Optimization of Biased Proportional Navigation

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An analylical treatment of the biased proportional navigalion (BPN) is carried oul with lhe aim of optimizing lhe bias parameter. It is shown that optimum biasing may lead to significantly more control-effort-efficient PN guidance in a wide variety of engagement situations, especially those involving higher target maneuvers. The performance of the BPN is compared with lhe standard (unbiased) proporlional navigation (PN) system for the general case of a maneuvering target, and the performance of the BPN is maximized to obtain the optimum bias value. The optimum bias is expressed through a simple algebraic equation which can be readily solved For lhe special (and very useful) case of the effective navigation constant being equal to 3, the equation reduces to a quadratic, leading to an explicit expression for lhe oplinium bias. Specific exaamples are provided to show the benefits of the BPN law clearly The higher control efficiency of the law is especially useful in extra-atmospheric interception, where the savings in control effort directly translates to a saving of propellent which forms part of the payload.

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I. NOMENCLATURE

Variables

- A Lateral acceleration (normal to velocity vector).
- K Acceleration bias coefficient.
- k Turn rate of target (A_T/V_T).
- L Rate bias constant.
- N Navigation constant.
- N' Effective navigation constant.
- p Bias parameter.
- r Range from pursuer to target.
- t Time.
- T Time to go.
- V Velocity.
- ΔV Cumulative velocity incremant.
- ϕ Angle between pursuer velocity vector and reference line.
- θ Line of sight (LOS) angle relative to reference line.
- β Angle between target velocity vector and reference line.
- $\dot{\theta}_B$ Rate bias.

Subscripts

- B Bias proportional navigation.
- c Constant bearing.
- f Final.
- i Initial.
- M Pursuer.
- ri Initial relative (velocity) along line of sight (LOS).
- T Target.
- o Optimum.

II. INTRODUCTION

Proportional navigation (PN) is a commonly used pursuit strategy for guided projectiles. In this strategy, the projectile turning rate is controlled to be proportional to the turn rate of line of sight (LOS) from the projectile to the target. Although PN guidance results in intercept under a wide variety of engagement conditions, its control-effortefficiency is not optimum in many situations especially in the case of maneuvering targets. Scope remains for improving the efficiency.

Variants have been suggested over the basic PN scheme to improve its efficiency. The biased PN (BPN) [1, 2] is one such scheme, in which a fixed angular rate is superimposed on the measured LOS rate before computing the commanded projectile turn rate (or lateral acceleration). Because of the introduction of an extra control parameter (i.e., the bias value), such a BPN may be made to achieve a given intercept with reduced total control effort. This is an important advantage for operations outside the atmosphere where lateral control forces are generated by the operation of control rockets, and the total control effort (integrated lateral force) determines the fuel requirement of the control engine(s). This fuel forms a part of the orbital payload which is at a high premium. For atmospheric flights, a reduction in control effort results in smaller pressure bottles in case of pneumatic actuators and smaller batteries in case of the modern all-electric actuators. The resulting space and weight saving could be very important in tactical applications.

To be able to take the best advantage of the BPN scheme, it is necessary to optimize the BPN performance with regard to the bias parameter. The performance of the BPN is maximized here to obtain the optimum bias value and the work reported here may be considered an extension of the earlier work by Brainin and McGhee [2]. The efficiency of BPN is explicitly compared relative to the PN which is more realistic as compared with normalization (as in [2]) with respect to control effort required by a single-impulse guidance scheme, which is not practical. Further, while a numerical approach was taken for the optimization in the earlier work, the optimization process here has been carried out analytically to the extent of obtaining a simple algebraic equation for the optimum bias parameter, which can be solved in real time even on small airborne computers. For the special and important case of the effective navigation constant being equal to 3, the equation is quadratic and the optimum bias parameter is obtained in closed form. To be able to appreciate the advantages of BPN in terms of physical parameters, examples are provided which clearly illustrate the savings in total control effort achieved by using a properly optimized BPN.

III. DEFINITION OF BIASED PROPORTIONAL NAVIGATION

Consider a target T and a pursuer M as points in a plane moving with constant speeds V_T and V_M, respectively, as shown in Fig. 1. The line MT from the pursuer to the target is the line of sight (LOS) which is inclined at an angle θ with respect to a reference line. If the pursuer velocity vector V_M makes an angle ϕ with the reference line, then the standard PN law is defined as

$$\dot{\phi} = N\dot{\theta} \tag{1}$$

where N is called the navigation constant. In this work, we use a modified form of (1) as follows [2]:

$$\dot{\phi} = N(\dot{\theta} - \dot{\theta}_B) \tag{2}$$

where $\dot{\theta}_B$ is a rate bias on the LOS turn rate. Equation (2) defines the BPN law. The BPN law (2) reduces to the standard PN law (1) when $\dot{\theta}_B$ equals zero.

IV. SOLUTION OF BIASED PROPORTIONAL NAVIGATION

We consider the case of pursuit against a target maneuvering with a constant lateral acceleration

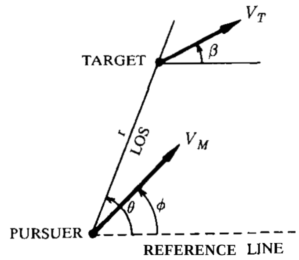


Fig. 1. Geometry of PN for maneuvering target.

A_T. The governing differential equations of motion, considering the geometry only, are obtained by resolving velocity components of the target and the pursuer along and normal to the LOS.

$$\dot{r} = V_T\cos(\theta - \beta) - V_M\cos(\theta - \phi) \tag{3}$$

$$r\dot{\theta} = -V_T\sin(\theta - \beta) + V_M\sin(\theta - \phi) \tag{4}$$

where $k = A_T/V_T$ represents the turn rate of the target, and $\beta = kt$.

The equations (3) and (4) for the pursuer motion under PN are not solvable in closed form. Here PN equations are linearized to make analytical treatment possible. Considering the homing trajectory to be a perturbation over a collision course, we can write (Fig. 1.)

$$\phi = \phi_c + \Delta\phi. \tag{5}$$

Assuming $\Delta\phi$ and θ to be small, (3) and (4) may be readily combined to yield an equation in $\dot{\theta}$ only.

$$\ddot{\theta}(t_f - t) - 2\dot{\theta} = \frac{V_M}{V_{ri}}(\cos\phi_c) + \frac{V_T}{V_{ri}}(\cos\beta_i)\dot{\beta} \tag{6}$$

where t is the time from launch, V_{ri} is the initial target-pursuer relative velocity along LOS, and $t_f = r_i/V_{ri}$ is the final intercept time. Using the BPN law (2), (6) reduces to

$$\ddot{\theta}(t_f - t) - (2 - N')\dot{\theta} - N'\dot{\theta}_B = \frac{V_T}{V_{ri}}(\cos\beta_i)\dot{\beta} \tag{7}$$

where

$$N' = \frac{NV_M\cos\phi_c}{V_{ri}} = \text{effective navigation constant.} \tag{8}$$

Equation (7) describes the behavior of the rate of change of LOS angle ($\dot{\theta}$) and can be integrated to give

$$\dot{\theta} = \left[\dot{\theta}_i - \frac{1}{N'-2}\left[\frac{V_T(\cos\beta_i)\dot{\beta}}{V_{ri}} + N'\dot{\theta}_B\right]\right]\left(\frac{t_f - t}{t_f}\right)^{N'-2} + \frac{1}{N'-2}\left[\frac{V_T(\cos\beta_i)\dot{\beta}}{V_{ri}} + N'\dot{\theta}_B\right] \tag{9}$$

where $\dot{\theta}_i$ = initial value of LOS angular rate. The expression (9) represents the LOS turn rate for pursuit against a maneuvering target under the BPN law.

V. PURSUER LATERAL ACCELERATION

The lateral acceleration A_M of the pursuer under the BPN law is obtained as

$$A_{MB} = V_M\dot{\phi} = NV_M(\dot{\theta} - \dot{\theta}_B). \tag{10}$$

Substituting **(9)** in *(10)* and rearranging, we get

$$A_{MB} = \frac{N'}{(N'-2)\cos\phi_c}\left[a - b\left(\frac{T}{T_i}\right)^{N'-2}\right] - \frac{bN'}{(N'-2)\cos\phi_c}\left[p - \left(\frac{T}{T_i}\right)^{N'-2}\right] \tag{11}$$

where

T = time to go = $t_f - t$

T_i = initial value of $T = t_f$

$$a = A_T + 2V_{ri}\dot{\theta}_B \tag{12}$$

$$b = A_T - (N'-2)V_{ri}\dot{\theta}_i + N'V_{ri}\dot{\theta}_B \tag{13}$$

$p = a/b$ = bias parameter.

From *(12)* and *(13)* and the definition of p, the rate bias $\dot{\theta}_B$ is expressed as

$$\dot{\theta}_B = \frac{1-p}{(pN'-2)V_{ri}}A_T + \frac{p(N'-2)}{pN'-2}\dot{\theta}_i \tag{14}$$

$$= KA_T/v_{ri} + L\dot{\theta}_i \tag{15}$$

where K = acceleration bias coefficient $= (1-p)/(N'p-2)$, and L = rate bias coefficient $= p(N'-2)/(pN'-2)$.

VI. CONTROL EFFORT

The cumulative velocity increment ΔV (which determines the total control effort) necessary for interception is defined for any pursuer trajectory as

$$\Delta V = \int_0^{T_i} |A_M|\,dT. \tag{16}$$

Two cases must be considered for computing the cumulative velocity increment ΔV.

***Case* I: $(0 \leq p \leq 1)$.**

In this case exactly one change of sign of A_M occurs in the interval $[0, T_i]$. This is apparent from (11), since p is a fraction between 0 and **1**, and $(T/T_i)^{N'-2}$ decreases monotonically from **1** to 0 for $N' > 2$. The acceleration reversal occurs at

$$T_r = T_i(a/b)^{1/(N'-2)} = T_i p^{1/(N'-2)}. \tag{17}$$

Then ΔV_B is given by (see Appendix)

$$\Delta V_B = \frac{M_i}{T_i}\frac{2N'}{N'-1} \times \left| (N'-2)p^{(N'-1)/(N'-2)} + \frac{|1 + (N'-2)p^{(N'-1)/(N'-2)} - (N'-1)p|}{N'p-2} \right| \tag{18}$$

where

$$M_i = \frac{1}{\cos\phi_c}\left|V_{ri}T_i^2\dot{\theta}_i + \frac{A_T}{2}T_i^2\right|. \tag{19}$$

***Case* II: $p \leq 0$ or $p \geq 1$.**

Since T/T_i has a minimum value of zero and maximum value of unity, the right-hand side (RHS) of *(11)* will remain unipolar during the entire pursuit if $N' > 2$. Thus, the lateral acceleration A_M never changes sign during the pursuit, and ΔV_B is given by (see Appendix)

$$\Delta V_B = \frac{M_i}{T_i}\frac{2N'(N'p-p-1)}{(N'-1)(N'p-2)}. \tag{20}$$

By making use of **(18)** or (20), the cumulative velocity increment ΔV_B for BPN for any value of the bias parameter p and effective navigation constant N' can be readily obtained as long as $N' > 2$, which includes most useful values of N'.

VII. OPTIMUM BIASING OF PROPORTIONAL NAVIGATION

The foregoing treatment provides a mechanism (through the introduction of a rate bias) of controlling the total control effort necessary for achieving a given mission. To make the best use of this freedom, it is necessary to optimize the rate bias to achieve a minimum control effort. Such an optimization is carried out below for the two cases considered in Section VI.

***Case* I: $0 \leq p \leq 1$.**

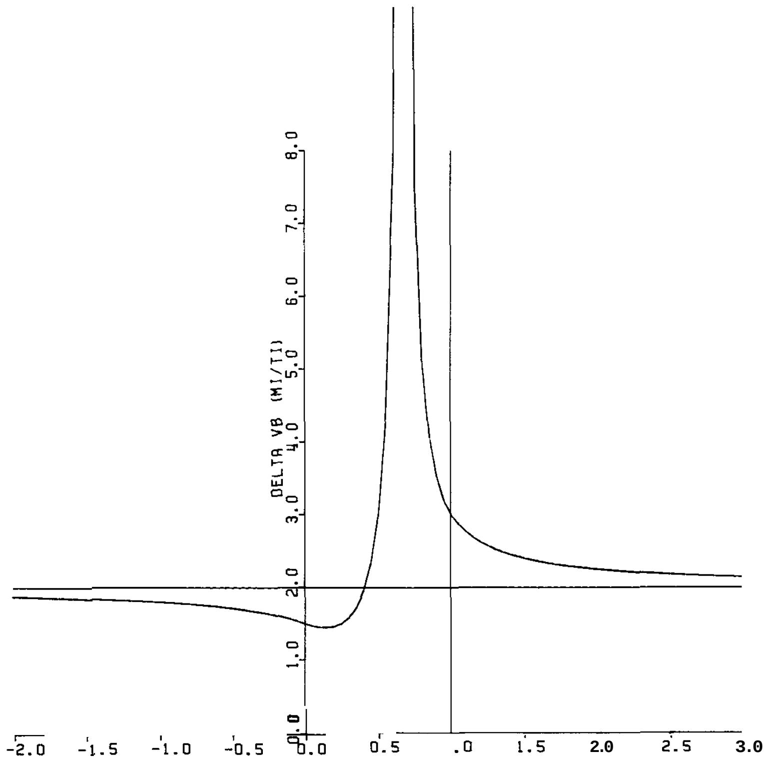
To minimize ΔV_B with respect to p, we first examine the quantity within the inner modulus in **(18)**, i.e.,

$$F \triangleq 1 + (N'-2)p^{(N'-1)/(N'-2)} - (N'-1)p. \tag{21}$$

It can be seen that $F = 1$ for $p = 0$ and $F = 0$ for $p = 1$, and

$$\frac{\partial F}{\partial p} = (N'-1)(p^{1/(N'-2)} - 1). \tag{22}$$

For $N' > 2$, the factors $(N'-1)$ and $(N'-2)$ are always positive. Also, since $0 \leq p \leq 1$, the quantity $p^{1/(N'-2)}$ is always a fraction and hence $(p^{1/N'-2)} - 1)$



$$T_i \times \left| \frac{(N'-2)p^{(N'-1)/(N'-2)} + 1 + (N'-2)p^{(N'-1)/(N'-2)} - (N'-1)p}{N'p - 2} \right|$$

$$p_o^{(N'-1)/(N'-2)} - 2\frac{N'-1}{N'}p_o^{(1/(N'-2))} + \frac{N'-2}{2N'} = 0 \qquad (24)$$

Equation **(24)** now expresses the optimum bias parameter p_o as a simple algebraic equation with coefficients dependent on N'. In general, **(24)** involves

For the important special case of **N'** $= 3$, **(24)** reduces to the quadratic form

$$6p_o^2 - 8p_o + 1 = 0 \qquad (25)$$

$$p_o 3/2 - 1.5 p_o 1/2 + 0.25 = 0 \qquad (27)$$

which is a simple cubic in $(p_o)^{1/2}$.

Case II: $p \leq 0$ or $p \geq 1$.

The velocity increment ΔV_B here is given by (20). The gradient of ΔV_B is given by

$$\frac{\partial}{\partial p}(\Delta V_B) = \frac{M_i}{T_i}\frac{2N'}{(N'-1)}\frac{(2-N')}{(N'p-2)^2}. \quad (28)$$

For $N' > 2$, this gradient is negative for all values of p, and hence ΔV_B is a monotonic function and has no distinct optimum. However, (28) indicates the existence of asymptotic stationary points for $p \to \pm\infty$, at which $\Delta V_B \to 2M_i/T_i$, from (20).

Global Optimum Biasing

To facilitate visualization of the function behavior, the dimensionless quantity $\Delta V_B(M_i/T_i)$ is plotted in Fig. 2 for $N' = 3$ with p varying from -2 to +3. For $0 \le p \le 1$, the formula (18) is used, and outside this domain (20) is used. It is seen that for $p \le 0$, the asymptotic stationary point represents a maximum and that for $p \ge 1$, it represents a minimum. Since ΔV_B at $p = 0$ has a value $(M_i/T_i)[N'/(N'-2)]$ which is less than the asymptotic value $2M_i/T_i$ for all $N' > 2$, and since the gradient of ΔV_B at $p = 0$ is $[N'(2-N')]/[2(N'-1)]$ (from (28)), which is negative for all $N' > 2$, it is clear from Fig. 2 that the global minimum of ΔV_B is within the domain $0 \le p \le 1$, and occurs at the optimum rate bias parameter p_o given by (24).

VIII. EXAMPLES

For reasons of generality, a nondimensional rate bias parameter p has been used in the formulation of the paper. It has also resulted in improving the tractability of the problem. However, the transformation used in the nondimensionalization has resulted in a certain blurring of the physical insight into the behavior of the BPN system. To be able to visualize the potential benefits of the biasing in clearer focus, two specific examples are provided below.

The first example considers an air-to-air tactical situation with a target speed of 300 m/s, a pursuer speed of 900 m/s, an initial pursuer-target separation of 5000 m, and an initial LOS angle of 60 deg. Table I shows the optimum bias parameter p_o computed from (24) and the optimum rate bias $\dot{\theta}_{Bo}$ from (14) for a range of realistic values of the effective navigation constant N'. This computation requires knowledge of the target maneuver A_T and the initial LOS rate $\dot{\theta}_i$ (or, equivalently, initial heading error $\Delta\phi_i$). Here two values of target maneuver are considered, $1g$ and 4 g, and the results are presented, respectively, in sections A and B of Bble I. An initial heading error of 15 deg is assumed for both cases, which is equivalent to an initial LOS rate of 41 mrad/s. The cumulative velocity increment ΔV_B required for intercept using BPN is computed from (18). For comparison, the cumulative velocity increment ΔV_{PN} for standard PN is also obtained from (18) with $\dot{\theta}_B = 0$ and is tabulated alongside.

TABLE I

N'	p_o	$\dot{\theta}_{Bo}$,mrad/s	ΔV_B,m/s	ΔV_{PN},m/s
A. Target Maneuver $A_T = 1.0g$				
21	0.787	8.472	281.532	282.799
25	0.341	5.326	264.305	265.713
3.0	0.140	2.939	248.338	256.837
3.5	0.062	1.437	236.487	252.806
4.0	0.029	0.515	227.445	250.814
5.0	0.007	-0.236	214.803	249.398
B. Target Maneuver $A_T = 4.0g$				
21	0.787	6.018	121.122	361.850
25	0.341	3.027	113.711	355.965
3.0	0.140	0.759	106.841	355.238
3.5	0.062	-0.669	101.743	357.167
4.0	0.029	-1.488	97.852	360.040
5.0	0.007	-2.259	92.413	366.412

Note: Optimum bias parameter and cumulative velocity increment requirement for BPN for air-to-air engagement. Requirement for standard PN provided for comparison. $V_T = 300$ m/s, $V_M = 900$ m/s, $r_i = 5000$ m, $\theta_i = 60$ deg, $\Delta\phi_i = 15$ deg (i.e., $\dot{\theta}_i = 41$ mrad/s).

The optimum rate bias $\dot{\theta}_{Bo}$ exhibits a strong dependence on the effective navigation constant N' for a given level of target maneuver. The most important observation from Bble I, however, concerns the cumulative velocity increments. At relatively low target maneuvers, such as in Bble IA, the advantage of an optimally biased PN over the standard PN is negligible, but assumes somewhat greater significance for larger values of N'. However, for stronger target maneuvers, a dramatic saving in control effort is achievable by BPN over standard PN. For example, in Table IB, for $N' = 2.1$ the optimum BPN requires only a third of the cumulative velocity increment demanded by the standard PN. For $N' = 5.0$, the control requirement of BPN is only a quarter of that of standard PN.

The second example corresponds to an engagement scenario in extra-atmospheric space. The initial target-pursuer separation in 185 km and the relative initial closing speed is 9000 m/s. These values are the same as those used for illustration in [1]. Here also, an initial LOS angle of 60 deg is assumed, as also an initial heading error of 15 deg, corresponding to an initial LOS rate of 1.11 mrad/s. In Table 11, in addition to the cumulative velocity increments ΔV_{Bo} and ΔV_{PN}, the quantity of propellent required for effecting these velocity increments is also presented. The latter quantity is computed assuming, as in [1], an initial interceptor weight of 270 kg and a liquid propellent with a specific impulse of 300 s. Either the cumulative velocity increment or the propellent requirement can be taken as a measure of the required control effort.

In Bble IIA and B, target maneuver A_T values of 0.5 g and 1.0 g are considered, respectively. It is apparent that in space pursuit scenarios, significant

TABLE II

N'	p_o	$\dot{\theta}_{Bo}$,mrad/s	ΔV_B,m/s	ΔV_{PN},m/s	BPN	PN
		A. Target Maneuver $A_T = 0.5\,g$				
					Propellant, Kg	
2.1	0.787	0.217	252.618	259.350	22.209	22.775
2.5	0.341	0.133	237.160	256.259	20.904	22.516
3.0	0.140	0.069	222.833	255.976	19.689	22.492
3.5	0.062	0.029	212.199	256.952	18.783	22.574
4.0	0.029	0.005	204.085	258.343	18.089	22.691
5.0	0.007	−0.016	192.742	261.360	17.116	22.944
		B. Target Maneuver $A_T = 1.0\,g$				
					Propellant, Kg	
2.1	0.787	0.183	170.233	298.789	15.175	26.066
2.5	0.341	0.101	159.817	301.038	14.271	26.253
3.0	0.140	0.039	150.162	305.004	13.431	26.581
3.5	0.062	−0.001	142.996	309.188	12805	26.927
4.0	0.029	−0.023	137.529	313.217	12.327	27.259
5.0	0.007	−0.044	129.884	320.453	11.657	27.855
		C. Target Maneuver $A_T = 4.0\,g$				
					Propellant, Kg	
2.1	0.787	−0.021	324.073	12A7.662	28.153	93.295
2.5	0.341	−0.091	304.243	1113.895	26.518	85.078
3.0	0.140	−0.143	285.863	1027.031	24.993	79.539
3.5	0.062	−0.176	272.221	978.743	23.854	76.388
4.0	0.029	−0.195	261.813	949.254	22982	74.438
5.0	0.007	−0.213	247.261	917.392	21.758	72.310

Note: **Optimum bias parameter and cumulative velocity increment and propellant requirement for BPN for extra-atmospheric engagement. Requirement for standard PN provided for comparison. $V_{ri} = 9000$ m/s, $r_i = 185$ km, $\theta_i = 60$ deg, $\Delta\phi_i = 15$ deg (i.e., $\dot{\theta}_i = 1.11$ mrad/s).**

control effort can be saved by employing optimum BPN even for relatively low target maneuver. Thus, for $A_T = 0.5\,g$, a 25 percent propellent saving is possible for $N' = 5.0$ and for $A_T = 1.0\,g$, the saving is as high as about 60 percent over the standard PN. In Table IIC, a high target maneuver of 4 g is deliberately included, keeping in view the possible space-based pursuit-evasion applications of the near future. For such target maneuvers, the propellent saving is by a factor better than 1:3 for all N'.

IX. CONCLUSIONS

In this paper a BPN has been studied from the point of view of control effort requirement. It has been shown that with optimal choice of the rate bias, it is possible to effect large savings in control effort required for intercepting maneuvering targets.

An analytical optimization of the BPN problem has been carried out in terms of a nondimensional rate bias parameter, resulting in a simple algebraic equation for the optimum value of the parameter from a minimum-control-effort point of view. The equation can be easily solved in real time even in the simple on-board computers of small projectiles. For the special but very useful case of $N' = 3$, the solution for the optimum rate bias parameter is explicit.

Two examples have been provided to concretely illustrate the gains possible by using an optimal BPN over the standard PN. The examples concern both atmospheric and extra-atmospheric pursuits. It has been shown that for highly maneuvering targets, the optimal BPN may require a total control effort as low as a quarter of the effort necessary for PN without bias. Such savings can be extremely valuable especially in extra-atmospheric engagements where maneuvers are carried out at the direct expense of propellent which forms part of the precious payload.

APPENDIX

The cumulative velocity increment ΔV is defined as

$$\Delta V = \int_0^{T_i} |A_M|\, dT \tag{A1}$$

where the pursuer lateral acceleration A_M is given for the biased case by (14) as

$$A_{MB} = \frac{bN'}{(N'-2)\cos\phi_c}\left[p - \left(\frac{T}{T_i}\right)^{N'-2}\right].$$

Hence

$$\int A_{MB}\,dT = \frac{bN'}{(N'-2)\cos\phi_c}\left[p - \left(\frac{T}{T_i}\right)^{N'-2}\right] dT$$
$$= \frac{N'b}{(N'-2)\cos\phi_c}\left[pT - \frac{T_i}{N'-1}\left(\frac{T}{T_i}\right)^{N'-1}\right] \tag{A2}$$

Two cases must be considered.

Case I: $0 \le p \le 1$.

In this case exactly one change of sign of A_{MB} occurs in the interval $[0, T_i]$. The acceleration reversal occurs at

$$T_r = T_i\, p^{1/(N'-2)} \tag{A3}$$

and ΔV_B is given by

$$\Delta V_B = \left|\int_0^{T_r} A_{MB}\,dT\right| + \left|\int_{T_r}^{T_i} A_{MB}\,dT\right|$$
$$= \left|\frac{N'b}{(N'-2)\cos\phi_c}\left[pT - \frac{T_i}{N'-1}\left(\frac{T}{T_i}\right)^{N'-1}\right]_0^{T_r}\right|$$
$$+ \left|\frac{N'b}{(N'-2)\cos\phi_c}\left[pT - \frac{T_i}{N'-1}\left(\frac{T}{T_i}\right)^{N'-1}\right]_{T_r}^{T_i}\right| \tag{A4}$$

The parameter b is given by (13) as

$$b = A_T - (N' - 2)V_{ri}\dot{\theta}_i + N'V_{ri}\dot{\theta}_B$$

$$= A_T - (N' - 2)V_{ri}\dot{\theta}_i + \frac{N'}{2}(a - A_T), \quad \text{using (12)}$$

$$= \frac{N'}{2}bp - (N' - 2)V_{ri}\dot{\theta}_i + A_T\frac{2 - N'}{2}, \quad \text{since } p = a/b$$

$$= \frac{-2(N' - 2)V_{ri}\dot{\theta}_i}{(2 - N'p)} + A_T\frac{2 - N'}{2 - N'p}. \tag{A5}$$

Substituting the value of b from (A5) in (A4) and rearranging,

$$\Delta V_B = \left|\frac{2N'}{(N' - 1)\cos\phi_c}\frac{1}{(N'p - 2)}\left(\frac{A_T}{2} + V_{ri}\dot{\theta}_i\right) \times [pT_r(N' - 1) - T_i p^{(N'-1)/(N'-2)}]\right|$$
$$+ \left|\frac{2N'}{(N' - 1)\cos\phi_c}\frac{1}{(N'p - 2)}\left(\frac{A_T}{2} + V_{ri}\dot{\theta}_i\right) \times [pT_i(N' - 1) - T_i - pT_r(N' - 1) + T_i p^{(N'-1)/(N'-2)}]\right|$$
$$= \left|\frac{2N'T_i}{(N' - 1)\cos\phi_c}\left(\frac{A_T}{2} + V_{ri}\dot{\theta}_i\right)\right| \times \left[\left|\frac{1}{(N'p - 2)}[p(N' - 1)\frac{T_r}{T_i} - p^{(N'-1)/(N'-2)}]\right| + \left|\frac{1}{(N'p - 2)}[p(N' - 1) - 1 - p\frac{T_r}{T_i}(N' - 1) + p^{(N'-1)/(N'-2)}]\right|\right]. \tag{A6}$$

Using the definition of T_r from (A3)

$$p(N' - 1)\frac{T_r}{T_i} = (N' - 1)p^{(N'-1)/(N'-2)}. \tag{A7}$$

Using (A7) in (A6) and rearranging

$$\Delta V_B = \frac{M_i}{T_i}\frac{2N'}{(N' - 1)}\left[\left|\frac{(N' - 2)}{(N'p - 2)}p^{(N'-1)/(N'-2)}\right| + \left|\frac{1 + (N' - 2)p^{(N'-1)/(N'-2)} - (N' - 1)p}{(N'p - 2)}\right|\right] \tag{A8}$$

where

$$M_i = \frac{1}{\cos\phi_c}\left|V_{ri}T_i^2\dot{\theta}_i + \frac{A_T}{2}T_i^2\right|. \tag{A8a}$$

Since $|(m/n)| = |m|/|n|$ and $|m| + |n| = |m + |n||$, we have

$$\Delta V_B = \frac{M_i}{T_i}\frac{2N'}{(N' - 1)} \times \left|(N' - 2)p^{(N'-1)/(N'-2)} + \frac{|1 + (N' - 2)p^{(N'-1)/(N'-2)} - (N' - 1)p|}{(N'p - 2)}\right|. \tag{A9}$$

Case II: $p \geq 1$ or $p \leq 0$.

Using (A2) in (A1)

$$\Delta V_B = \left|\frac{N'b}{(N' - 2)\cos\phi_c}\left[pT - \frac{T_i}{N' - 1}\left(\frac{T}{T_i}\right)^{N'-1}\right]_0^{T_i}\right|$$
$$= \left|\frac{N'b}{(N' - 2)\cos\phi_c}\left[pT_i - \frac{T_i}{N' - 1}\right]\right|. \tag{A10}$$

Using (A5) in (A10) and rearranging, we get

$$\Delta V_B = \left|\frac{2N'(N'p - p - 1)}{(N' - 1)(N'p - 2)}\frac{T_i}{\cos\phi_c}\left[V_{ri}\dot{\theta}_i + \frac{A_T}{2}\right]\right|$$
$$= \left|\frac{2N'(N'p - p - 1)}{(N' - 1)(N'p - 2)}\right|\frac{M_i}{T_i}, \quad \text{using (A8a)}$$
$$= \frac{M_i}{T_i}\frac{2N'(N'p - p - 1)}{(N' - 1)(N'p - 2)}, \tag{A11}$$

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