Hermite polynomials for signal reconstruction from zero-crossings

Part 1 : One-dimensional signals

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Abstract: Generalised Hermite polynomials are employed for the reconstruction of an unknown signal from a knowledge of its zero-crossings, under certain conditions on its spatial/spectral width, but dispensing with the assumption of bandlimitedness. A computational implementation of the proposed method is given for one-variable (or one-dimensional) signals, featuring an application of simulated annealing for optimal reconstruction.

1 Introduction

The present paper deals with the problem of reconstruction of signals of one variable (for instance, time or space, termed, in the literature, 'one-dimensional signals') from zero-crossing information. This problem of reconstruction belongs to the class of so-called inverse problems, and is of relevance to optics, acoustics, crystallography and vision, among many other fields.

In the case of time-signals, the zero-crossing information is believed to be important in a situation where a signal has been subjected to a nonlinear distortion, like hard-clipping, which leaves the zero-crossings of the signal intact. The main question (which will be formulated more precisely below) is whether (and under what conditions) it is possible to reconstruct the original signal from a knowledge of its zero-crossings only.

The literature on this problem contains results which, in a majority of cases, seem to be *ad hoc*. In view of the constraint on space, only the most relevant references are briefly mentioned here. For more details and a critical analysis of the available results, see Reference 7.

Perhaps the first paper on the reconstruction of one-dimensional signals from zero-crossings is due to Requicha [1]. The algorithm of Voelcker and Requicha [2] uses a bandpass operator, and the authors use the contraction-mapping argument in the attempt to establish that the solution is unique. Logan [3] attempts to establish that a one-dimensional bandpass signal (under constraints on the bandwidth) can be uniquely recovered from its zero-crossings if there are no complex zeros common to the signal and its Hilbert transform.

Poggio *et al.* [4,5] consider the problem of a unique representation (within a multiplicative constant) of a

periodic one-dimensional bandpass function by means of its zero-crossings. They use the zero-crossings of the signal filtered by a Gaussian filter with the variance parameter, α , which controls the width of the filter. Their result is that the map of all these zero-crossings determines the signal uniquely if the filtered signal can be represented by a finite polynomial.

The recent report of Hummel and Moniot [6] deals with the reconstruction problem in 'scale-space', by which is meant a one-parameter family of data sets obtained from the Laplacian of a Gaussian-filtered version of the image. As in References 4 and 5, the width of the filter is controlled by the variance parameter, α . Hummel and Moniot [6] deal with the possibility of reconstruction given the zero-crossings at multiple scales of resolution in the so-called scale-space (in which the zero-crossing locations are plotted against the variance parameter, α). They conclude that 'reconstruction is possible, but can be unstable'. In addition, they suggest the inclusion of gradient data along the zero-crossings in the representation, and demonstrate that the reconstruction is then stable. However, the class of functions considered by Hummel and Moniot is inadequate to deal with the general zero-crossing problem. (See, for details, References 7 and 8). Moreover, this class is included in the class introduced in Section 2.

In this paper, we propose what seems to be a novel solution to the reconstruction problem, using in part the results of de Bruijn [9] on the characterisation of uncertainty in signals without the constraint of strict bandlimitedness. (See also Reference 10 in the context of uncertainty analysis).

2 New framework for reconstruction

Most signals encountered in nature are (i) neither periodic nor made up of a set of elementary singular functions, which are confined to infinitely small spatial/time intervals; and (ii) infinite neither in spatial nor in spectral extent. Therefore, for convenience in analysis and synthesis, they are to be represented by (nonsingular) elementary functions which are confined both along the spatial

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and spectral axes. It may be noted here that, in the literature on signal processing, one- and two-variable functions are called one- and two-dimensional signals, respectively.

If a one-variable signal (such as a speech signal as a function of time) is considered in a time-window of a finite size (as is normally done in speech signal analysis), its Fourier transform cannot have finite width. The duality property in Fourier analysis tells us that confining the spectrum of the speech signal to a finite part in the frequency domain forces, in turn, the signal to have infinite extent (or width) in the time domain.

Similarly, if we consider an image as a function of two (space) variables in the (finite) field of view of a camera or of the human visual system, its Fourier transform cannot be confined to a finite part of the (two-variable) spectral domain. Moreover, it is believed, on the basis of empirical findings, that the human visual system does have characteristics which exhibit this type of finiteness in both the spatial and spectral domains.

On the contrary, recall that the assumption of finite bandwidth is the foundation on which most of the results of both one- and two-dimensional signal processing are based, including those of reconstruction from zero-crossing information.

In the paper, we propose the use of a window of a certain 'effective width' for each of the two domains. If, as in the case of the speech-signal analysis, a time-frame is to be considered, this could be imbedded in the effective width. Similarly, the finite physical image could be contained in the region bounded by the effective widths in the x - and y -directions. Of course, the effective width is determined by the nature of the signal, and is in fact related to the uncertainty, in its characterisation [10, 19 and 21].

An analogy with the complex-frequency Fourier transform is perhaps appropriate here. It is the positive part of the frequency that is of practical relevance, whereas the negative part is necessitated by the mathematical framework involving complex exponential functions.

Similarly, the framework proposed in this paper is applicable, in general, to situations in which the 'time/space' window can be imbedded in an 'effective-width' window, thereby facilitating the development of a consistent mathematical theory.

2.1 Information content in zero-crossings

A reference has already been made to the significance of reconstruction of a one-variable signal from zero-crossing information. Apart from this, a virtue of signal representation by zero-crossings is that it eliminates dependence on dynamic response range (gain), as the zeros are independent of signal amplitude. It should, however, be noted that this necessitates high accuracy in the localisation of the zeros.

As far as images (or functions of two variables) are concerned, a scheme suggested for image representation (see, for instance, Marr [17]) in the models of biological vision is the encoding of the extrema in the blurred spatial derivatives of the image. The blurring is modelled by a Gaussian function. Furthermore, such representations are useful, and contain significant information because they identify image regions marked by abrupt changes in brightness, such as those that arise at the edges of the objects and occlusion boundaries. Mathematically, the extrema in the blurred spatial derivatives are the loci of zero-crossings. Fig. 1 shows a typical scan-line of the Laplacian of a Gaussian-blurred image.

According to Marr [17], the zero-crossings of the scan-line shown in Fig. 1 are points where important information (pertaining to the original image) is contained. When

Fig. 1 Typical scan-line of a Laplace-Gaussian filtered image

we deal with images (as two-variable functions), the zero-crossing contours result in something similar to line drawings. And we know how perceptually rich are line drawings, such as cartoons. Therefore, the zero-crossing scheme is suggested as a plausible approach to compact information encoding in biological visual systems.

2.2 Space of one-variable signals under consideration

We assume that the signals under consideration are defined over $(-\infty, \infty)$ in both the time/spatial and spectral domains. In what follows, t is an independent variable which, in practice, could be interpreted as either time or space. Let $f(t)$ be a real-valued function of $t \in \mathcal{R}$, with the Fourier transform

$$F(jw) = \int_{-\infty}^{\infty} f(t) \exp(-jw t) dt$$

The two functions, $f(t)$ and $F(jw)$, form a Fourier integral pair. The classical uncertainty principle says that they cannot both be of short duration or be highly concentrated.

One interpretation of this is obtained by defining the spread of the function as follows. We say that f is essentially concentrated [10] on a measurable set X_1 if there is a function $g(t)$ vanishing outside X_1 such that

$$\|f - g\| \leq \delta$$

where $\|\cdot\|$ is a suitable norm and δ is a (prespecified, arbitrarily) small positive quantity.

Similarly, we say that $F(jw)$ is essentially concentrated on a measurable set W_1 if there is a function $G(jw)$ vanishing outside W_1 with

$$\|F(jw) - G(jw)\| \leq \delta$$

for some suitable norm $\|\cdot\|$ and positive δ .

Then the function $f(t)$ is essentially zero outside an interval of length X_1 , and $F(jw)$ is essentially zero outside an interval of length W_1 . Under these conditions, the uncertainty principle leads to the inequality, $X_1 W_1 \geq 1$.

A similar inequality for the uncertainty principle can be obtained by defining the time/spatial and spectral spreads of the function as follows. Let

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |f(t)|^2 dt \\ &= \int_{-\infty}^{\infty} |F(w)|^2 dw / (2\pi) \end{aligned} \quad (1)$$

which is nothing but the well-known Parseval relation. (See Appendix, Section 6.1). Then the effective spreads arounds t_0 and w_0 are defined by (see Gabor [18])

$$X_2 = \sqrt{\left\{ \int_{-\infty}^{\infty} (t - t_0)^2 |f(t)|^2 dt / E \right\}} \quad (2a)$$

$$W_2 = \sqrt{\left\{ \int_{-\infty}^{\infty} (w - w_0)^2 |F(w)|^2 dw / (2\pi E) \right\}} \quad (2b)$$

The inequality in this case reads

$$X_2 \cdot W_2 \geq 1 \quad (2c)$$

For the framework proposed in the paper, use of the spreads X , and W_2 turns out to be more convenient in the analysis of the problem of reconstruction.

2.3 Possible elementary functions

Many types of function could be used as basis functions to represent the one-variable signals of interest to us. However, for the class of inverse problems under study, it is desirable that the basis functions possess the following properties.

(i) The set of functions should be independent and constitute a complete and orthogonal basis (so that the representation is unique for all the signals in the space under consideration, and the coefficients of expansion are easily computed).

(ii) The spatial-spectral spread constraints imposed on the signal should be expressible in terms of the coefficients of expansion (so as to facilitate a solution to the inverse problem).

Some possible basis functions are:

- (a) Legendre polynomials
- (b) Laguerre polynomials
- (c) Bessel functions
- (d) Chebyshev polynomials
- (e) Prolate spheroidal functions
- (f) Gabor functions
- (g) Hermite polynomials

A few comments are in order, regarding the suitability of these functions. Legendre and Chebyshev polynomials are defined over the (normalised) interval $[-1, 1]$. Their Fourier transforms have an infinite spread. As a consequence, they do not permit the use of the uncertainty inequalities to determine the coefficients when the original signal is not explicitly given.

The Laguerre polynomials are defined over $[0, \infty)$, and have Fourier transforms with infinite spread. However, it is found that the uncertainty inequalities cannot be easily reduced to a form involving only the coefficients of expansion.

As far as Bessel functions are concerned, the derivation of the Fourier transforms for the basis functions, and the determination of the coefficients of expansion from explicit or implicit information about the signal are extremely complicated.

Prolate spheroidal functions [16, 20] are a special class of functions, explicitly designed for optimum signal expansion. However, when considered in the context of reconstruction of signals from zero-crossing information, they lead us to the solution of transcendental equations, which are extremely difficult to solve.

Gabor [18] suggested the expansion of a signal into a discrete set of properly shifted and modulated versions of an elementary signal. The Gabor elementary functions, $\Gamma_{mn}(x)$, as they are normally called, are defined by

$$\Gamma_{mn}(x) = g(x - mD) \exp(jnW_r x)$$

where m, n are integers, D is a shift parameter in the spatial domain, W_r is the shift parameter in the spectral domain, $W_r \cdot D \leq 2\pi$, and $g(\cdot)$ is a Gaussian function. They have the main advantage in achieving the lowest bound on the 'joint entropy', defined as the product $X \cdot W_2$, where X , and W_2 are defined by eqns. 2a and 2b, respectively. Hence, a representation using the Gabor elementary functions provides the best spectral information for every point along the signal variation. Further, if the condition of optimal information cell size, $W_r \cdot D = 2\pi$, is satisfied, the set of functions $\{\Gamma_{mn}(x)\}$ is complete.

Under these conditions, the signal under consideration is expressed by the Gabor elementary functions using a set of weighting coefficients

$$f(x) = \sum_{m, n=-\infty}^{m, n=\infty} \alpha_{mn} \Gamma_{mn}(x)$$

However, the Gabor elementary functions are not orthogonal, as a consequence of which, the computation of the coefficients, α_{mn} , turns out to be difficult. More significantly, however, given the implicit information, such as zero-crossings, about a signal, the extraction of the coefficients in the Gabor framework leads to transcendental equations, which are difficult to solve.

Interestingly, when the complex part of a Gabor elementary function is expanded, it is seen to consist of even and odd polynomials of x . Hence, a Gabor elementary function can be treated as a special form of a sequence of Hermite polynomials. And this observation motivates us to consider Hermite polynomials (or, rather, their generalised version) in their own right.

2.4 Generalised Hermite polynomials

2.4.7 Preliminaries: In contrast to what has been said above, Hermite polynomials seem to be ideal for handling the problems of reconstruction of signals from partial information. They are defined over $(-\infty, \infty)$, and have certain special characteristics which facilitate the reconstruction of signals from zero-crossings. (For information on the classical Hermite polynomials, see Reference 11.) Here, we introduce a generalised version of these polynomials to facilitate their application to the reconstruction problem under space-bandwidth constraints.

The generalised Hermite polynomials [7a] under consideration are parametrised by a , and generated by the recipe

$$H_n(t, \sigma) = (-1)^n \exp(t^2/2)(d^n/dt^n) \exp(-t^2) |_{t=t/\sqrt{\sigma}}$$

for

$$n = 0, 1, 2, 3, \dots$$

and governed by the second-order differential equation

$$\sigma^2(d^2y/dt^2) + (a - t^2 + 2n\sigma)y = 0 \quad (3)$$

for $n = 0, 1, 2, \dots$, and $\sigma > 0$

The first few polynomials are

$$H_0(t, a) = \exp[-t^2/(2 \cdot a)]$$

$$H_1(t, \sigma) = (2t/\sqrt{\sigma}) * H_0(t, \sigma)$$

$$H_2(t, \sigma) = [(4t^2/\sigma) - 2] * H_0(t, \sigma)$$

$$H_3(t, a) = [(8t^3/\sigma\sqrt{\sigma}) - 12t/\sqrt{\sigma}] * H_0(t, a)$$

$$H_4(t, a) = [(16t^4/\sigma^2) - (48t^2/\sigma) + 12] * H_0(t, \sigma)$$

These polynomials are orthogonal over the interval

$(-\infty, \infty)$. It is known [12] that these form a complete basis for the class C of real functions, $\phi(t)$, defined on the infinite interval $(-\infty, \infty)$, which are piecewise-continuous in every finite subinterval, $[-a, a]$, and satisfy the condition

$$\int_{-\infty}^{\infty} (1+t^2) \exp(-t^2/\sigma) \phi^2(t) dt < \infty$$

An important property of these polynomials is that they are Fourier-transformable, and their transforms are given by

$$\tilde{H}_n(j\omega, \sigma) = (-j)^n H_n(\omega\sigma, \sigma)$$

As a consequence, the Fourier transforms of the polynomials also satisfy a recursive relation which enables us to determine, quite simply, the effective spectral width of the signal in terms of the coefficients of expansion. This property facilitates the reconstruction of signals from partial information.

Another property is that the complex extension of $H_n(t, \sigma)$, for $n = 0, 1, 2, 3, \dots$, is an entire function of order 2 and type $(1/2\sigma)$, by virtue of which the function which is expressed by a series involving $H_n(t, \sigma)$ (or, rather, its complex extension) is also an entire function [13] of order 2 and type $(1/2\sigma)$, and signals of a class more general than those considered in the standard finite bandwidth theory can be analysed or synthesised. More importantly, the number of zeros of the signals under consideration does not need to obey the Shannon rule applicable to bandlimited signals. That is, the number of zeros in a unit interval is not proportional to the bandwidth. For instance, the simplest signal in the present framework is $[\exp(-x^2/2\sigma)]$, which does not have any zeros in the interval $(-\infty, \infty)$. On the contrary, according to classical theory, it should have an infinite number of zeros because the bandwidth is infinite. See Remarks 1 and 2 below.

We denote the class of functions which can be represented by the generalised Hermite polynomials by C_H .

The parameter σ controls the effective width of the signal in both the spatial and frequency domains; the smaller the value of σ , the smaller the spatial width and the greater the spectral width, and *vice versa*. A function in C_H is completely specified by the coefficients, τ_n , in the expansion eqn. 4 (see later). Alternatively, such a function could have, at the most, as many zeros (real and complex) as the highest degree of the polynomial part of the Hermite function in the expansion. As a consequence, the function is completely determined by these zeros.

Remark 1: Note that the number of these zeros is independent of the (essential) bandwidth, which is controlled by σ . At this point, it is interesting to note, that the results of the literature on the Nyquist rate for the zeros of deterministic functions or those of Rice [14] (on the zeros of random functions) cannot be employed because, in the present framework, the assumption of an ideal bandlimited/bandpass function is no longer valid.

Remark 2: In the literature on signal reconstruction, the class of signals is restricted to the class of entire functions of order 1 and type β , which implies that signals of bandwidth 2β (in the strict, classical sense) are considered. (See, for instance, Temes *et al.* [19, p. 200].) For the sake of completeness, a brief explanation follows.

We introduce the complex variable, z , defined by $z = x + jy$, and, further, define the class of bandlimited

signals of type β as follows:

$$B_\beta = \left\{ f(z) \mid f(x) = (1/2\pi) \int_{-\beta}^{\beta} \exp(j\mu x) F(\mu) d\mu; F \in L_2(-\beta, \beta) \right\}$$

According to the Paley-Wiener theorem [13, p. 103], the entire function $f(z)$ is of exponential type β and belongs to L_2 on the real axis if and only if

$$f(z) = (1/2\pi) \int_{-\beta}^{\beta} \exp(jzt) F(t) dt$$

where $F \in L_2(-\beta, \beta)$. Thus a square integrable exponential function is always bandlimited and *vice versa*. Moreover, there are some interesting results regarding the zeros of a bandlimited function and their distribution. (See, for details, Boas [13], and also Temes *et al.* [19, p. 202].) Noting that, for real bandlimited signals, zeros occur as either real zero-crossings or complex conjugate pairs, the density of the zeros is given by

$$\lim_{T \rightarrow \infty} \{\text{zero count}/T\} = 2\beta \text{ zeros/unit interval}$$

In other words, the zero-density rate of a bandlimited signal is equal to the Nyquist rate.

Remark 3: Wyner [21] deals with the problem of dimensionality of signals in L_2 , and generalises a result due to Slepian *et al.* [14]. Let $C_S(T, W, n)$, with $T, W > 0$, and $0 < \eta < 1$, represent the class of signals which satisfy $x(t) = 0$ for $|t| > (T/2)$, and

$$\int_{-(T/2)}^{(T/2)} x^2(t) dt \leq 1$$

These signals are time-limited to an interval to length T , and approximately bandlimited to W , where η is the measure of the amount of energy outside the band. The result of Slepian *et al.* [14] amounts to the following: as $\eta \rightarrow 0$, and the product $WT \rightarrow \infty$, the space of $C_S(T, W, \eta)$ is approximately $2WT$ -dimensional; in other words, $2WT$ (explicit) numbers/coefficients are required to uniquely define the signal. In contrast, the result of the present paper (on zero-crossing-based reconstruction) concerns implicit sampling, and the 'dimensionality' (in the terminology of Wyner [21]) of the signal is merely the (finite) degree of the polynomial (in the Hermite expansion) used for its representation minus the number of constraints imposed on the signal (in terms of space-bandwidth product or equivalent).

In what follows, Σ_n denotes summation with respect to n , which ranges, unless otherwise indicated, from $n = 0$ to $n = \infty$. Let the L_2 -norm squares of these polynomials be denoted by k_n^2 , for $n = 0, 1, 2, 3, \dots$, and those of their Fourier transforms by $\tilde{k}_n^2$.

For a real function, $f(t)$, defined on the infinite interval, we define formally the series

$$f(t) = \Sigma_n \tau_n H_n(t, \sigma), \quad -\infty < t < \infty \quad (4)$$

where the coefficients τ_n are calculated from the relation

$$\tau_n = (1/k_n^2) \int_{-\infty}^{\infty} \exp(-t^2/2\sigma) f(t) dt \quad (5)$$

for $n = 0, 1, 2, 3, \dots$. If $f(t)$ is piecewise-smooth in every finite interval $[-a, a]$, and if the integral

$$\int_{-\infty}^{\infty} \exp(-t^2/\sigma) f^2(t) dt$$

is finite, the series of eqn. 4 with coefficients calculated from eqn. 5 converges to $f(t)$ at every continuity point of $f(t)$ [11]. It is important to note here that the signals represented by eqn. 4 are not polynomial functions of finite degree, but their zeros are, however, those of mere polynomial functions and hence are finite in number.

Below, we assume, for simplicity, that $t_0 = 0$, and $w_0 = 0$. The following relations can be established [7, 8] (see Appendix, Section 6.1):

$$E = \sum_n \tau_n^2 k_n^2 \quad (6)$$

$$\int_{-\infty}^{\infty} t^2 f^2(t) dt = (\sigma/4) \sum_n [2(n+1)\tau_{n+1} + \tau_{n-1}]^2 k_n^2 \quad (7)$$

(where τ_n values with negative subscripts are set to zero), from which we obtain the effective spatial width as

$$X_2 = \sqrt{\{(\sigma/4) \sum_n [2(n+1)\tau_{n+1} + \tau_{n-1}]^2 k_n^2 / (\sum_n \tau_n^2 k_n^2)\}} \quad (8)$$

Similarly, the effective spectral width is given by

$$W_2 = \sqrt{\{(1/4\sigma) \sum_n [2(n+1)\tau_{n+1} - \tau_{n-1}]^2 k_n^2 / (\sum_n \tau_n^2 k_n^2)\}} \quad (9)$$

The effective space-bandwidth product is given by

$$\begin{aligned} X_2 \cdot W_2 &= (1/4) \sqrt{\{[\sum_n [2(n+1)\tau_{n+1} + \tau_{n-1}]^2 k_n^2] \\ &\quad \times [\sum_n [2(n+1)\tau_{n+1} - \tau_{n-1}]^2 k_n^2] \\ &\quad \div [\sum_n \tau_n^2 k_n^2]\}} \quad (10a) \end{aligned}$$

and the effective space-bandwidth ratio by

$$\begin{aligned} X_2/W_2 &= \sqrt{\{\sigma \sum_n [2(n+1)\tau_{n+1} + \tau_{n-1}]^2 k_n^2 \\ &\quad \div [\sum_n [2(n+1)\tau_{n+1} - \tau_{n-1}]^2 k_n^2]\}} \quad (10b) \end{aligned}$$

From eqn. 10b, it can be verified that the ratio of the two summations (contained in the curly brackets) can be expressed as a ratio of positive definite quadratic forms in the coefficients, τ_n , and hence has a finite upper value and a finite lower value (dependent on the maximum and minimum values of the two matrices of the quadratic forms). After simplifying eqn. 10b, it can be shown that

$$L_1 \leq (X_2/W_2)/\sigma \leq L_2 \quad (11)$$

where L_1 and L_2 are, respectively, the (square root of the) lower and upper values of the ratio of the quadratic forms in eqn. 10b. Note that L_1 and L_2 are dependent on the number of coefficients in the expansion, but are independent of σ [7]. From eqn. (8), if X_2 is finite, then from eqn. (9), W_2 is also finite for $\sigma > 0$. An important consequence of this is that the upper bound on uncertainty is finite, in contrast with the results of the literature based on the assumption of band-limited functions for which the upper bound on uncertainty is infinite.

2.4.2 Reconstruction theorem: As indicated above, the generalised Hermite polynomials are solutions of the differential eqn. 3. (See, for comparison, Szego [11].)

The reconstruction proposed here can be thought of as determining a function f , of independent variables t and σ , which can be expanded as a linear combination of the solutions of eqn. 3.

What we, in fact, propose to do is to expand the unknown function in terms of these polynomials. The problem of reconstruction is thereby transformed to the determination of the coefficients of this expansion, subject to the 'constraint' of the zero-crossings. The reconstruction problem can then be posed as follows. Given the signum function, S_g , or the zero-crossings of the unknown function f , find it as an expansion in terms

of the solutions to eqn. 10, such that

$$\text{signum}[f_a(t)] = S_g(t)$$

or the zero-crossings of $f_a(t)$ are the same as those of $f(t)$.

It turns out that the solution to this problem is non-unique if we are to deal with functions of class C_H , without any constraints. However, by imposing some reasonably practical constraints, it is possible to guarantee uniqueness analytically.

Further, in practice, the input data may be noisy. In this case, we need to consider approximate reconstruction. As a result, we are led to the minimisation of a suitable norm of the error

$$\text{signum}[f_a(t)] - S_g(t)$$

in the class of functions C_H . It turns out that this reduces, in the present framework, to one of minimisation in a parameter space, one element of which governs the effective spread of the function in both the spatial and spectral domains. We present a numerical implementation of an algorithm for reconstruction. Note that a minimisation of the norm of the error cannot be attempted by standard variational theory in view of the discontinuous nature of the error function.

Remark 4: If prolate spheroidal functions [15] are chosen to represent the unknown function, uniqueness conditions are more difficult to establish [7].

In what follows, we deal with unmodulated signals, i.e. signals for which $w_0 = 0$. It is assumed that the zero-crossing points (real or complex) are given as solutions to an equation of a polynomial of degree N . However, if only the signum function of the unknown signal is given, it is possible to arrive at the zero-crossing points as those where the signum function changes sign. In this case, we need to assume that there are no other zero-crossing points, i.e. that the unknown signal has only real zero-crossing points.

Let the polynomial equation be given by

$$\begin{aligned} t^N + \alpha_{N-1}t^{N-1} + \alpha_{N-2}t^{N-2} \\ + \dots + \alpha_2t^2 + \alpha_1t + \alpha_0 = 0 \quad (12) \end{aligned}$$

If the signal is represented by

$$\begin{aligned} f(t) = (t^N + \alpha_{N-1}t^{N-1} + \alpha_{N-2}t^{N-2} + \dots + \alpha_2t^2 \\ + \alpha_1t + \alpha_0) \exp(-t^2/2\sigma) \quad (13) \end{aligned}$$

where σ is unknown, it has the same zeros as those of eqn. 12. However, the signal is not unique. In fact, there are, theoretically, an infinite number of signals with the same zero-crossings, owing to the arbitrary nature of σ in eqn. 13. The differences amongst them show up only in their spatial and spectral widths. For instance, Fig. 2A shows plots of two different signals (analytically created) whose zero-crossings are exactly the same. Their Fourier transform magnitudes (scaled to highlight differences) are shown in Fig. 2B. Apparently, without some additional constraints on the reconstructed signal, it is impossible to realise a unique reconstruction. For the sake of clarity, we consider just two possible cases.

Case 1: If the effective spreads in the spatial and spectral domains are given explicitly as X_s and W_s , then, from eqn. 8 and 11, we obtain

$$(X_s/W_s)/L_2 \leq \sigma \leq (X_s/W_s)/L_1$$

There is no unique value of σ satisfying this inequality. However, we can choose either the maximum or the

minimum value of σ leading to a signal with maximal or minimal physical spread, respectively. Note that both

Fig. 2A Two different signals with the same zero-crossings

Fig. 2B Fourier transform magnitudes of signals in Fig. 2A

these values of σ are unique, and are dependent only on the order of the matrix of the quadratic forms in eqn. 10b. We insert this value of σ in eqn. 13 to get the reconstructed signal.

Case 2: The effective spread is given in only one (say, spatial) domain. To deal with this case, we represent the reconstructed function, $f_d(t)$, by

$$f_d(t) = \sum_{i=\phi}^N \tau_i H_i(t, \sigma) \quad (14)$$

where the values of τ_i , $i = \phi, 1, 2, \dots, N$, and σ are unknown. In effect, there are $(N+2)$ unknown parameters. A minor simplification of the righthand side of eqn. 14 gives

$$f_d(t) = \{[\tau_0 + \tau_1(2t/\sqrt{\sigma}) + \tau_2[(4t^2/\sigma) - 2] + \tau_3[(8t^3/\sigma\sqrt{\sigma}) - (12t/\sqrt{\sigma})] + \dots] \times \exp(-t^2/2\sigma)\} \quad (15)$$

in which we identify the coefficient of t^n as, say, $c_n(\sigma)$, for $n = 0, 1, 2, 3, \dots, N$. Comparison of the polynomial part of eqn. 15 with eqn. 12 yields the relation

$$c_n(\sigma) = \alpha_i \quad i = 0, 1, 2, 3, \dots, N \quad (16)$$

As the effective spatial spread is given as X_s , eqn. (8) gives

$$\Sigma_n [2(n+1)\tau_{n+1} + \tau_{n-1}]^2 k_n^2 = (4/\sigma)(\Sigma_n \tau_n^2 k_n^2) X_s^2 \quad (17)$$

Eqns. 16 and 17 can be combined to yield an equation of degree $(N+1)$ for the unknown σ . After solving for σ ,

each positive value of σ can be used to determine the other unknown values of τ_i from eqn. 16, which when substituted in eqn. 14 yield the reconstructed function. Note that there are at most $(N+1)$ values satisfying the constraints of the reconstruction problem. Of these, the only meaningful ones are those corresponding to $\sigma > 0$.

As a result, similarly to Case 1, the uniqueness of the reconstructed function (i.e. unique values for the unknown parameters, τ_i and σ in eqn. 14) requires that an additional constraint be imposed. This can take the form, for instance, of minimal or maximal spread of the signal, which can be characterised, respectively, by the smallest or the largest positive value of σ satisfying eqns. 16 and 17.

It is observed that the uniqueness in reconstruction, as formulated above, is rather indirect and somewhat unsatisfactory. It appears that there is no better solution analytically, unless we are given information about the derivative of the signal. This we analyse as Case 3 below:

Case 3: Suppose, in addition to the zero-crossing information given by eqn. 12 (as in Case 2 above), we know just one zero-crossing point of the derivative of the unknown signal. Let this point be specified as z_d . Then, it is possible to determine the σ value uniquely. On differentiating the assumed reconstruction signal (13) with respect to t , and setting it to zero at $(t = z_d)$, we get:

$$\begin{aligned} \sigma = \{ & t(t^N + \alpha_{N-1}t^{N-1} + \alpha_{N-2}t^{N-2} \\ & + \dots + \alpha_2t^2 + \alpha_1t + \alpha_0) \\ & \div [Nt^{N-1} + (N-1)\alpha_{N-1}t^{N-2} \\ & + (N-2)\alpha_{N-2}t^{N-3} + \dots + 2\alpha_2t + \alpha_1] \} \\ & (t=z_d) \end{aligned}$$

This seems to be the only possibility for a unique determination of σ .

Remark 5: Note that the reconstruction is unique only upto a scale factor. That is, if the original signal is $f(x)$, the reconstructed signal could be $K * f(x)$, where K , the gain factor, is a real number. This is because the effective spreads X_s and W_s are not affected by gain factors.

The main results of this paper can be summarised as a theorem.

Theorem 1: Let the zeros of the unknown signal be given by eqn. 12. Assume that the signal is unmodulated, and belongs to the class of signals which can be represented by the generalised Hermite polynomials in the series of the form in eqn. 4.

(i) Suppose that the effective spatial and spectral spreads, as defined by eqns. 2a and 2b, are given as X_s and W_s , respectively. Then, the reconstructed signal is given by eqn. 13, with σ obtained from the inequality 11 as explained in Case 1.

(ii) Suppose the signal has an effective spatial width, X_s . There are at most $(N+1)$ solutions to the problem of signal reconstruction. However, if the spread of the signal is constrained to be minimum (or maximum), the signal can be reconstructed uniquely (up to a scale factor) by solving the set of algebraic equations, eqns. 16 and 17. (See Appendix, Section 6.3, for a proof of the theorem.)

Corollary I: If the zero-crossing locations or, equivalently, the coefficients of the polynomial eqn. 12 are not exact and are affected by noise, the reconstructed signal is well behaved, in the sense that the error in the recon-

structions is linear in the perturbations of the zero crossing information. (See Reference 7.)

Corollary 2: If, in addition to the zero-crossing information [in the form of the polynomial equation (eqn. 12) or equivalent to it], we are given one zero of the derivative of the unknown signal, then σ can be determined uniquely, and the reconstructed signal is given by eqn. 13.

Remark 6: As far as bandpass signals are concerned, the simplest case is the one in which the signal can be represented by

$$f_{mod}(t) = f(t) \cos \mu t = \sum \tau_n H_n(t, \sigma) \cos \mu t \quad (18)$$

[where $f(t)$ is as defined in eqn. 4] which corresponds to functions analysed by Logan [3] and others. The zeros of $f_{mod}(t)$ in eqn. 18 are given by the zeros of $f(t)$ and of $\cos \mu t$. Given the signum function of $f_{mod}(t)$, standard Fourier analysis gives the dominant frequency, which can be identified with μ . The zeros of $f_{mod}(t)$ contributed by $\cos \mu t$ are then removed, thereby reducing the problem to the case considered above.

The following result plays a useful role in the theoretical analysis of related problems, as also in a computer implementation of the above theorem. This result is a generalisation of the one usually meant for entire functions of order 1, as found, for instance, in Boas [13, pp. 23–24].

Lemma 1: If $f(z)$ is an entire function of order 2, and is represented by the expansion

$$f(z) = \sum_{n=0}^{\infty} \tau_n H_n(z, \sigma) \exp(-z^2/2\sigma)$$

(where τ_n, σ are real) which is real for real z , and has only real zeros, then the zeros of $df(z)/dz$ are also real and are interlaced with (or separated by) the zeros of $f(z)$. (See Reference 7, also Reference 13, p. 23.)

Remark 7: This lemma can be used to check whether the signal after reconstruction has only real zeros or not. To this end, the derivative of the reconstructed signal is checked for its zeros. If these are interlaced with the zeros of $f(z)$, it implies that the original signal has only real zeros. In the numerical implementation, too, this result can be used to guide the choice of the optimal parameters in reconstruction.

Remark 8: A two-dimensional version of the above theorem is derived in Reference 7 by converting the zero-crossing contours into a constraint equation. Such an approach totally dispenses with any reference to the ‘infinite zero-crossing points’. This is distinct from the results of the literature on two-dimensional signal reconstruction for which ‘infiniteness’ of the zero-crossing points or the irreducibility of two-variable functions plays a crucial role.

2.4.3 Limitations of the proposed solution: To bring out the mathematical peculiarities of the reconstruction problem and the limitations of the proposed solution, we now present an example.

Suppose the unknown signal $f(t) \in C_H$ has an effective bandwidth W_s . Let it be represented by eqn. 3. Assuming that $f^2(t) \in C_H$, consider the related signals $f_1(t) = [f^2(t) - 1]$ and $f_2(t) = [f^2(t) + 1]$ which are also essentially bandlimited. Note that f_1 has zeros at those points where

$f(t) = 1$ or -1 . If the signum function of $f_1(t)$ is specified, it is assumed that the zeros of $[f(t) - 1]$ and $[f(t) + 1]$ are real and simple. The theorem can be applied to the reconstruction of $f_1(t)$. The bandwidth of the signal is controlled by σ' in the representation

$$f_1(t) = \sum \tau'_n H_n(t, \sigma') \exp(-t^2/2\sigma')$$

The coefficients τ'_n are chosen so as to have the zeros of $f_1(t)$ the same as the zeros of $[f(t) - 1]$ and $[f(t) + 1]$ together. Still, the reconstructed signal is arbitrary, unless σ' is fixed by some additional constraint, in the manner made explicit in the statement of the theorem. Note that the class of signals under consideration is not amenable to an application of the results of the literature. Furthermore, if the effective spreads in both the spatial and spectral domains are specified, then σ' can be computed directly. As a result, the signal is uniquely reconstructed.

On the other hand, the signal $f_2(t)$ does not have real zeros, and hence no information on its signum function can be given. However, if the complex zeros of the signal can be specified as solutions to the polynomial eqn. 11, the proposed procedure can still be applied.

For limitations with respect to the algorithmic implementation, see below.

3 Computer implementation

Given the exact locations of the N zeros of the signal, a solution of the $(N + 1)$ -th degree polynomial in σ can be obtained by standard numerical procedures. However, it is not always the case that complete and correct information is available on the zero-crossings. In such instances, it would be desirable to compute ‘approximate’ solutions. Moreover, the error functional defined for quantifying the difference between the given signum function and the signum function of the reconstructed signal is nonconvex. Therefore, we have to have recourse to techniques such as simulated annealing [15] (see Appendix, Section 6.2) to compute the unknown parameter values of the function being reconstructed. For an implementation of reconstruction on the basis of the theorem, a numerical strategy has been devised, using simulated annealing, to compute an approximate solution starting from the given (desired) signum function and specifications such as effective spatial width.

It is assumed that the effective spatial width of the signal is X_s , and it is desired that the actual physical spread is minimal (or maximal). Let the approximate signal be represented by eqn. 13, and the number of total iterations in one optimisation operation be limited by I . The reconstruction algorithm is given in the Appendix, Section 6.4.

To illustrate the efficacy of the proposed generalised Hermite representation and the numerical optimisation procedure, we give just one example (Figs. 3A and B). The reconstruction is based on zero-crossing information (in the form of the signum function), a specification of the effective spatial width and minimality of the physical spread. (Recall that the reconstruction is unique only up to a scale factor.)

Computer studies conducted on these lines have shown that, in the case of one-dimensional signals, the optimisation technique for signal reconstruction based on zero-crossings works well.

Remark 9: The reconstruction algorithm, as implemented on a Motorola 68020 processor, when applied to a signal described by 512 points, takes a couple of hours.

However, the same algorithm can be run on a multiprocessor machine (with each processor starting from a different set of parameter values), thereby reducing the

Fig. 3A Overlay of original (—) and reconstructed (---) signals

Fig. 3B Fourier transform magnitudes of original (—) and reconstructed (---) signals

computation time. Nevertheless, it is very unlikely that a realtime solution to a practical reconstruction problem can be achieved.

4 Conclusions

A new approach to the problem of reconstruction of signals, given partial information of the type of zero-crossing points has been proposed. The superiority of the results lies in the fact that the traditional assumption of strict bandlimitedness has been dispensed with. Computational implementation shows that the proposed procedure of employing simulated annealing works well. Analysis related to reconstruction from one-bit phase or from Fourier magnitude information can also be accomplished using the generalised Hermite polynomials. Results on these and on the two-dimensional reconstruction problem will appear elsewhere.

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6 Appendix

6.1 Hermite coefficient relations

The generalised Hermite polynomials generated by the function eqn. 1 can be shown to satisfy the recurrence relation

$$tH_n(t, \sigma) = (\sqrt{\sigma/2})[H_{n+1}(t, \sigma) + 2nH_{n-1}(t, \sigma)] \quad n = 0, 1, 2, 3, \dots, N \quad (19)$$

In the Fourier domain, denoting the Fourier transform of $H_n(t, \sigma)$ by $\tilde{H}_n(j\omega, \sigma)$, the corresponding recurrence relation is

$$j\omega\tilde{H}_n(t, \sigma) = [1/(2\sqrt{\sigma})][2n\tilde{H}_{n-1}(j\omega, \sigma) - \tilde{H}_{n+1}(j\omega, \sigma)] \quad n = 0, 1, 2, 3, \dots, N \quad (20)$$

Let

$$k_n^2 = \int_{-\infty}^{\infty} H_n^2(t, \sigma) dt$$

and

$$\tilde{k}_n^2 = \int_{-\infty}^{\infty} |\tilde{H}_n(j\omega, \sigma)|^2 d\omega$$

Then it is easy to verify that

$$\tilde{k}_n^2 = (\sqrt{\sigma})k_n^{*2} \quad (21)$$

Assuming that the function $f(t)$ belongs to the class of functions C introduced in Section 2, $f(t)$ can be expanded in terms of the Hermite functions; i.e.

$$f(t) = \sum_n \tau_n H_n(t, \sigma) \quad (22)$$

The standard result is

$$E_1 = \int_{-\infty}^{\infty} f^2(t) dt = \sum_n \tau_n^2 k_n^2 \quad (23)$$

Let

$$tf(t) = \sum_n \mu_n H_n(t, \sigma)$$

By the use of the recurrence relation, eqn. 19, we obtain

$$\mu_n = (\sqrt{\sigma/2})[2(n+1)\tau_{n+1} + \tau_{n-1}] \quad (24)$$

Similarly, let $jwF(jw) = \sum_n \tilde{\mu}_n \tilde{H}_n(jw, \sigma)$. By the use of the recurrence relation, eqn. 20, we obtain

$$\tilde{\mu}_n = [1/(2\sqrt{\sigma})][2(n+1)\tau_{n+1} - \tau_{n-1}] \quad (25)$$

As a consequence, we have,

$$\int_{-\infty}^{\infty} t^2 f^2(t) dt = \sum_n \mu_n^2 k_n^2 = (\sigma/4) \sum_n [2(n+1)\tau_{n+1} + \tau_{n-1}]^2 k_n^2 \quad (26)$$

(where τ_n values with negative subscripts are set to **zero**), from which we obtain the effective spatial width as

$$X_2 = \sqrt{\{(\sigma/4) \sum_n [2(n+1)\tau_{n+1} + \tau_{n-1}]^2 k_n^2 / (\sum_n \tau_n^2 k_n^2)\}} \quad (27)$$

Note that the smaller the value of σ , the smaller the spatial spread. Similarly, the effective spectral width is given by

$$W_2 = \sqrt{\{(1/4\sigma) \sum_n [2(n+1)\tau_{n+1} - \tau_{n-1}]^2 \tilde{k}_n^2 / (\sum_n \tau_n^2 \tilde{k}_n^2)\}} \quad (28)$$

In contrast with the spatial spread, the smaller the value of σ , the greater the spectral spread.

6.2 Simulated annealing

Annealing is a manufacturing process in which a molten metal is cooled extremely slowly to produce a stress-free solid. In mathematical terms, this means achieving a configuration of atoms with energy near or at the global minimum.

Simulated annealing, introduced by Kirkpatrick [16] and others, is a strategy for combinatorial optimisation problems, such as that of minimising a function of many variables. When it is applied to nonphysical optimisation problems, we employ an analogue of 'controlled cooling operations', to achieve the desired solution. For instance, in the present problem of reconstruction, we seek to find a 'configuration' of parameters, σ , and τ_i , for $i = 0, 1, 2, \dots, N$ that minimises an error function. We are interested in iterative improvement strategies, which attempt to perturb some existing, suboptimal solution (for the parameters) in the direction of a better, smaller error solution. However, this strategy becomes easily trapped in the local minima of the error function of the type used in this paper. One scheme to overcome this limitation is simply to try many initial sets of parameters, improve each set, and ultimately use the best answer found. However, a good solution cannot be guaranteed.

Simulated annealing offers a strategy similar to the iterative improvement, but with a major difference. Annealing allows perturbations to move uphill, on the 'surface' of the error function, in a controlled fashion. Consequently, it is possible to jump out of local minima and potentially fall into a more promising downhill path. The control is exercised through the parameter T (pseudotemperature) in the annealing algorithm.

The main idea, then, is to propose some perturbation in the parameter set $\{\sigma, \tau_i | i = 0, 1, \dots, N\}$, and evaluate the change, δE , in the error function. If $\delta E < 0$, the new configuration has low 'energy', and is accepted as the start point for the next 'move' or perturbation. However,

if $\delta E > 0$, the move may still be permitted under certain conditions. The new, higher energy configuration is acceptable, as moderated by the current temperature. The probability of accepting a move which results in $\delta E > 0$, at temperature T , is given by

$$\text{Prob}[\text{accept}] = \exp(-\delta E/T)$$

In practice, however, this probabilistic acceptance is achieved by generating a uniform random number, u , in $[0, 1]$, and comparing it against $\text{Prob}[\text{accept}]$. Only if $\text{Prob}[\text{accept}] > u$, is the move accepted. Thus, it is likely that very probable moves may be rejected, and very improbable moves may be accepted, at least occasionally. By successively lowering the temperature, and running the algorithm, we can simulate the material coming to equilibrium at each newly reduced temperature.

Initially, the effective temperature parameter is high enough to permit an aggressive random search of the parameter space. A typical stopping criterion is to terminate annealing when the error reduction over three/four successive temperatures is sufficiently small, e.g. less than 1%.

The algorithm as actually implemented is given in Section 6.4.

6.3 Proof of theorem

The signal under consideration has N zeros (real or complex) specified by the polynomial eqn. 12. If the zeros are real, and only the signum function of the signal $f(t)$ is given, the polynomial part of eqn. 12 can be constructed.

Let the attempted reconstruction function be denoted by $f_a(t)$, and expanded in terms of the generalised Hermite functions $H_n(t, \sigma)$, to obtain eqn. 15, where it is understood that τ_i for $i = N+1, N+2, N+3, \dots$, are all zero. The unknown parameters are σ , and τ_i , for $i = 0, 1, 2, \dots, N$.

Matching the corresponding coefficients of t in eqns. 12 and 15 (in expanded form), we obtain

$$\begin{aligned} \tau_N &= [\alpha_N \sigma^{(N/2)}/2^N] \\ \tau_{N-1} &= [\alpha_{N-1} \sigma^{(N-1)/2}/2^{(N-1)}] \\ \tau_1 &= (q_1)\alpha_1 + (q_3)\alpha_3 + \dots \end{aligned}$$

and

$$\tau_0 = (q_0)\alpha_0 + (q_2)\alpha_2 + \dots \quad (29)$$

where the coefficients denoted by q_0, q_1, q_2 , and q_3 are functions of σ and $\sqrt{\sigma}$.

If we can fix σ , the values of the coefficients τ_n are unique. Theoretically, fixing any one of the $(N+2)$ unknown coefficients given by the $(N+1)$ eqns. 25 will guarantee unique values for the rest. However, this procedure is not (physically) as meaningful as fixing σ .

If both X_s and W_s are specified, σ obeys inequality 11. The maximum and minimum eigenvalues of the numerator and denominator matrices of eqn. 10b can be computed. Correspondingly, choice of the lower value L_1 leads to the maximal value of σ . Similarly, the choice of the upper value L_2 leads to the minimal value of σ . These values are unique, once the degree of the polynomial is fixed (corresponding to the number of real zeros given). This proves the first part of the theorem.

As far as the second part is concerned, only a knowledge of X_s is to be assumed. Solve for σ from eqns. 29 and 16, leading to an $(N+1)$ -degree equation in σ [which, therefore, has $(N+1)$ solutions]. However, if we impose the constraint of positivity and minimality (or maximality) on σ to pick the relevant root, we can

compute the unknown coefficients, τ_i ($i = 0, 1, 2, \dots, N$) from eqn. 29, and substitute them into eqn. 14 to obtain a unique (up to a scale factor) $f_a(t)$. The theorem is proved.

6.4 Reconstruction algorithm

Program Reconstruction [input = $\text{sgn } f(t)$ and S_e ; output = τ_n, σ, μ].

Begin

Step 0: Find the dominant frequency $\mu^{(0)}$ of $\text{sgn } f(t)$ by a Fourier analysis. Set $\mu = \mu^{(0)}$.
Solve for the minimal (maximal) positive root of the algebraic equation in σ obtained by combining eqns. 29 and 16. Let this value be given by σ_c .
Choose coefficients τ_n randomly.
Let these be denoted by $\tau_n^{(0)}$.
Set $\tau_c = \tau_n^{(0)}$ and $\sigma = \sigma_c$.
Set $f_a(t) = \sum \tau_n H_n(t, \sigma) \cos \mu t$.
Set $i = 1$.

Step 1: Compute $\text{sgn } f_a^{(i)}(t)$, S_{ea}^i .

Find the errors,

$$e_1^{(i)} = \sum_{\text{interval}} |\text{sgn } f_a^{(i)}(t) - \text{sgn } f(t)|,$$

and

$$e_2^{(i)} = |S_{ea}^{(i)} - S_e|$$

Step 2: If $e_1^{(i)} >$ prespecified value, and $e_2^{(i)} >$ prespecified value and $i < I$ then

perturb the parameters $\tau_n^{(i)}$, $\sigma^{(i)}$, and $\mu^{(i)}$ in a simulated annealing scheme such that σ is always > 0 , and is the minimal (maximal) root of the algebraic equation obtained by combining eqns. 29 and 16.

Set $i = i + 1$;

Go to Step 1.

else if $e_1^{(i)} >$ prespecified value, and

$i > I$ then set $N = N + 1$ and $i = 1$;

else go to Step 3.

Step 3: Check whether the value of σ so obtained is the minimal (maximal) positive root of the algebraic equation obtained by combining eqns. 29 and 16.

[* check whether the roots of $f'(z)$ are interlaced with the given zeros of $f(z)$].

If this is true,

tabulate the values of τ_n , σ , and μ .

Goto Step 4.

else set $i = I$ and goto Step 1.

Step 4: STOP

End