Hermite polynomials for signal reconstruction from zero-crossings

Part 1 : One-dimensional signals

Y.V. Venkatesh

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Abstract: Generalised Hermite polynomials are employed for the reconstruction of an unknown signal from a knowledge of its zero-crossings, under certain conditions on its spatial/spectral width, but dispensing with the assumption of bandlimitedness. **A** computational implementation of the proposed method is given for one-variable (or one-dimensional) signals, featuring an application of simulated annealing for optimal reconstruction.

1 Introduction

The present paper deals with the problem of reconstruction of signals of one variable (for instance, time or space, termed, in the literature, 'one-dimensional signals') from zero-crossing information. This problem of reconstruction belongs to the class of so-called inverse problems, and is of relevance to optics, acoustics, crystallography and vision, among many other fields.

In the case of time-signals, the zero-crossing information is believed to be important in a situation where a signal has been subjected to a nonlinear distortion, like hard-clipping, which leaves the zero-crossings of the signal intact. The main question (which will be formulated more precisely below) is whether (and under what conditions) it is possible to reconstruct the original signal from a knowledge of its zero-crossings only.

The literature on this problem contains results which, in a majority of cases, seem to be ***ad hoc***. In view of the constraint on space, only the most relevant references are briefly mentioned here. For more details and **a** critical analysis of the available results, see Reference 7.

Perhaps the first paper on the reconstruction of one-dimensional signals from zero-crossings is due to Requicha [1]. The algorithm of Voelcker and Requicha [2] uses a bandpass operator, and the authors use the contraction-mapping argument in the attempt to establish that the solution is unique. Logan **[3]** attempts to establish that a one-dimensional bandpass signal (under constraints on the bandwidth) can be uniquely recovered from its zero-crossings if there are no complex zeros common to the signal and its Hilbert transform.

Poggio *et al.* **[4,** 5] consider the problem of a unique representation (within a multiplicative constant) of a periodic one-dimensional bandpass function by means of its zero-crossings. They use the zero-crossings of the signal filtered by a Gaussian filter with the variance parameter, a, which controls the width of the filter. Their result is that the map of all these zero-crossings determines the signal uniquely if the filtered signal can be represented by a finite polynomial.

The recent report of Hummel and Moniot [6] deals with the reconstruction problem in 'scale-space', by which is meant a one-parameter family of data sets obtained from the Laplacian of a Gaussian-filtered version of the image. **As** in References **4** and 5, the width of the filter is controlled by the variance parameter, a. Hummel and Moniot [6] deal with the possibility of reconstruction given the zero-crossings at multiple scales of resolution in the so-called scale-space (in which the zero-crossing locations are plotted against the variance parameter, a). They conclude that 'reconstruction is possible, but can be unstable'. In addition, they suggest the inclusion of gradient data along the zero-crossings in the representation, and demonstrate that the reconstruction is then stable. However, the class of functions considered by Hummel and Moniot is inadequate to deal with the general zero-crossing problem. (See, for details, References 7 and 8). Moreover, this class is included in the class introduced in Section 2.

In this paper, we propose what seems to be a novel solution to the reconstruction problem, using in part the results of de Bruijn [9] on the characterisation of uncertainty in signals without the constraint of strict bandlimitedness. (See also Reference 10 in the context of uncertainty analysis).

2 New framework for reconstruction

Most signals encountered in nature are (i) neither periodic nor made up of a set of elementary singular functions, which are confined to infinitely small spatial/time intervals; and (ii) infinite neither in spatial nor in spectral extent. Therefore, for convenience in analysis and synthesis, they are to be represented by (nonsingular) elementary functions which are confined both along the spatial

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and spectral axes. It may be noted here that, in the literature on signal processing, one- and two-variable functions are called one- and two-dimensional signals, respectively.

If a one-variable signal (such as a speech signal as a function of time) is considered in a time-window of a finite size (as is normally done in speech signal analysis), its Fourier transform cannot have finite width. The duality property in Fourier analysis tells us that confining the spectrum of the speech signal to a finite part in the frequency domain forces, in turn, the signal to have infinite extent (or width) in the time domain.

Similarly, if we consider an image as a function of two (space) variables in the (finite) field of view of a camera or of the human visual system, its Fourier transform cannot be confined to a finite part of the (two-variable) spectral domain. Moreover, it is believed, on the basis of empirical findings, that the human visual system does have characteristics which exhibit this type of finiteness in both the spatial and spectral domains.

On the contrary, recall that the assumption of finite bandwidth is the foundation on which most of the results of both one- and two-dimensional signal processing are based, including those of reconstruction from zero-crossing information.

In the paper, we propose the use of a window of a certain 'effective width' for each of the two domains. If, as in the case of the speech-signal analysis, a time-frame is to be considered, this could be imbedded in the effective width. Similarly, the finite physical image could be contained in the region bounded by the effective widths in the x- and y-directions. Of course, the effective width is determined by the nature of the signal, and is in fact related to the uncertainty, in its characterisation [10, 19 and 21].

An analogy with the complex-frequency Fourier transform is perhaps appropriate here. It is the positive part **of** the frequency that is of practical relevance, whereas the negative part is necessitated by the mathematical framework involving complex exponential functions.

Similarly, the framework proposed in this paper is applicable, in general, to situations in which the 'time/space' window can be imbedded in an 'effective-width' window, thereby facilitating the development of a consistent mathematical theory.

2.1 Information content in zero-crossings

A reference has already been made to the significance of reconstruction of a one-variable signal from zero-crossing information. Apart from this, a virtue of signal representation by zero-crossings is that it eliminates dependence on dynamic response range (gain), as the zeros are independent of signal amplitude. It should, however, be noted that this necessitates high accuracy in the localisation of the zeros.

As far as images (or functions of two variables) are concerned, a scheme suggested for image representation (see, for instance, Marr [17]) in the models of biological vision is the encoding of the extrema in the blurred spatial derivatives of the image. The blurring is modelled by a Gaussian function. Furthermore, such representations are useful, and contain significant information because they identify image regions marked by abrupt changes in brightness, such as those that arise at the edges of the objects and occlusion boundaries. Mathematically, the extrema in the blurred spatial derivatives are the loci of zero-crossings. Fig. 1 shows a typical scan-line of the Laplacian of a Gaussian-blurred image. According to Marr [17], the zero-crossings of the scan-line shown in Fig. 1 are points where important information (pertaining to the original image) is contained. When

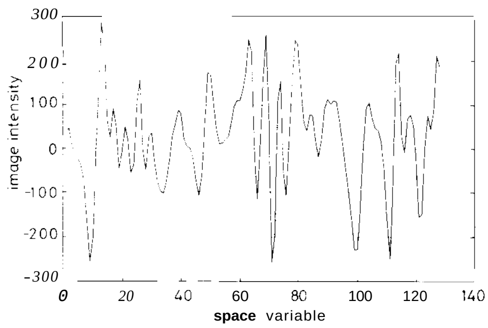

Fig. 1 *Typical scan-line of a Laplace-Gaussian filtered image*

we deal with images (as two-variable functions), the zero-crossing contours result in something similar to line drawings. And we know how perceptually rich are line drawings, such as cartoons. Therefore, the zero-crossing scheme is suggested as a plausible approach to compact information encoding in biological visual systems.

2.2 Space of one-variable signals under consideration

We assume that the signals under consideration are defined over $(-\infty, \infty)$ in both the time/spatial and spectral domains. In what follows, t is an independent variable which, in practice, could be interpreted as either time or space. Let $f(t)$ be a real-valued function of $t \varepsilon \mathscr{R}$, with the Fourier transform

$$F(jw) = \int_{-\infty}^{\infty} f(t) \exp(-jwt)\, dt$$

The two functions, $f(t)$ and $F(jw)$, form a Fourier integral pair. The classical uncertainty principle says that they cannot both be of short duration or be highly concentrated.

One interpretation of this is obtained by defining the spread of the function as follows. We say that f is essentially concentrated [10] on a measurable set X, if there is a function $g(t)$ vanishing outside X_1 such that

$$\|f - g\| \leqslant \delta$$

where $\|\cdot\|$ is a suitable norm and 6 is a (prespecified, arbitrarily) small positive quantity.

Similarly, we say that $F(jw)$ is essentially concentrated on a measurable set W_1 if there is a function $G(jw)$ vanishing outside W_1 with

$$\|F(jw) - G(jw)\| \leqslant \delta$$

for some suitable norm $\|\cdot\|$ and positive δ.

Then the function $f(t)$ is essentially zero outside an interval of length X_1, and $F(jw)$ is essentially zero outside an interval of length W_1. Under these conditions, the uncertainty principle leads to the inequality, $X_1 W_1 \geqslant 1$.

A similar inequality for the uncertainty principle can be obtained by defining the time/spatial and spectral spreads of the function as follows. Let

$$E = \int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(w)|^2\, dw/(2\pi) \qquad (1)$$