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Flat connections, geometric invariants and energy of harmonic functions on compact Riemann surfaces

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Abstract. A geometric invariant is associated to the space of flat connections on a G-bundle over a compact Riemann surface and is related to the energy of harmonic functions.

Keywords. Principal G-bundle; flat connections; Chern-Simons forms; energy of maps; harmonic maps.

Introduction

This work grew out of an attempt to generalize the construction of Chern–Simons invariants. In this paper, we associate a geometric invariant to the space of flat connection on a $SU(2)$-bundle on a compact Riemann surface and relate it to the energy of harmonic functions on the surface.

Our set up is as follows. Let $G = SU(2)$ and M be a compact Riemann surface and $E \to M$ be the trivial G-bundle. (Any $SU(2)$-bundle over M is topologically trivial). Let $\mathscr{C}$ be the space of all connections and $\mathscr{F}$ the subspace of all flat connections on this G-bundle. We endow on $\mathscr{C}$ the Frechet topology and the subspace topology on $\mathscr{F}$.

Given a loop $\sigma: S^1 \to \mathscr{F}$, we can extend σ to the closed unit disc $\tilde{\sigma}: D^2 \to \mathscr{C}$ since $\mathscr{C}$ is contractible. On the trivial G-bundle $E \times D^2 \to M \times D^2$ we define a *tautological* connection form ϑ^σ as follows

$$\vartheta^\sigma|_{(e,t)} = \tilde{\sigma}(t) \forall (e,t) \in E \times D^2.$$

Clearly restriction of ϑ^σ to the bundle $E \times \{t\} \to M \times \{t\}$ is $\tilde{\sigma}(t) \forall t \in D^2$. Let $K(\vartheta^\sigma)$ be the curvature form of ϑ^σ. Evaluation of the second Chern polynomial on this curvature form $K(\vartheta^\sigma)$ gives a closed 4-form on $M \times D^2$, which when integrated along D^2 yields a 2-form on M. This 2-form is closed since $\dim M = 2$ and thus defines an element in $H^2(M,R) \approx R$. It is seen that this class is independent of the extension of σ. We thus have a map

$$\chi: \Omega(\mathscr{F}) \to H^2(M,R) \approx R$$

where $\Omega(\mathscr{F})$ is the loop-space of $\mathscr{F}$.

We assume that the genus of $M \geqslant 2$. The energy $E(f)$ of any smooth function $f: M \to G$ is defined using the Poincare metric on M and the bi-invariant metric on $G = SU(2)$ given by the Killing form.

Any smooth function $f:M \to G$ defines a flat connection $\omega_f = f^*(\mu)$ on the trivial bundle $M \times G \to M$, where μ is the Maurer–Cartan form on G. By a result of Hitchin ([H]), the loop in $\mathscr{C}$ is given by

$$\sigma_f(t) = \frac{1}{2}(\omega_f + (\cos t)\omega_f + (\sin t)(*\omega_f)) \text{ for } t\in[0,2\pi],$$

where $*:\Lambda^1(M,\mathscr{G}) \to \Lambda^1(M,\mathscr{G})$ is the Hodge star operator, is actually a loop in $\mathscr{F}$ if and only if f is harmonic. ($\mathscr{G}$ is the Lie-Algebra of G).

The main result of this paper is

Theorem *If $f:M \to G$ is a harmonic map, then*

$$\chi(\sigma_f) = -\frac{1}{4\pi}E(f).$$

1. Construction of the basic geometric invariant

In this paper we suppose M is a compact Riemann surface of genus $g \geqslant 2, G = SU(2)$ with Lie algebra $\mathscr{G} = su(2)$ and $\pi:E \to M$ is the trivial G-bundle on M. $\mathscr{C}$ is the space of connections and $\mathscr{F}$ is the subspace of all flat connections on $E \to M$. D^2 is the closed unit disc in R^2 and $\partial D^2 = S^1$ is the unit circle. $\Omega(\mathscr{F}) = \mathrm{Map}(S^1, \mathscr{F})$ is the loop-space of $\mathscr{F}$. Given a loop $\sigma:S^1 \to \mathscr{F}$ we extend σ to $\tilde{\sigma}:D^2 \to \mathscr{C}$ ($\mathscr{C}$ is contractible). On the trivial bundle $E \times D^2 \to M \times D^2$, let ϑ^σ be the tautological connection defined in the introduction. Let $K(\vartheta^\sigma)$ be the curvature 2-form of the connection ϑ^σ. Let C_2 be the second Chern polynomial on $\mathscr{G}$. For the Lie algebra $\mathscr{G} = su(2)$, C_2 is essentially the determinant. More particularly $C_2(A) = -(1/4\pi^2)\det(A)$ for $A \in su(2)$ (cf. [KN], Chap. XII). Now an easy computation shows that

$$C_2(A) = \frac{1}{8\pi^2}\text{trace}(A^2) \text{ for } A\in\mathscr{G}.$$

Evaluation of C_2 on $K(\vartheta^\sigma)$ gives a closed 4-form $\overline{C_2(K(\vartheta^\sigma))}$ on $E \times D^2$ which projects to a closed 4-form $C_2(K(\vartheta^\sigma))$ on $M \times D^2$. Integrating $C_2(K(\vartheta^\sigma))$ along D^2 yields a closed 2-form on M (dim $M = 2$) and thus defines a cohomology class in $H^2(M,R)$ i.e.

$$\left\{\int_{D^2} C_2(K(\vartheta^\sigma))\right\} \in H^2(M,R) \approx \mathrm{R}.$$

We outline the proof of the following lemma (cf. [G], § 1 and [GS], § 2, 3).

Lemma 1.1. *$\int_{D^2} C_2(K(\vartheta^\sigma))$ is independent of the extension of $\sigma:S^1 \to \mathscr{C}$ to $\tilde{\sigma}:D^2: \to \mathscr{C}$.*

Proof. Let $\tilde{\sigma}_1, \tilde{\sigma}_2$ be two extensions of σ with corresponding connection forms $\vartheta_1^\sigma, \vartheta_2^\sigma$ and curvature forms $K(\vartheta_1^\sigma), K(\vartheta_2^\sigma)$ on the bundle $E \times D^2 \to M \times D^2$. On $E \times D^2$ we have

$$\mathrm{d}TC_2(\vartheta_1^\sigma) = \overline{C_2(K(\vartheta_1^\sigma))}$$

$$\mathrm{d}TC_2(\vartheta_1^\sigma) = \overline{C_2(K(\vartheta_2^\sigma))}$$

where $TC_2(\vartheta_1^\sigma)$, $TC_2(\vartheta_2^\sigma)$ are the Chern–Simons secondary forms with respect to $\vartheta_1^\sigma, \vartheta_2^\sigma$ respectively (cf. [CS, 3]). We can easily check that $\overline{C_2(K(\vartheta_1^\sigma))} - \overline{C_2(K(\vartheta_2^\sigma))}$ is an exact form on E (cf. [G, 1]). Since $\pi^*: H^2(M,R) \to H^2(E,R)$ is an isomorphism it follows that $\{C_2(K(\vartheta_1^\sigma))\} = \{C_2(K(\vartheta_1^\sigma))\} \in H^2(M,R)$ and this proves the lemma.

We thus have a map

$$\Omega(\mathscr{F}) \xrightarrow{\chi} H^2(M,R) \approx R$$

$$\sigma \mapsto \chi(\sigma) = \left\{ \int_{D^2} C_2(K(\vartheta^\sigma)) \right\}$$

where $\Omega(\mathscr{F})$ is the loop-space of $\mathscr{F}$. It is easy to check that $\chi(\sigma \circ \sigma') = \chi(\sigma) + \chi(\sigma')$ where $\sigma \circ \sigma'$ is the composite of two loops in $\mathscr{F}$. We call this map χ the geometric invariant.

2. Energy of functions and a class of special loops

We recall the definition of energy of a function. Let X and Y be Riemannian manifolds. Given a smooth map $f: X \to Y$, the energy density of f is a function $e(f): X \to R$ defined by

$$e(f)(x) = \|\mathrm{d}f(x)\|^2$$

where $\|\mathrm{d}f(x)\|$ denotes the Hilbert–Schmidt norm of the differential $\mathrm{d}f(x) \in T_x^*(x) \otimes T_{f(x)}(Y)$. If X is compact and oriented, the energy of f, denoted by $E(f)$ is given by

$$E(f) = \left(\int_M e(f)(x)\mathrm{d}x \right)^{1/2}$$

where $\mathrm{d}x$ is the volume form of X with respect to its Riemannian metric. f is *harmonic* if it is a critical point of the energy functional.

Using the Poincare metric on the compact Riemann surface of genus $\geqslant 2$ and the bi-invariant metric on $G = SU(2)$ given by the Killing form, we can define the energy $E(f)$ of a smooth function $f: M \to G$ by the above formula.

Any smooth function $f: M \to G$ defines a flat connection $\omega_f = f^*(\mu)$ on the trivial bundle $E \to M$ where

$$\mu = \begin{pmatrix} i\mu_1 & \mu_2 + i\mu_3 \\ -\mu_2 + i\mu_3 & -i\mu_1 \end{pmatrix}$$

is the Maurer–Cartan form on G. In the case of the trivial bundle $E \to M$, clearly the space of all connections $\mathscr{C}$ can be identified with the space $\Lambda^1(M, \mathscr{G})$ of all $\mathscr{G}$-valued 1-forms on M. For any smooth function $f: M \to G$, consider the loop in $\mathscr{C}$ given by $\sigma_f(t) = \frac{1}{2}(\omega_f + (\cos t)\omega_f + (\sin t)(*\omega_f))$ for $t \in [0, 2\pi]$, where $*: \Lambda^1(M, \mathscr{G}) \to \Lambda^1(M, \mathscr{G})$ is the Hodge star operator. By a result of Hitchin ([H]), we know that $\sigma_f([0, 2\pi]) \subset \mathscr{F}$ iff f is harmonic, i.e. σ_f is a loop in $\mathscr{F}$ iff f is harmonic.

3. Relation between the geometric invariant and the energy of harmonic maps

We prove the following result

Theorem 3.1. *If $f:M\to G$ is a harmonic map, then $\chi(\sigma_f) = -\frac{1}{4\pi}E(f)$.*

Proof. At the outset we show that the closed 2-form which represents $\chi(\sigma_f)\in H^2(M,R)$ is $\frac{1}{2\pi}(*\omega_1\Lambda\omega_1 + *\omega_2\Lambda\omega_2 + *\omega_3\Lambda\omega_3)$ where

$$\omega_f = f^*\mu = \begin{pmatrix} i\omega_1 & \omega_2 + i\omega_3 \\ -\omega_2 + i\omega_3 & -i\omega_1 \end{pmatrix}.$$

We extend the loop σ_f in $\mathscr{F}$ to a map $\tilde{\sigma}_f: D^2\to\mathscr{C}$ in an obvious way. We drop the suffix f and simply use σ and $\tilde{\sigma}$ in the computations that follow.

Let (s,t) be the polar coordinates on $D^2 = \{(s,t), 0\leqslant s\leqslant 1, 0\leqslant t\leqslant 2\pi\}$.
Set $\tilde{\sigma}(s,t) = s\sigma(t)$. We now compute the curvature $K(\vartheta^\sigma)$ of the connection form ϑ^σ on the bundle $E\times D^2\to M\times D^2$.

$$\begin{aligned} K(\vartheta^\sigma) &= d\vartheta^\sigma + \tfrac{1}{2}[\vartheta^\sigma, \vartheta^\sigma], \\ &= d\vartheta^\sigma + \vartheta^\sigma\wedge\vartheta^\sigma, \\ &= d_E\vartheta^\sigma + d_{D^2}\vartheta^\sigma + \vartheta^\sigma\wedge\vartheta^\sigma, \\ &= d_{D^2}^{\vartheta^\sigma} + K(\tilde{\sigma}(s,t)), \end{aligned}$$

where $K(\tilde{\sigma}(s,t)$ is the curvature of $\tilde{\sigma}(s,t))$ and d_E and d_{D^2} are respectively the exterior differentials on E and D^2.

If we set

$$\sigma(t) = \begin{pmatrix} i\alpha(t) & \beta(t) + i\gamma(t) \\ -\beta(t) + i\gamma(t) & -i\alpha(t) \end{pmatrix}$$

as a form on M for each $t\in S^1$, then after a straightforward calculation (see [G], Lemma 4.1), it follows that $\int_{D^2} C_2(K(\vartheta^\sigma))$ is cohomologous to the form

$$\frac{1}{4\pi^2}\int_{S^1}\left(\frac{d}{dt}\alpha(t)\wedge\alpha(t) + \frac{d}{dt}\beta(t)\wedge\beta(t) + \frac{d}{dt}\gamma(t)\wedge\gamma(t)\right)dt$$

Now

$$\omega = f^*\mu = \begin{pmatrix} i\omega_1 & \omega_2 + i\omega_3 \\ -\omega_2 + i\omega_3 & -i\omega_1 \end{pmatrix}$$

so that

$$\sigma(t) = \begin{pmatrix} i(\omega_1 + \cos t\omega_1 + \sin t*\omega_1) & (\omega_2 + \cos t\omega_2 + \sin t*\omega_2) + \\ & i(\omega_3 + \cos t\omega_3 + \sin t*\omega_3) \\ -(\omega_2 + \cos t\omega_2 + \sin t*\omega_2) + & -i(\omega_1 + \cos t\omega_1 + \sin t*\omega_2) \\ i(\omega_3 + \cos t\omega_3 + \sin t*\omega_3) & \end{pmatrix}$$

i.e.

$$\alpha(t) = (\omega_1 + \cos t\omega_1 + \sin t{*}\omega_1)$$

$$\beta(t) = (\omega_2 + \cos t\omega_2 + \sin t{*}\omega_2)$$

$$\gamma(t) = (\omega_3 + \cos t\omega_3 + \sin t{*}\omega_3)$$

Now

$$\frac{d}{dt}\alpha(t)\Lambda\alpha(t) = ((-\sin t)\omega_1 + \cos t{*}\omega_1)\Lambda(\omega_1 + \cos t{*}\omega_1 + \sin t{*}\omega_1)$$

$$= -\sin^2 t\omega_1\Lambda{*}\omega_1 + \cos^2 t{*}\omega_1\Lambda\omega_1$$

$$= {*}\omega_1\Lambda\omega_1.$$

Similarly

$$\frac{d}{dt}\beta(t)\wedge\beta(t) = {*}\omega_2\wedge\omega_2$$

$$\frac{d}{dt}\gamma(t)\wedge\gamma(t) = {*}\omega_3\wedge\omega_3.$$

It follows that $\int_{D^2} C_2(K(\vartheta^\sigma))$ is cohomologous to the form

$$\frac{1}{4\pi^2}\int_{S^1}({*}\omega_1\wedge\omega_1 + {*}\omega_2\wedge\omega_2 + {*}\omega_3\wedge\omega_3)dt$$

$$= \frac{1}{2\pi}({*}\omega_1\wedge\omega_1 + {*}\omega_2\wedge\omega_2 + {*}\omega_3\wedge\omega_3).$$

Thus the closed 2-form on M representing $\chi(\sigma_f)\in H^2(M,R)$ is $\frac{1}{2\pi}({*}\omega_1\wedge\omega_1 + {*}\omega_2\wedge\omega_2 + {*}\omega_3\wedge\omega_3)$.

To prove that $\chi(\sigma_f) = -\frac{1}{4\pi}E(f)$, we check using local coordinates that the forms

$$\frac{1}{2\pi}\left(\frac{d}{dt}\alpha(t)\wedge\alpha(t) + \frac{d}{dt}\beta(t)\wedge\beta(t) + \frac{d}{dt}\gamma(t)\wedge\gamma(t)\right)$$

and $-\frac{1}{4\pi}e(f)(m)dm$ (dm is the volume form on M) are equal at any arbitrary point.

Since any left translation in G is an isometry, for any $m\in M$, $\|df(m)\| = \|d(L_{f(m)^{-1}}\circ f)(m)\|$ where $L_{f(m)^{-1}}:G\to G$ is left translation by $f(m)^{-1}$. We can therefore assume that f maps some point $m\in M$ to the identity element in G, i.e. $f(m)=1$.

Since we intend to use local coordinates to prove the equality of forms, we can go to the universal cover D^2 of M with Poincare metric and assume $f:D^2\to G$ and $f(m)=1$ for some fixed $m\in D^2$. Since there exist an isometry of D^2 which maps the origin to m, we can assume $f(0)=1$ and check equality of forms at the origin.

At the origin we have

$$\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right\rangle = 1 = \left\langle\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right\rangle$$

and

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = 0$$

where $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ are the usual coordinate vector fields. Let dx and dy be the dual 1-forms. Clearly at the origin $*dx = dy$ and $*dy = -dx$. Since $dm = dx \wedge dy$ we have

$$e(f)(m)dm\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = e(f)(m).$$

We prove that

$$\frac{1}{2\pi}(*\omega_1 \wedge \omega_1 + *\omega_2 \wedge \omega_2 + *\omega_3 \wedge \omega_3)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = -\frac{1}{4\pi}e(f)(m).$$

If $\omega_j = a_j dx + b_j dy$ ($1 \leqslant j \leqslant 3$, a_j, b_j are functions on D^2) then $*\omega_j = a_j dy - b_j dx$ for $1 \leqslant j \leqslant 3$ so that $*\omega_j \wedge \omega_j = -(a_j^2 + b_j^2)dx \wedge dy$ for $1 \leqslant j \leqslant 3$

$$\Rightarrow \frac{1}{2\pi}(*\omega_1 \wedge \omega_1 + *\omega_2 \wedge \omega_2 + *\omega_3 \wedge \omega_3) = -\frac{1}{2\pi}(a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2)dx \wedge dy$$

For $f: D^2 \to SU(2)$ with $f(0) = 1$

$$\|df(0)\|^2 = \left\| df(0)\left(\frac{\partial}{\partial x}\right)\right\|^2 + \left\| df(0)\left(\frac{\partial}{\partial y}\right)\right\|^2$$

$$= \left\|\frac{\partial f(0)}{\partial x}\right\|^2 + \left\|\frac{\partial f(0)}{\partial y}\right\|^2$$

By definition of Maurer–Cartan form

$$\frac{\partial f(0)}{\partial x} = \mu\left(\frac{\partial f(0)}{\partial x}\right) =$$

$$\begin{pmatrix} i\mu_1\left(\frac{\partial f(0)}{\partial x}\right) & \mu_2\left(\frac{\partial f(0)}{\partial x}\right) + i\mu_3\left(\frac{\partial f(0)}{\partial x}\right) \\ -\mu_2\left(\frac{\partial f(0)}{\partial x}\right) + i\mu_3\left(\frac{\partial f(0)}{\partial x}\right) & -i\mu_1\left(\frac{\partial f(0)}{\partial x}\right) \end{pmatrix}$$

The pairing $(A, B) \mapsto \mathrm{trace}(AB)$ for $A, B \in su(2)$ gives the Killing form on $su(2)$ so that

$$\left\|\frac{\partial f(0)}{\partial x}\right\|^2 = \mathrm{trace}\left(\frac{\partial f(0)}{\partial x}\frac{\partial f(0)}{\partial x}\right)$$

$$= 2\left\{\left(\mu_1\left(\frac{\partial f(0)}{\partial x}\right)\right)^2 + \left(\mu_2\left(\frac{\partial f(0)}{\partial x}\right)\right)^2 + \left(\mu_3\left(\frac{\partial f(0)}{\partial x}\right)\right)^2\right\}.$$

Similarly

$$\left\|\frac{\partial f(0)}{\partial x}\right\|^2 = 2\left\{\left(\mu_1\left(\frac{\partial f(0)}{\partial y}\right)\right)^2 + \left(\mu_2\left(\frac{\partial f(0)}{\partial y}\right)\right)^2 + \left(\mu_3\left(\frac{\partial f(0)}{\partial y}\right)\right)^2\right\}.$$

$$\Rightarrow \|df(0)\|^2 = 2\left\{\sum_{j=1}^{3}\left(\mu_j\left(\frac{\partial f(0)}{\partial x}\right)\right)^2 + \left(\mu_j\left(\frac{\partial f(0)}{\partial y}\right)\right)^2\right\}.$$

Noting that $f^*\mu_j = \omega_j (1 \leqslant j \leqslant 3)$ we have

$$\omega_j\left(\frac{\partial}{\partial x}\right)(0) = (f^*\mu_j)\left(\frac{\partial}{\partial x}\right)(0) = \mu_j\left(\frac{\partial f}{\partial x}(0)\right).$$

Now

$$\omega_j\left(\frac{\partial}{\partial x}\right) = (a_j \mathrm{d}x + b_j \mathrm{d}y)\left(\frac{\partial}{\partial x}\right) = a_j.$$

Therefore

$$\omega_j\left(\frac{\partial f(0)}{\partial x}\right) = a_j \quad (1 \leqslant j \leqslant 3).$$

Similarly

$$\omega_j\left(\frac{\partial f(0)}{\partial y}\right) = b_j \quad (1 \leqslant j \leqslant 3).$$

Thus

$$\|\mathrm{d}f(0)\|^2 = 2\{a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_3^2 + b_3^2\} = e(f)(m).$$

Therefore we have

$$(-(*\omega_1 \wedge \omega_1 + *\omega_2 \wedge \omega_2 + *\omega_3 \wedge \omega_3)) = \frac{1}{2} e(f)(m) \mathrm{d}x \wedge \mathrm{d}y.$$

In other words

$$\left(\frac{1}{2\pi}(*\omega_1 \wedge \omega_1 + *\omega_2 \wedge \omega_2 + *\omega_3 \wedge \omega_3)\right) = -\frac{1}{4\pi} e(f)(m) \mathrm{d}m$$

Consequently $\chi(\sigma_f) = -\frac{1}{4\pi} E(f)$ and the theorem follows.

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