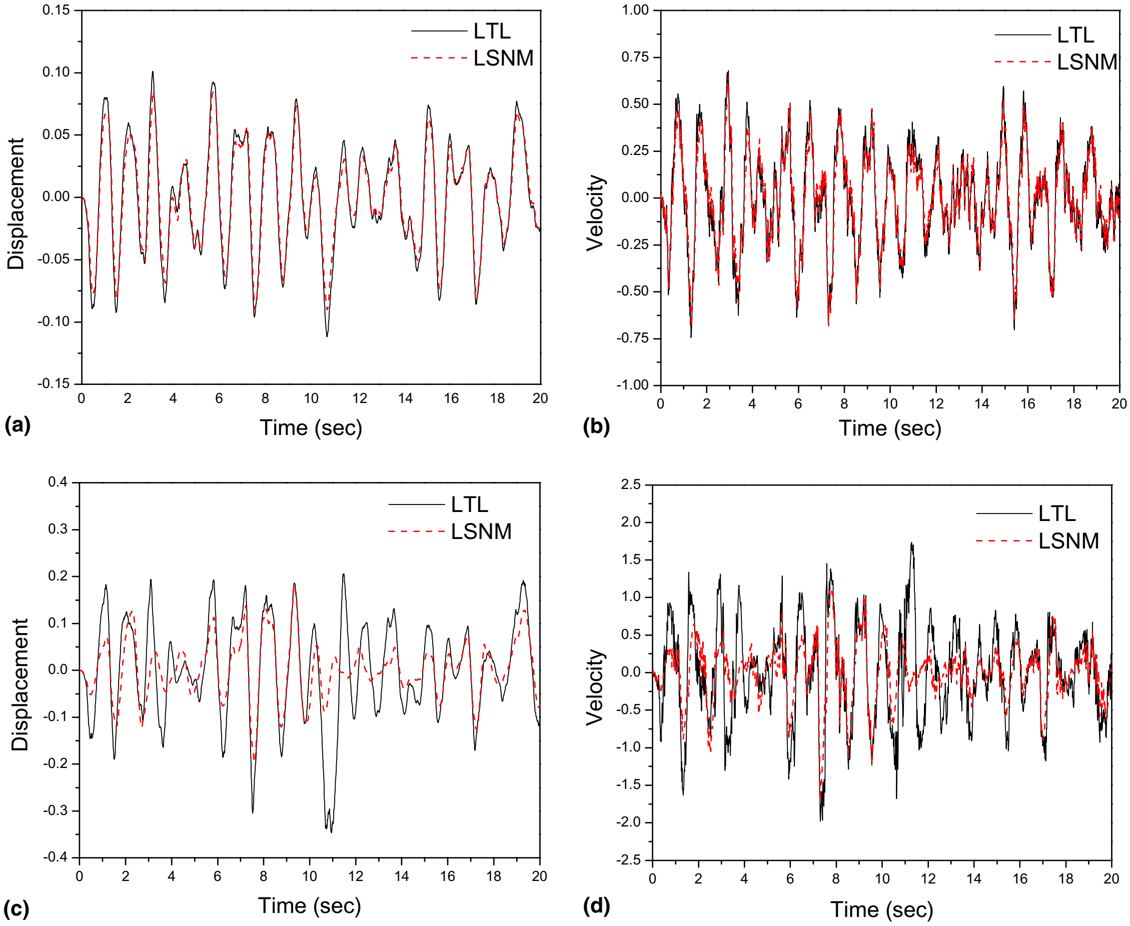

Fig. 5. Duffing oscillator under combined additive and multiplicative noises: (a) displacement histories $\varepsilon_5 = 1.0$; (b) velocity histories $\varepsilon_5 = 1.0$; (c) displacement histories $\varepsilon_5 = 10.0$; (d) velocity histories $\varepsilon_5 = 10.0$: $\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 0.0$, $\varepsilon_4 = 0.5$.

Unlike the case of the new LTL approach, the system coefficient matrix A $(X_{1,i}, X_{2,i}, t_i)$ is a function of the unknown, discretized state variables $X_{1,i}$ and $X_{2,i}$. Accordingly, for the old LTL case, the FSM is given by

$$\Phi(x_i, t) = \exp\{[M(X_i, t_i)](t - t_{i-1})\}, \tag{47}$$

where the $2n \times 2n$ coefficient matrix, M, has the form

$$[M] = \begin{bmatrix} [0] & [I] \\ [C] & [C_1] \end{bmatrix}. \tag{48}$$

One readily observes the need to form the above matrix for every t_i, $i \in Z^+$. Such a laborious process of repeated evaluations of the FSM is the main reason for the old LTL approach to become computationally expensive, especially as compared with its new counterpart. This point will be further elaborated through an example problem in the next section.

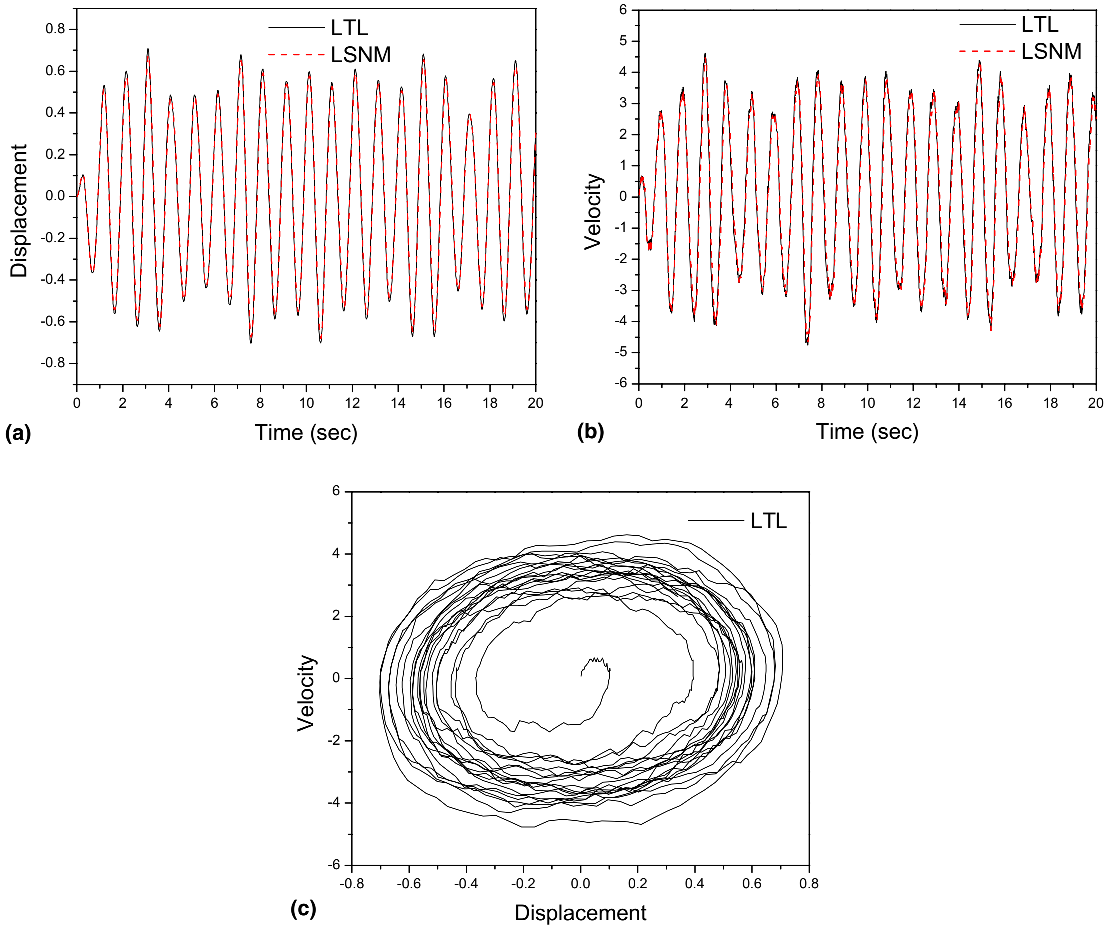

Fig. 6. Duffing oscillator under combined deterministic excitation and weak additive noise: (a) displacement histories; (b) velocity histories and (c) phase plot (LTL) ($\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 0.2$, $\varepsilon_4 = 1.0$, $\varepsilon_5 = 0.0$).

6. Numerical illustrations

The focus of the present work is on the proposition and theoretical error estimates of a new variant of the stochastic transversal linearization scheme. Presently, a limited numerical illustration on the stochastic response of an SDOF oscillator is provided here.

The problem considered a single-degree-of-freedom (SDOF) hardening Duffing equation driven by combined additive and multiplicative noises, in addition to a deterministic periodic excitation. The equation of motion is written in an incremental form as

$$\begin{aligned} \mathrm{d}x_1(t) &= x_2(t)\,\mathrm{d}t \\ \mathrm{d}x_2(t) &= (-2\pi\varepsilon_1 x_2 - 4\pi^2\varepsilon_2(1 + x_1^2)x_1 + 4\pi^2\varepsilon_3\cos(2\pi t))\,\mathrm{d}t + \varepsilon_4\,\mathrm{d}W_1(t) + \varepsilon_5 x_1\,\mathrm{d}W_2(t). \end{aligned} \tag{49}$$

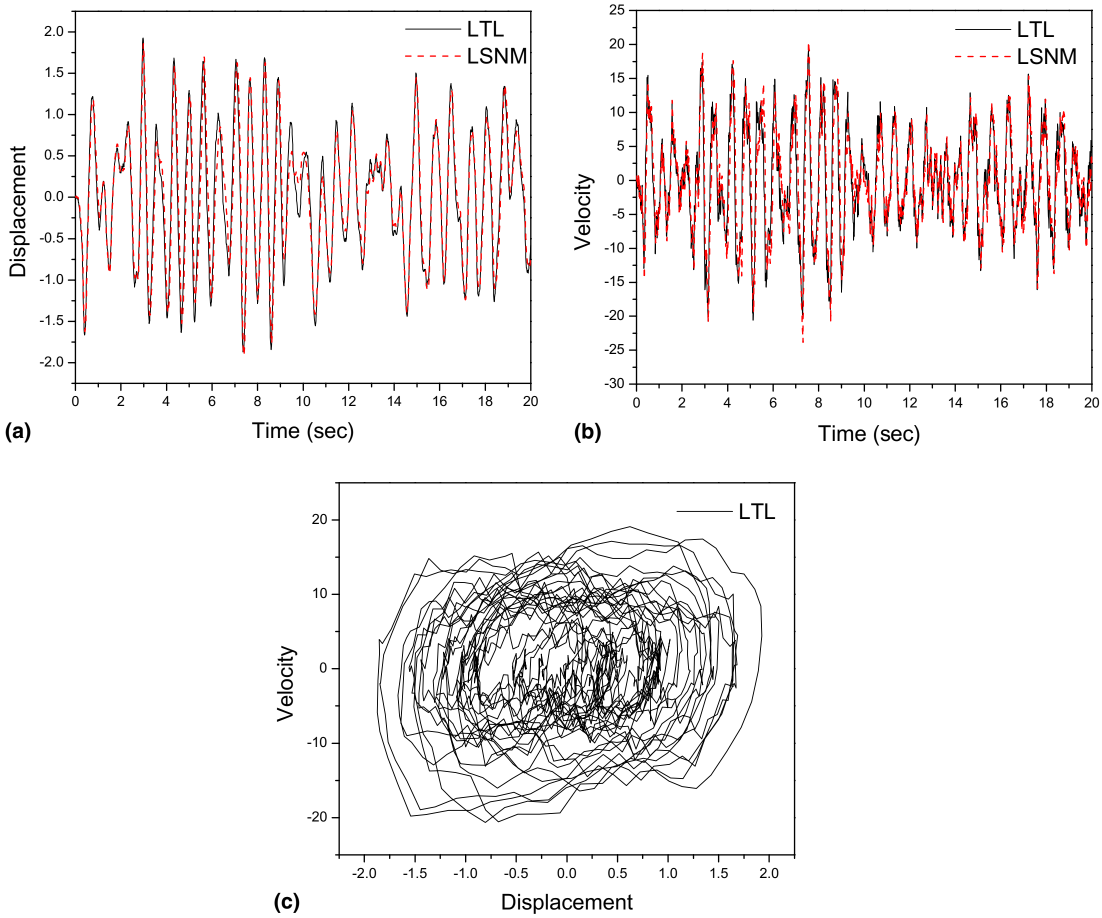

Fig. 7. Duffing oscillator under combined deterministic excitation and strong additive noise: (a) displacement histories; (b) velocity histories and (c) phase plot (LTL) ($\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 0.2$, $\varepsilon_4 = 15.0$, $\varepsilon_5 = 0.0$).

The corresponding local closed form solution takes the form

$$\overline{X}(t_i) = [\Phi(t_i, t_{i-1})]\left\{X_{i-1} + \int_{t_{i-1}}^{t_i} [\Phi^{-1}(s, t_{i-1})]\bar{\psi}(s)\,\mathrm{d}s + \int_{t_{i-1}}^{t_i} [\Phi^{-1}(s, t_{i-1})]\,\mathrm{d}\overline{W}(s)\right\}, \tag{50}$$

where $\overline{X}(t_i) = \{\bar{x}_{1,i}, \bar{x}_{2,i}\}^T$ is the linearized solution based on the initial condition $\widehat{X}_{i-1} = \{x_{1,i-1}, x_{2,i-1}\}^T$, $\bar{\psi}(s) = \{0, 4\pi^2\varepsilon_3\cos(2\pi t) - 4\pi^2\varepsilon_2 x_{1,i}^3\}^T$ is the deterministic force vector, $\mathrm{d}\overline{W}(t) = \{0, \varepsilon_4\,\mathrm{d}W_1(t) + \varepsilon_5 x_{1,i}\,\mathrm{d}W_2(t)\}^T$ is the (modulated) Wiener vector and $[\Phi(t_i,t_{i-1})]$ is the fundamental solution matrix corresponding to the linear part of the vector field. The superscript 'T' stands for vector transposition. The fundamental solution matrix is constructed via the following matrix exponentiation:

$$[\Phi(t_i, t_{i-1})] = \exp\{[A](t_i - t_{i-1})\}, \tag{51}$$

where the 2×2 coefficient matrix $[A]$ is given by:

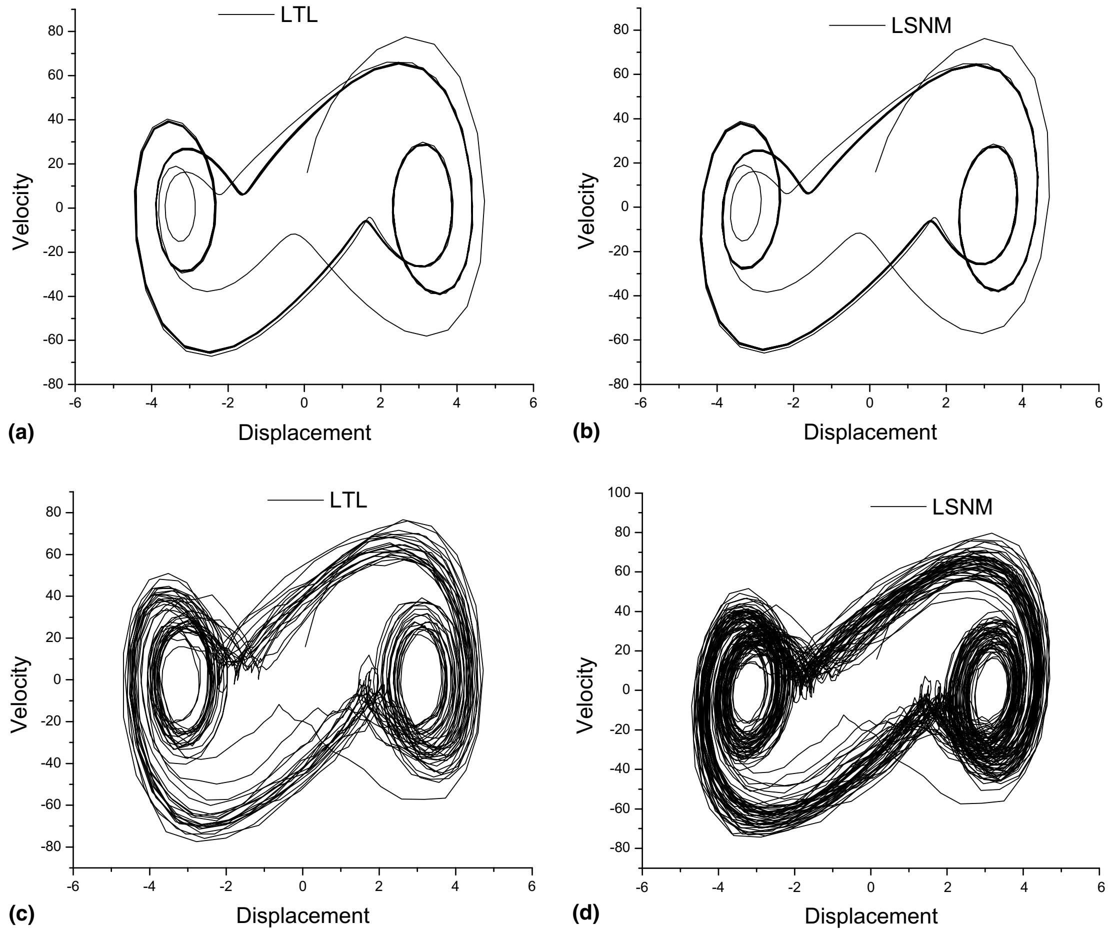

Fig. 8. A chaotic attractor under additive noise: (a) phase plot via LTL, $\varepsilon_4 = 0.5$; (b) phase plot via LSNM, $\varepsilon_4 = 0.5$; (c) phase plot via LTL, $\varepsilon_4 = 20.0$; (d) phase plot via LSNM, $\varepsilon_4 = 20.0$: $\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 41.0$, $\varepsilon_5 = 0.0$.

$$[A] = \begin{bmatrix} 0 & 1 \\ -4\pi^2\varepsilon_2 & -2\pi\varepsilon_1 \end{bmatrix}. \tag{52}$$

Now, the LTL-based solution, $\bar{x}_1(t_i)$, for the displacement component is written in the long hand over i th interval as

$$\begin{aligned}\bar{x}_1(t_i) &= \Phi_{11}(t_i, t_{i-1})x_{1,i-1} + \Phi_{12}(t_i, t_{i-1})x_{2,i-1} + \Phi_{11}(t_i, t_{i-1})\int_{t_{i-1}}^{t_i} \Phi_{12}^{-1}(s, t_{i-1})\{4\pi^2\varepsilon_3\cos(2\pi s) \\ &\quad - 4\pi^2\varepsilon_2 x_{1,i}^3\}\,\mathrm{d}s + \Phi_{12}(t_i, t_{i-1})\int_{t_{i-1}}^{t_i} \Phi_{22}^{-1}(s, t_{i-1})\{4\pi^2\varepsilon_3\cos(2\pi s) - 4\pi^2\varepsilon_2 x_{1,i}^3\}\,\mathrm{d}s \\ &\quad + \left[\Phi_{11}(t_i, t_{i-1})\int_{t_{i-1}}^{t_i} \Phi_{12}^{-1}(s, t_{i-1})\,\mathrm{d}\widehat{W}(s) + \Phi_{12}(t_i, t_{i-1})\int_{t_{i-1}}^{t_i} \Phi_{22}^{-1}(s, t_{i-1})\,\mathrm{d}\widehat{W}(s)\right] \\ &= \Psi(t_i, t_{i-1}, x_{1,i-1}, x_{2,i-1}, x_{1,i}),\end{aligned} \tag{53}$$

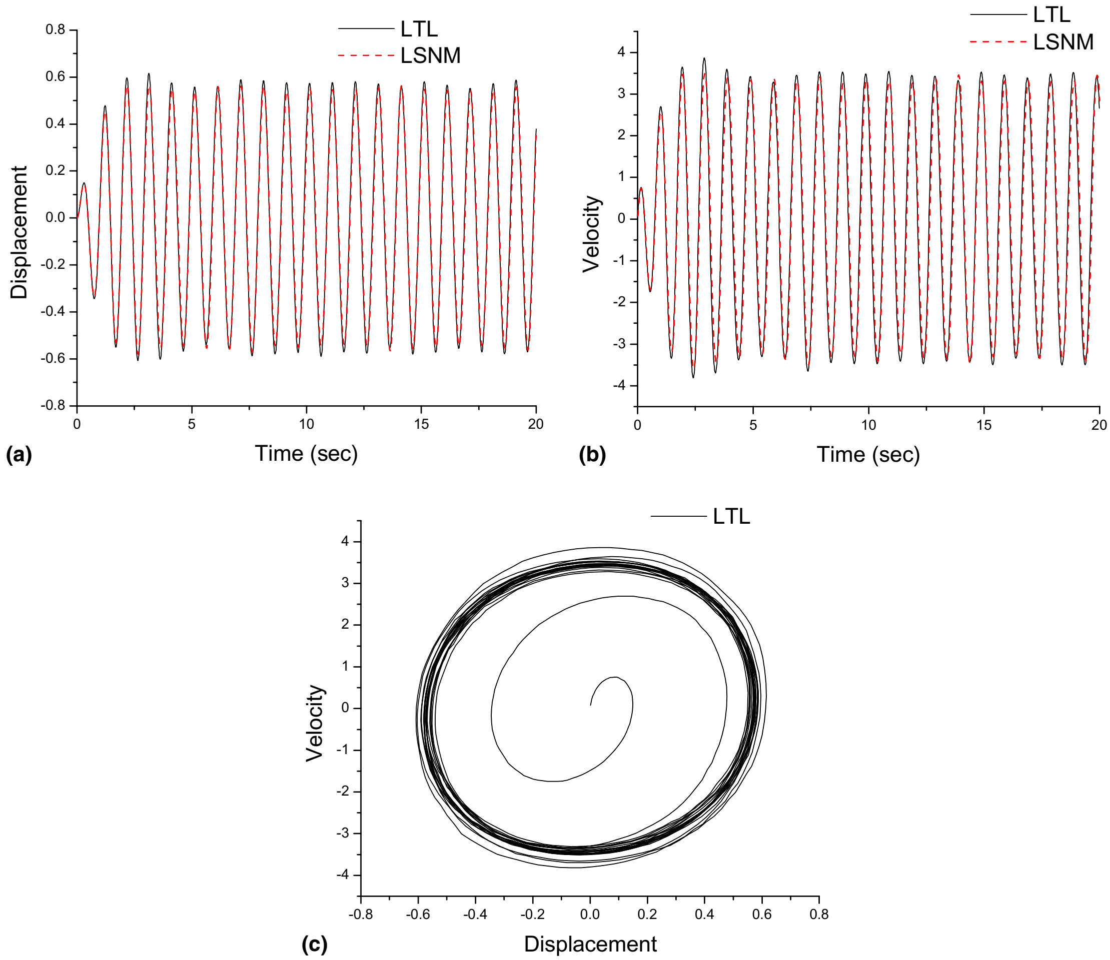

Fig. 9. Duffing oscillator under deterministic excitation and weak multiplicative noise: (a) displacement histories; (b) velocity histories and (c) phase plot (LTL) ($\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 0.2$, $\varepsilon_4 = 0.0$, $\varepsilon_5 = 0.5$).

where

$$\mathrm{d}\widehat{W}(t) = \varepsilon_4\,\mathrm{d}W_1(t) + \varepsilon_5 x_{1,i}\,\mathrm{d}W_2(t). \tag{54}$$

Imposition of the constraint condition (11) (i.e., $\bar{x}_{1,i} = x_{1,i}$ for the present case) leads to a non-linear algebraic equation to determine $x_{1,i}$. For the present set of problems, it is readily verified that a 3-term expansion (Eqs. (39) and (40)) is enough to maintain the same error orders, as derived in the Section 3 on error analysis. A consistent time step size h =0.01 sec has been adopted in all the following numerical results. To check the accuracy and limitation of the proposed LTL technique, comparisons of results have been made with lower order stochastic Newmark method (LSNM, [18]) under different combination of periodic, additive and multiplicative loads. It may be noted that comparisons of LTL solutions with those obtained with other popular schemes, especially the stochastic Heun or Euler schemes, are not provided here as the transversal linearization schemes have already been shown to have a higher accuracy over a given time step.

To begin with, Fig. 1 provides a set of comparisons of displacement and velocity histories obtained via the old LTL method and the new one proposed herein for a few combinations of additive and multiplicative noise intensities. In Figs. 2–4, displacement and velocity histories of LTL-based solutions of the oscillator under weak, medium and strong intensities of additive white-noise inputs are shown and compared with a lower order stochastic Newmark method (LSNM) of comparable accuracy [18]. No deterministic and multiplicative stochastic inputs are assumed to be acting on the oscillator in these examples. Comparisons of time histories obtained via LTL with those via LSNM appear to be quite close. Fig. 5(a–d) show displacement and velocity histories, obtained via LTL and LSNM, under combined additive and multiplicative excitations with the deterministic forcing amplitude parameter $\varepsilon_3 = 0$. The two methods are found to be in good agreement for a low multiplicative noise intensity $\varepsilon_5 = 1.0$, but vary considerably under a strong multiplicative noise input $\varepsilon_5 = 10.0$. The effects of low and strong additive noise intensities for a small deterministic sinusoidal input ($\varepsilon_3 = 0.2$) have been studied through time history plots

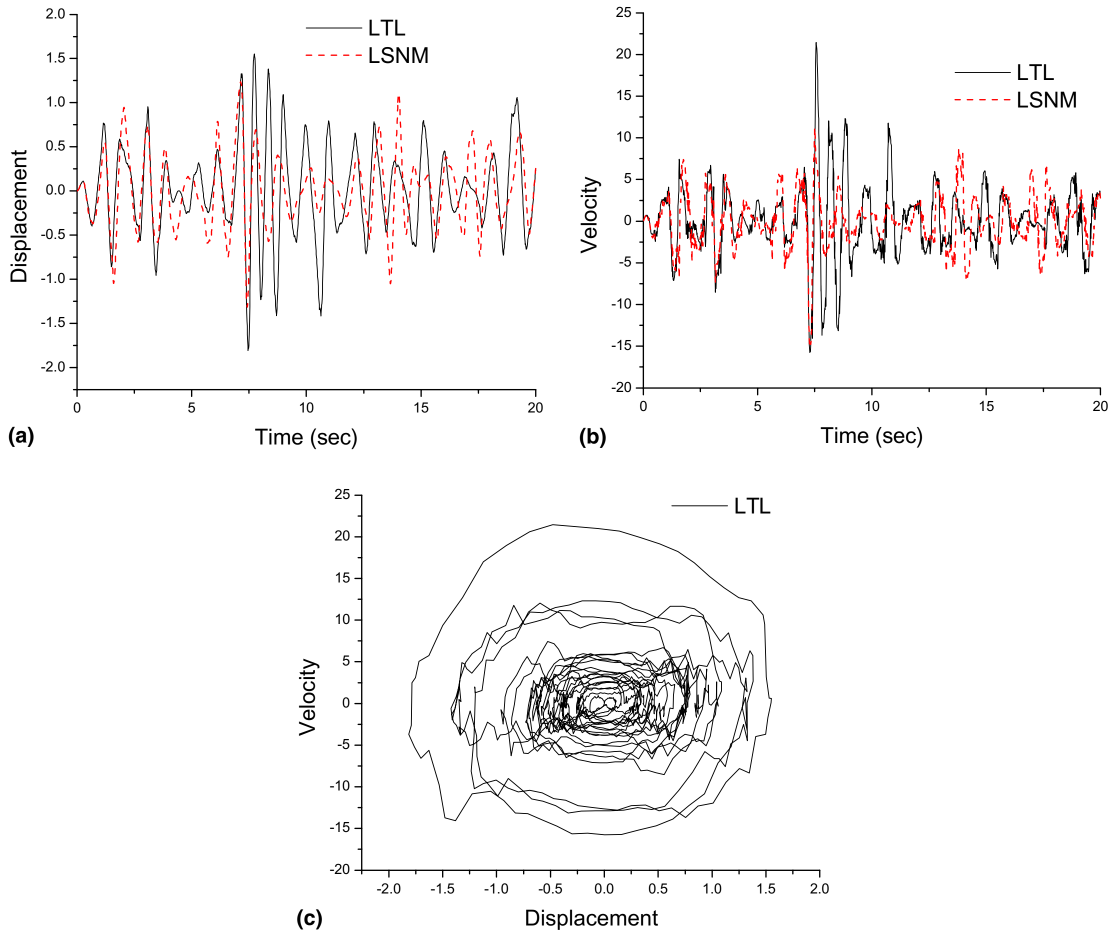

Fig. 10. Duffing oscillator under deterministic excitation and strong multiplicative noise: (a) displacement histories; (b) velocity histories and (c) phase plot (LTL) ($\varepsilon_1 = 0.25$, $\varepsilon_2 = 1.0$, $\varepsilon_3 = 0.2$, $\varepsilon_4 = 0.0$, $\varepsilon_5 = 15.0$).

Table 2
A comparison of CPU times required in the strong stochastic LTL solution via new and old approaches while integrating a Duffing oscillator; $\varepsilon_1 = 0.25$, $\varepsilon_2 = 0.5$, $\varepsilon_3 = 1.0$ with different combination of additive and multiplicative noise intensities ε_4 and ε_5

ε_4	ε_5	CPU times required for different time step size, h					
		$h = 0.001$ s		$h = 0.01$ s		$h = 0.02$ s	
		Old LTL	New LTL	Old LTL	New LTL	Old LTL	New LTL
0.01	0.00	55.53	2.85	6.14	0.33	3.34	0.16
0.10	0.00	56.18	2.86	6.15	0.32	3.34	0.16
0.25	0.00	55.91	2.91	6.15	0.32	3.34	0.16
0.25	0.25	55.86	2.91	6.09	0.32	3.39	0.16

in Figs. 6 and 7. The comparison through both the methods consistently yield satisfactory results. For a strong deterministic input, phase plots under weak and strong additive noise intensity have been shown in Fig. 8(a–d). The effects of weak and strong multiplicative noise intensities under small deterministic amplitude are shown in Figs. 9 and 10. The two methods achieve quite close results under weak multiplicative intensity, whereas, under strong multiplicative noise, LTL results err substantially compared to LSNM. Finally, in order to bring out the relative computational efficiency of the new LTL method vis-à-vis its old counterpart, Table 2 shows a typical comparison of the CPU times consumed by the two methods during path wise integration of the Duffing equation.

7. Conclusions

A new, efficient and simpler version of the stochastic locally transversal linearization (LTL) procedure is proposed and theoretically explored for accurate path wise integration of non-linear stochastic engineering dynamical systems. Detailed estimates for local and global error orders for displacement and velocity components are provided. These estimates are based on implicit Ito–Taylor expansions in terms of the LTL-based and original vector fields. The presently developed technique is notably higher in accuracy than several other existing algorithms, such as the Heun scheme or similar other schemes based on stochastic Runge–Kutta and computationally faster than the existing LTL methods. The other competing algorithms, based on stochastic Taylor expansions of displacement and velocity vectors and thereby leading (formally) to similar accuracy levels, involves extremely cumbersome (and, sometimes, nearly impossible) computations of multiple stochastic integrals (MSI-s). The LTL methodology however effectively avoids such complexities. A computationally attractive feature of the proposed LTL technique over its previously developed counterparts lies with the fact that the fundamental solution matrix (FSM) and its inverse are calculated only once for all time steps. This is in contrast with a repeated computation of FSM in the earlier version of the stochastic LTL technique. A limited numerical implementation of the proposed algorithms is provided by obtaining sample path solutions of a single-degree non-linear dynamical system under additive and multiplicative white noise excitations. The comparisons of LTL and LSNM solutions are generally good and even under high intensity additive noise, both the methods yield very close solution. However, they appear to differ to an extent in some cases, especially when the intensities of multiplicative stochastic excitations are quite high.

It must be emphasized that the spirit of the transversal linearizaion method is quite consistent with the class of geometric integrators preserving the Lie algebraic structure of the evolving flow. For stochastically driven dynamical systems, such methods have hardly been explored in the literature. Since the LTL-based linearized system may be constructed in a non-unique way and the solution of the linearized system nearly always involves exponential transformation (for the derivation of the fundamental system matrices), it seems logical to explore the possibility of adapting the method for a geometric integration of stochastically driven oscillators. This observation points to an important research direction to be pursued shortly.

In most engineering applications, one is more interested in obtaining the expected values of certain functionals of displacement and velocity processes rather than their path wise solutions. From this point of view, a weak form of the LTL method is far more desirable as it is computationally more efficient and simpler to implement. The authors have already developed and computationally implemented a weak form of the LTL method and the development is being reported in a companion paper.

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