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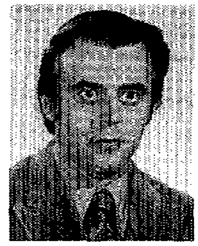

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L_2-Stability of Nonstationary Feedback Systems: Frequency-Domain Criteria

MALUR K. SUNDARESHAN AND M. A. L. THATHACHAR

Abstract—**A frequency-domain positivity condition is derived for linear time-varying operators in L_2 and is used to develop L_2-stability criteria for linear and nonlinear feedback systems. These criteria permit the use of a very general class of operators in L_2 with nonstationary kernels, as multipliers. More specific results are obtained by using a first-order differential operator with a time-varying coefficient as multiplier. Finally, by employing periodic multipliers, improved stability criteria are derived for the nonlinear damped Mathieu equation with a forcing function.**

I. Introduction

THE employment of an operator theoretic framework for stability studies has resulted in the development of the positivity theorem (due to Zames [1]) as a versatile tool for the input-output stability analysis of feedback systems. Many useful criteria for the L_2-stability of time-invariant nonlinear systems [2] and systems with isolated time-varying gains [3],[4], have been developed in the last few years, following this approach. However, the derivation of similar results in the more general situation of systems containing linear time-varying operators which do not admit a separation of time-variations, has not received much attention. Recalling that the application of the positivity theorem requires the open-loop to be factored into a composition of two positive operators, the unpopularity of this problem may be attributed to the difficulty of obtaining positivity conditions for arbitrary time-varying operators. Recently, by using an internal (state-space) description of the operator, Willems [5], Estrada and Desoer [6], and the authors [7] have obtained positivity conditions in the time-domain (and from these L_2-stability criteria [7] for feedback systems containing a time-varying linear part). These positivity conditions require the solution of certain associated Riccati equations and are not easy to check, except in a few simple cases.

This paper presents simpler frequency-domain conditions for the positivity and L_2-stability of time-varying systems, derived with the imposition of certain additional constraints of differentiability, null initial conditions, etc. The method draws inspiration from a recent paper due to Blodgett and Young [8], which gives an absolute stability

Manuscript received October 30, 1972; revised March 23, 1973. Paper recommended by J. C. Willems, Past Chairman, and R. A. Skoog, Chairman of the IEEE S-CS Stability, Nonlinear, and Distributed Systems Committee.
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criterion for a zero-input feedback system with a time-varying linear part and a Popov-type nonlinearity, using a multiplier $(1 + qs)$, $q > 0$. It should however be mentioned that, although [8] contains many valuable ideas, the main stability theorem appears to be incorrect. A corrected version of the stability criterion of [8] may be obtained as a special case of the criteria derived in this paper, which permit the use of more general time-varying multipliers.

II. Problem Formulation

Notations and Definitions

A certain familiarity with the notions of L_p-spaces will be assumed. Let R, R^+, and R^n denote, respectively, the real numbers, the nonnegative real numbers, and n-dimensional Euclidean space.

The concept of the extended L_2-space (L_{2e}) is defined by,

$$L_{2e} = \{x(\cdot):x_T(\cdot) \in L_2, \qquad \forall\, T \in R^+\} \tag{2.1}$$

where $x_T(\cdot)$ is the truncation of $x(\cdot)$, $x_T(t) = x(t)$, $\forall\, t \in [0,T]$, and is zero otherwise.

An operator H in L_2 (L_{2e}) is defined as a single-valued mapping of $L_2(L_{2e})$ into itself. H is said to be a "causal" operator in L_2 (L_{2e}) if $(Hx(\cdot))_T = (Hx_T(\cdot))_T$ $\forall\, x(\cdot) \in L_2$ (L_{2e}) and $\forall\, T \in R^+$.

Let $\mathcal{O}_E$ denote the class of linear causal operators H in L_{2e} with an external (input-output) description, i.e., if $H \in \mathcal{O}_E$, then there exists a map (the kernel of H) $h(\cdot,\cdot)$: $R^+ \times R^+ \rightarrow R$ such that,

$$y_H(t) = Hu_H(t) = \int_0^\infty h(t,\tau)u_H(\tau)\, d\tau \;\; \forall\, u_H(\cdot) \in L_{2e} \tag{2.2}$$

where $u_H(\cdot):R^+ \rightarrow R$ is the input to H, $y_H(\cdot):R^+ \rightarrow R$ is the output of H and $h(t,\tau) = 0\;\; \forall\, \tau > t$.

Let $\mathcal{O}_I$ denote the class of linear causal operators H in L_{2e} with an internal (state-space) description, i.e., if $H \in \mathcal{O}_I$, then there exists a quadruple $\{A_H(\cdot),b_H(\cdot), c_H(\cdot)$, and $d_H(\cdot)\}$ and a positive integer n, where $A_H(\cdot)$: $R^+ \rightarrow R^n \times R^n$, $b_H(\cdot):R^+ \rightarrow R^n$, $c_H(\cdot):R^+ \rightarrow R^n$ and $d_H(\cdot):R^+ \rightarrow R$, such that H is described by the dynamical equations,

$$\begin{aligned} \dot{x}_H(t) &= A_H(t)\, x_H(t) + b_H(t)\, u_H(t) \\ y_H(t) &= c_H'(t)\, x_H(t) + d_H(t)\, u_H(t) \end{aligned} \tag{2.3}$$

where $c_H'(t)$ denotes the transpose of $c_H(t)$,$x_H(\cdot):R^+ \rightarrow R^n$ is the state of H and $u_H(\cdot)$, $y_H(\cdot)$ are defined as earlier.

Let $\mathcal{O}_{IPA}$ be the subset of $\mathcal{O}_I$ consisting of those operators H which satisfy the following requirements:

1) The internal description of H is in the phase-variable canonical form.

2) The elements of $A_H(t)$ and $c_H(t)$ are n-times differentiable with respect to t, n being the dimension of the state-vector of H.

3) H is exponentially asymptotically stable with zero input, i.e., $\|x_H(t)\|_E \leqslant K \|x_H(t_0)\|_E \exp(-a(t - t_0))\;\; \forall\, t \geqslant t_0$, K, and a being positive constants and $\|\cdot\|_E$ denoting the Euclidean norm.

4) H has zero initial conditions, i.e., $x_H(0) = 0$.

Because of property 3 above, it is simple to observe that, $H \in \mathcal{O}_{IPA}$ implies $H \in \mathcal{O}_E$ with the kernel restricted by $|h(t,\tau)| \leqslant K \exp(-a(t - \tau))$, which in turn implies that $H:L_2 \rightarrow L_2$.

An operator $H \in \mathcal{O}_E$ is said to have "finite gain" if

$$\gamma(H) = \sup_{\substack{u_H(\cdot)\in L_{2e}, T\in R^+ \\ u_{HT}(\cdot) \neq 0}} \frac{\|(Hu_H(\cdot))_T\|}{\|u_{HT}(\cdot)\|} < \infty, \tag{2.4}$$

the norms indicated being the L_2-norms. Note that $H \in \mathcal{O}_{IPA} \Longrightarrow \gamma(H) < \infty$.

An operator $H \in \mathcal{O}_E$ is said to be "positive(e)" [strongly positive(e)] if the inequality,

$$\langle u_{HT}(\cdot),(Hu_H(\cdot))_T\rangle \geqslant \epsilon\, \langle u_{HT}(\cdot),u_{HT}(\cdot)\rangle, \qquad \forall\, u_H(\cdot) \in L_{2e} \tag{2.5}$$

and $\forall\, T \in R^+$ holds with $\epsilon = 0[\epsilon > 0]$. If in addition $H \in \mathcal{O}_I$, then $y_H(\cdot)$ given by (2.3) may be used in the place of $Hu_H(\cdot)$ in (2.5).

If H is a causal operator in L_2, then

$$H \text{ positive}(e) \Longleftrightarrow \langle u_H(\cdot),Hu_H(\cdot)\rangle \geqslant 0, \qquad \forall\, u_H(\cdot) \in L_2. \tag{2.6}$$

System Description

The system under consideration has the configuration as in Fig. 1 and is described by the input-output relations,

$$\begin{aligned} e_1(t) &= u_1(t) - w_2(t) \\ e_2(t) &= u_2(t) + w_1(t) \\ w_1(t) &= Ge_1(t) \text{ and } w_2(t) = Fe_2(t) \end{aligned} \tag{2.7}$$

with the following assumptions:

Assumption 1: $u_1(\cdot),u_2(\cdot) \in L_2$, and $e_1(\cdot),e_2(\cdot) \in L_{2e}$.

Assumption 2: $G \in \mathcal{O}_E \cap \mathcal{O}_I$.

Assumption 3: F is a time-invariant Popov-type nonlinear memoryless operator in L_{2e} defined by,

$$Fx(\cdot) = f(x(\cdot)), \qquad \forall\, x(\cdot) \in L_{2e} \tag{2.8}$$

$$f(\cdot):R \rightarrow R,\, f(0) = 0, \text{ and } xf(x) \geqslant 0, \qquad \forall\, x(\cdot) \in R.$$

Let us denote the class of operators F satisfying (2.8) by $\mathfrak{F}_P$.

The Main Problem

Find conditions on G and F which ensure that the system described by (2.7) is L_2-stable, i.e., $u_1(\cdot),u_2(\cdot) \in L_2 \Longrightarrow e_1(\cdot),e_2(\cdot) \in L_2$.

III. Solution of the Main Problem

In this section, it is proposed to provide a solution to the above problem by applying the positivity theorem [1],[2] after transforming the system [9] with the introduction of "multipliers" (as shown in Fig. 2). Recall that

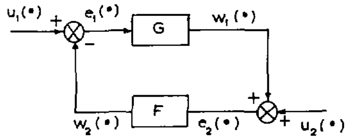

Fig. 1. The feedback system under consideration.

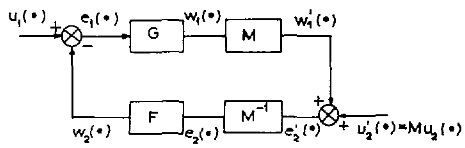

Fig. 2. System transformed with the introduction of multipliers.

the application of the theorem requires a factorization of the loop into a composition of two operators, one of which is strongly positive(e) with finite gain and the other is positive(e).

A Frequency-Domain Positivity Condition for Time-Varying Linear Operators

A criterion for the positivity(e) of linear time-varying operators $H \in \mathcal{O}_{IPA}$ will now be enunciated. This is a sufficient condition and requires the establishment of the nonnegativity of an even degree polynomial whose coefficients result from the minimization of certain combinations of $a_{Hi}(t)$ and $c_{Hi}(t)$ and their derivatives.

Lemma: Let $\{\alpha_{Hi}\}$, $i = 1,2,\cdots,n$ be an n-tuple of constants associated with an operator $H \in \mathcal{O}_{IPA}$ and defined by,

$$\alpha_{Hi} = \inf_{t \in R^+} \Big\{ c_{Hi}(t) a_{Hi}(t) + \sum_{j=1}^{n} \beta_{n+1-j,i-j} (c_{Hj}(t))^{(n+1+j-2i)} + \sum_{j<k=2}^{n} \beta_{k-j,i-j} [a_{Hj}(t) c_{Hk}(t) + c_{Hj}(t) a_{Hk}(t)]^{(k+j-2i)} \Big\} \tag{3.1}$$

where, $-a_{Hi}(t)$, $i = 1,2,\cdots,n$ are the elements of the nth row of $A_H(t)$, $c_{Hi}(t)$, $i = 1,2,\cdots,n$ are the elements of $c_H(t)$ and $\beta_{k,j}$ are the coefficients defined by,

$$\beta_{k,j} = \tfrac{1}{2}(-1)^{k+j}\left[\delta(j) + k\delta(j-1) + kh(j-2)\frac{(k-j-1)!}{(k-2j)!\,j!}\right] h(k-2j) \tag{3.2}$$

where

$$\delta(j) = \begin{cases} 1, & j = 0 \\ 0, & j \neq 0 \end{cases} \text{ and } h(j) = \begin{cases} 1, & j \geqslant 0 \\ 0, & j < 0 \end{cases}$$

(the superscripts within the brackets in (3.1) denote the order of the derivative with respect to time; terms with negative superscripts should be discarded).

Then H is positive(e) if 1)

$$d_H(t) \geqslant 0, \qquad \forall\, t \in R^+ \tag{3.3}$$

and 2) the frequency-domain inequality,

$$\sum_{i=1}^{n} \alpha_{Hi} \omega^{2(i-1)} \geqslant 0, \qquad \forall\, \omega \in R \tag{3.4}$$

are satisfied.

If, in addition to the above hypotheses, $d_H(t) \geqslant \epsilon > 0$ $\forall\, t \in R^+$, then H is strongly positive(e).

Proof:

Positivity(e) of H: Since $H \in \mathcal{O}_{IPA}$, $x_{Hi}(t) = 0$ at $t = 0$ $\forall\, i = 1,2,\cdots,n$ and $\lim_{t\to\infty} x_{Hi}(t) = 0$ $\forall\, i = 1,2,\cdots,n$.[1]

Now, since $H \in \mathcal{O}_{IPA} \Rightarrow H$ is a causal operator in L_2, it is sufficient to prove, in view of (2.6), that

$$\langle u_H(\cdot), y_H(\cdot)\rangle \geqslant 0, \qquad \forall\, u_H(\cdot) \in L_2. \tag{3.5}$$

Now, Left-hand side of (3.5)

$$= \langle u_H(\cdot), c_H'(\cdot)\, x_H(\cdot) + d_H(\cdot)\, u_H(\cdot)\rangle, \text{ from (2.3)}$$

$$= \langle u_H(\cdot), c_H'(\cdot)\, x_H(\cdot)\rangle + \langle u_H(\cdot), d_H(\cdot)\, u_H(\cdot)\rangle. \tag{3.6}$$

Second term on the right-hand side (RHS) of (3.6)

$$= \int_0^\infty d_H(t)\, u_H^2(t)\, dt \geqslant 0, \qquad \forall\, u_H(\cdot) \in L_2 \tag{3.7}$$

since $d_H(t) \geqslant 0$, $\forall\, t \in R^+$.

Further, since $H \in \mathcal{O}_{IPA}$, substituting for $u_H(\cdot)$ from (2.3), we have, first term on the RHS of (3.6)

$$= \langle b_H'(\cdot)\, \dot{x}_H(\cdot) - b_H'(\cdot) A_H(\cdot)\, x_H(\cdot), c_H'(\cdot)\, x_H(\cdot)\rangle$$

$$= \int_0^\infty \sum_{i=1}^{n} \dot{x}_{Hn}(t) c_{Hi}(t) x_{Hi}(t)\, dt + \int_0^\infty \sum_{i=1}^{n}\sum_{k=1}^{n} a_{Hi}(t) \cdot c_{Hk}(t) x_{Hi}(t)\, x_{Hk}(t)\, dt$$

(since $H \in \mathcal{O}_{IPA} \Rightarrow b_H'(\cdot) = [0\ 0\ \cdots\ 0\ 1]$).

$$= \sum_{i=1}^{n} \int_0^\infty c_{Hi}(t) x_{Hi}(t) \dot{x}_{Hn}(t)\, dt + \sum_{i=1}^{n} \int_0^\infty a_{Hi}(t) c_{Hi}(t) x_{Hi}^2(t)\, dt + \sum_{i<k=2}^{n} \int_0^\infty [a_{Hi}(t) c_{Hk}(t) + a_{Hk}(t) c_{Hi}(t)] x_{Hi}(t) x_{Hk}(t)\, dt. \tag{3.8}$$

We will next simplify the first and third terms on the RHS of (3.8) by repeated integration by parts. With this motive, let us now state a simple result.

Proposition: If $f(t)$ is a real-valued function differentiable k-times and if $y_i(t)$, $i = 1,2,\cdots$ are a family of time-functions satisfying 1) $\dot{y}_i(t) = y_{i+1}(t)$, $i = 1,2,\cdots$ and 2) $y_i(t_1) = y_i(t_2) = 0$, $\forall\, i = 1,2,\cdots$ and some $t_1, t_2 \in R^+$, then

$$\int_{t_1}^{t_2} f(t)\, y_i(t) y_{i+k}(t)\, dt = \int_{t_1}^{t_2} \sum_{j=0}^{k} \beta_{k,j} (f(t))^{(k-2j)} y_{i+j}^2(t)\, dt, \tag{3.9}$$

[1] This last assertion is a result of the following arguments. If $h_i(t,\tau)$ are the kernels relating the components of the state-vector to the input, i.e., $x_{Hi}(t) = \int_0^\infty h_i(t,\tau) u_H(\tau)\, d\tau$, $|x_{Hi}(t)| \leqslant \int_0^\infty |h_i(t,\tau)|\, |u_H(\tau)|\, d\tau \leqslant K \int_0^\infty e^{-a(t-\tau)} |u_H(\tau)|\, d\tau$ since $H \in \mathcal{O}_{IPA}$; noting that the last term is a convolution of two L_2-functions, a straightforward application of the Riemann–Lebesgue theorem gives the intended result.

where $\beta_{k,j}$ are the coefficients defined as in (3.2) (the superscript within the brackets in (3.9) denoting, as earlier, the order of the time-derivative).

The proof of this result is simply established by repeated integration by parts and is omitted as it is readily available in [8].

Now, simplifying the first and third terms on the RHS of (3.8) from an application of the above result[2], we have, RHS of (3.8)

$$
\begin{aligned}
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^\infty \beta_{n+1-j,i-j}(c_{Hj}(t))^{(n+1+j-2i)} x_{Hi}^2(t)\, dt \\
&\quad + \sum_{i=1}^{n} \int_0^\infty a_{Hi}(t) c_{Hi}(t) x_{Hi}^2(t)\, dt \\
&\quad + \sum_{i=1}^{n} \sum_{j<k=2}^{n} \int_0^\infty \beta_{k-j,i-j}[a_{Hj}(t)c_{Hk}(t) \\
&\quad + a_{Hk}(t)c_{Hj}(t)]^{(k+j-2i)} x_{Hi}^2(t)\, dt \\
&\geq \sum_{i=1}^{n} \int_0^\infty \alpha_{Hi} x_{Hi}^2(t)\, dt, \quad \alpha_{Hi} \text{ being defined as in (3.1)} \\
&= \frac{1}{2\pi} \sum_{i=1}^{n} \int_{-\infty}^{+\infty} \alpha_{Hi} |X_{Hi}(j\omega)|^2\, d\omega, \qquad (3.10)
\end{aligned}
$$

by an application of Parseval's theorem, where $X_{Hi}(j\omega)$ is the Fourier-transform of $x_{Hi}(t)$.

Now, since $x_{Hi}(t) = \dot{x}_{H(i-1)}(t)$, $\forall\ i = 1,2,\cdots,n$, $X_{Hi}(j\omega) = j\omega X_{H(i-1)}(j\omega)$ and by induction, $X_{Hi}(j\omega) = (j\omega)^{i-1} X_{Hi}(j\omega)$, $\forall\ i = 1,2,\cdots,n$.

Hence, on substitution, we have

$$
\text{RHS of (3.10)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_{H1}(j\omega)|^2 \left[\sum_{i=1}^{n} \alpha_{Hi} \omega^{2(i-1)} \right] d\omega \geq 0, \text{ from (3.4).} \qquad (3.11)
$$

Thus, combining (3.7) and (3.11), (3.5) results.

Strong positivity(e) of H: It is simple to observe that, H is strongly positive(e) if $(H - \epsilon E)$ is positive(e) for some $\epsilon > 0$, E being the identity operator in L_{2e}. Further, $H \in \mathcal{O}_{IPA} \Longrightarrow (H - \epsilon E) \in \mathcal{O}_{IPA}$ with the state-equation unchanged, but the output equation modified into, $\bar{y}_H(\cdot) = c_H'(\cdot) x_H(\cdot) + [d_H(\cdot) - \epsilon] u_H(\cdot)$. Hence, the positivity(e) of $(H - \epsilon E)$ may be established by working as before [repeating the steps from (3.5) to (3.11)] and noting that $d_H(t) \geq \epsilon$, $\forall\ t \in R^+$. Q.E.D.

Frequency-Domain Stability Criteria

The positivity condition obtained in the previous section makes the proof of the following stability criterion for linear systems a straightforward application of the positivity theorem.

Theorem 1: If there exists an operator $M \in \mathcal{O}_{IPA}$ such that M is invertible in L_2, $L = MG \in \mathcal{O}_{IPA}$ and the following conditions are satisfied:

1) $d_L(t) \geq \epsilon > 0$ and $d_M(t) \geq 0$, $\quad \forall\ t \in R^+$

2) $\sum_{i=1}^{n_L} \alpha_{Li} \omega^{2(i-1)} \geq 0$, $\quad \forall\ \omega \in R$

3) $\sum_{i=1}^{n_M} \alpha_{Mi} \omega^{2(i-1)} \geq 0$, $\quad \forall\ \omega \in R$,

where n_L and n_M are the dimensions, respectively, of L and M, $\{\alpha_{Li}\}$ and $\{\alpha_{Mi}\}$ are the constants associated, respectively, with L and M, being defined by (3.1).

Then the system described by (2.7) is L_2-stable for $F = E$.

A Few Remarks

Remark 1: In comparison with the existing results for the stability of similar systems, it may be observed that Theorem 1 permits the use of a very general class of time-varying multipliers, much like the results of [7]. However, it should be noted that, while [7] uses a minimal realization to give stability criteria in the time-domain which require the solution of certain matrix Riccati equations, the present results start with a canonical realization satisfying additional differentiability requirements, to yield simpler frequency-domain conditions. The noteworthy feature of the present results is the ease in checking the stability conditions, which has resulted from the imposition of certain additional constraints on the system.

Remark 2: A stability criterion for nonlinear systems, similar to the Theorem 1, may be formulated by using a time-varying multiplier M which is decomposible into $M = E + Z$ with $\|Z\| < 1$ (the norm used is the operator norm) and restricting the nonlinearity to be odd and monotonically nondecreasing. The statement of the result will be omitted, in view of the popularity (see Zames and Falb [2] or Willems and Brockett [10]) of such methods. It should, however, be mentioned that the conditions imposed by this criterion on the linear part of the system would be simpler and more explicit than the continuous-time version of the results of [10].

Remark 3: The difficult step in the application of the stability criteria of the present type lies in ensuring that the composite operator $L \in \mathcal{O}_{IPA}$ (in fact, this is the only difficult step; once such a representation is obtained, the determination of the constants $\{\alpha_{Li}\}$ and $\{\alpha_{Mi}\}$, although at first glance appears to be difficult, is quite simple irrespective of the dimensions of G and M). Even in this step, the difficult part is only to ensure a phase-variable realization of L, since with such a representation being ensured, the other requirements result in a simple manner from the imposition of suitable restrictions on the describing elements. This difficulty is greatly reduced by the availability of well-established algorithms [11],[12] which ensure for any uniformly controllable and differentiable realization of L, the existence of a unique nonsingular $n_L \times n_L$ matrix, which transforms the realization of L to the phase-variable form.

[2] Note that all the requirements of the proposition are fulfilled since $H \in \mathcal{O}_{IPA}$ implies $a_{Hi}(t)$ and $c_{Hi}(t)$ are differentiable n-times and $x_{Hi}(t) = 0$ at $t = 0$ and $t = \infty$. Further, $x_{Hi}(t)$ being the state-vector components of H, $\dot{x}_{Hi}(t) = x_{H(i+1)}(t)$, $\forall\ i = 1,2,\cdots,(n-1)$.

IV. Simplified Criteria Using First-Order Multipliers

The Case of the Time-varying Popov Multiplier

Although it is evident that the stability criteria given in the previous section, which permit the use of a very general class of time-varying operators in L_2 as multipliers, are more general than the existing results, the advantage of using a time-varying multiplier is yet to be decisively demonstrated. In this section, we propose to do this by using a simple first-order differential operator as the multiplier and derive a stability criterion which will be compared with the existing results that employ similar but time-invariant multipliers. As an application of the results derived, L_2-stability conditions for the nonlinear damped Mathieu equation are obtained in a subsequent section and compared with the results of the earlier investigators.

The multipliers that will be considered in this section are operators $M \in \mathcal{O}_E$ having a decomposition $M = Q + D$ where Q is a time-varying gain defined by $Qx(t) = q(t)\,x(t)$, $\forall\, x(\cdot) \in L_{2e}$, $q(t)$ being a nonnegative function on R^+ differentiable almost everywhere, and D, the differential operator in L_{2e}, $Dx(t) = \dot{x}(t)$, $\forall\, x(\cdot) \in L_{2e}$. Let us denote the class of such operators by $\mathfrak{M}_P$. It is simple to observe that $M \in \mathfrak{M}_P \Longrightarrow M^{-1}$ exists as an operator in L_{2e}. One particular point needs to be emphasized when the use of multipliers $M \in \mathfrak{M}_P$ is contemplated. Since $M = Q + D$ is not a bounded operator in L_2, the familiar introduction of M into the loop (see Fig. 2) will change the nature of the inputs to the system (note that $Mu_2 \in L_2$). This difficulty may however be overcome, following Zames [1], by restricting $u_2(\cdot)$ to be a fixed function $u_{2f}(\cdot)$ in L_2 such that $\dot{u}_{2f}(\cdot) \in L_2$.

Theorem 2: If $G \in \mathcal{O}_{IPA}$ and there exists an operator $M \in \mathfrak{M}_P$ satisfying the following conditions:

1) $MG: L_2 \rightarrow L_2$.
2) $d_G(t) \geqslant 0, \quad \forall\, t \in R^+$.
3) $q(t)\, d_G(t) + \frac{1}{2}\dot{d}_G(t) + c_{Gn}(t) \geqslant \epsilon > 0, \qquad \forall\, t \in R^+$.
4) $\sum_{i=1}^{n} \alpha_i \omega^{2(i-1)} \geqslant 0, \qquad \forall\, \omega \in R$,

where α_i are the constants defined by,

$$\alpha_i = \inf_{t \in R^+} \Big\{ [q(t)c_{Gi}(t) + \dot{c}_{Gi}(t) + c_{G(i-1)}(t) - c_{Gn}(t)a_{Gi}(t)]a_{Gi}(t) + \sum_{k=1}^{n} \beta_{n+1-k,i-k}[q(t)c_{Gk}(t) + \dot{c}_{Gk}(t) + c_{G(k-1)}(t) - c_{Gn}(t)a_{Gk}(t)]^{(n+1+k-2i)} + \sum_{j<k=2}^{n} \beta_{k-j,i-j}[a_{Gj}(t)(q(t)c_{Gk}(t) + \dot{c}_{Gk}(t) + c_{G(k-1)}(t) - c_{Gn}(t)a_{Gk}(t)) + a_{Gk}(t)(q(t)c_{Gj}(t) + \dot{c}_{Gj}(t) + c_{G(j-1)}(t) - c_{Gn}(t)a_{Gj}(t))]^{(k+j-2i)} \Big\}$$

$\beta_{k,j}$ being the coefficients defined as in (3.2), n being the dimension of G and $c_{G0}(t) \equiv 0$.

Then the system described by (2.7), with the additional assumption $u_2(\cdot) \equiv u_{2f}(\cdot) \ni \dot{u}_{2f}(\cdot) \in L_2$, is L_2-stable for all $F \in \mathfrak{F}_P$.

The proof of this theorem is omitted due to space requirements. The general pattern of development, however, follows very closely the proof of the lemma in Section II, in establishing the strong positivity(e) of MG. A complete proof may be found in [13].

Discussion of the Result

1) Although Theorem 2 contains a frequency-domain inequality similar to those appearing in Theorem 1, the conditions are explicitly on the elements describing G (note in comparison that Theorem 1 imposes conditions on the composite operator $L = MG$) and hence, are far simpler to check. The complexity of the expression for the coefficients $\{\alpha_i\}$ need not be a serious drawback, since the evaluation of these is very simple even for systems of large dimensions.

2) The advantage of using a time-varying multiplier is illustrated in the following example.

Example: Consider the system with the linear part G governed by the nonautonomous differential equation,

$$\ddot{y} + 2\dot{y} + \mathcal{P}(t)y = \psi(u)$$
$$\psi(u) = \ddot{u} + 2\dot{u} + (\mathcal{P}(t) - 1)u \qquad (4.1)$$

where $\mathcal{P}(t)$ is a gain, the bounds on whose rate of variation are to be determined for the stability of the feedback loop comprising of a nonlinear operator $F \in \mathfrak{F}_P$ in cascade with G.

Representing G in the phase-variable canonical form, we have the describing quadruple,[3]

$$A_G(t) = \begin{bmatrix} 0 & 1 \\ -\mathcal{P}(t) & -2 \end{bmatrix} \quad b_G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$c_G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } d_G(t) = 1.$$

Using Brockett's result [14], it may be observed that $G \in \mathcal{O}_{IPA}$ if $0 < \epsilon' < \mathcal{P}(t) \leqslant 11.5$.

For a more restricted form of the system (i.e., with $d_G(t) \equiv 0$ and no inputs into the feedback loop), Blodgett and Young [8] have proved absolute stability when the maximum variation of $\mathcal{P}(t)$ is restricted to $\dot{\mathcal{P}}(t) \leqslant 4\mathcal{P}(t)$.

An application of Theorem 2 results in the following conditions for L_2-stability:

$$1)\ 0 < \epsilon \leqslant q(t) \leqslant 2, \qquad \forall\, t \in R^+ \qquad (4.2)$$
$$2)\ \dot{\mathcal{P}}(t) \leqslant 2[q(t)\,\mathcal{P}(t) + \tfrac{1}{2}\ddot{q}(t) - \dot{q}(t)]. \qquad (4.3)$$

[3] The consideration of the particular form of $\psi(u)$ in (4.1) is merely to facilitate a simple phase-variable realization of G.

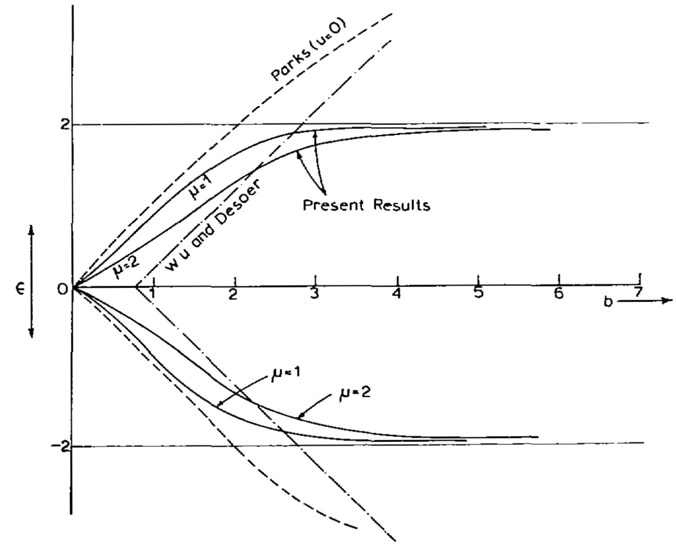

Fig. 3. Stability regions for the Mathieu equation from (4.9).

Now, choosing $q(t) = (2 + e^{-\lambda t} - e^{-\mu t})$, $\lambda, \mu \in R^+ \ni \lambda \geqslant \mu$, we get,

$$\dot{\mathcal{P}}(t) \leqslant [4\mathcal{P}(t)\{1 + \tfrac{1}{2}(e^{-\lambda t} - e^{-\mu t})\} + (\lambda^2 e^{-\lambda t} - \mu^2 e^{-\mu t}) + 2(\lambda e^{-\lambda t} - \mu e^{-\mu t})] \quad (4.4)$$

which is better than the constraint $\dot{\mathcal{P}}(t) \leqslant 4\mathcal{P}(t)$ since $\lambda \geqslant \mu$. An observation of the fact that a choice of $q(t) \equiv 2$ would yield the bound $\dot{\mathcal{P}}(t) \leqslant 4\mathcal{P}(t)$ and the improvement has resulted from the inclusion of the exponential functions, clearly reveals the advantage of using a time-varying element in the multiplier.

Periodic Multipliers for Systems Described by Mathieu Equation

As another application of Theorem 2, an L_2-stability criterion for the nonlinear damped Mathieu equation with a forcing function will be derived in this section. The stability regions on the parameter plane are compared with the available criteria and improvements are demonstrated through the use of periodic multipliers.

Consider the feedback system with $F \in \mathfrak{F}_P$ and the linear part governed by the Mathieu equation,

$$\ddot{y} + 2\dot{y} + \mathcal{P}(t)y = \psi(u); \quad \mathcal{P}(t) = (b + \epsilon \cos \mu t) \quad (4.5)$$

where $\psi(u)$ has the same form as in (4.1) and b, ϵ, and μ are finite constants with $b > 0$, $\mu > 0$. The problem of interest is to obtain the stability region in the b-ϵ parameter plane.

An application of Theorem 2 results in the following conditions for L_2-stability:

$$1)\ 0 < \epsilon' \leqslant q(t) \leqslant 2, \qquad \forall\, t \in R^+ \quad (4.6)$$

$$2)\ \Big\{q(t)(b + \epsilon \cos \mu t) + \tfrac{1}{2}\ddot{q}(t) - \tfrac{1}{2}\frac{d}{dt}(b + \epsilon \cos \mu t + 2q(t))\Big\} \geqslant 0, \qquad \forall\, t \in R^+. \quad (4.7)$$

An additional condition to be satisfied to ensure that $G \in \mathcal{O}_{IPA}$ is $0 < (b + \epsilon \cos \mu t) \leqslant 11.5$; this will be satisfied if b and ϵ are restricted by $0 < (b \pm \epsilon) \leqslant 11.5$. [Note however that this bound does not make use of the fact that the time varying coefficient is periodic with period π/μ; more relaxed bounds, for larger values of μ, may be obtained by using Willems' [15] result for periodic systems.]

In the following, two different choices of $q(t)$ are made and corresponding stability regions are obtained in order to demonstrate that different choices of $q(t)$ often lead to improved results.

Choice No. 1: Let

$$q(t) = q_1(t) = \left(1 - \frac{\epsilon}{2}\cos \mu t\right), \qquad -2 < \epsilon < +2 \quad (4.8)$$

so as to satisfy (4.6). Substituting this in (4.7) and simplifying, we have

$$b \geqslant \frac{|\epsilon|[2 + |\epsilon| + \mu^2/2]}{[2 + |\epsilon|]}, \qquad -2 < \epsilon < +2. \quad (4.9)$$

Choice No. 2: Let

$$q(t) = q_2(t) = \left(1 + \frac{\epsilon}{\mu}\sin \mu t\right), \qquad -\mu < \epsilon < +\mu. \quad (4.10)$$

Note that (4.6) is satisfied. Substituting in (4.7) and simplifying, we have

$$b \geqslant \frac{\epsilon^2}{2[\mu - |\epsilon|]}, \qquad -\mu < \epsilon < +\mu. \quad (4.11)$$

Comparison with the Existing Criteria: Figs. 3 and 4 show the stability regions in the $b - \epsilon$ plane given by (4.9) and (4.11), respectively, for different values of μ and those obtained by the criteria of Parks [16] and Wu and Desoer [17].

1) For small values of μ, (4.9) gives approximately $b \geqslant |\epsilon|$ and thus we get Parks' condition within the indicated range of ϵ. Even for $\mu = 1$, it may be seen from Fig. 3

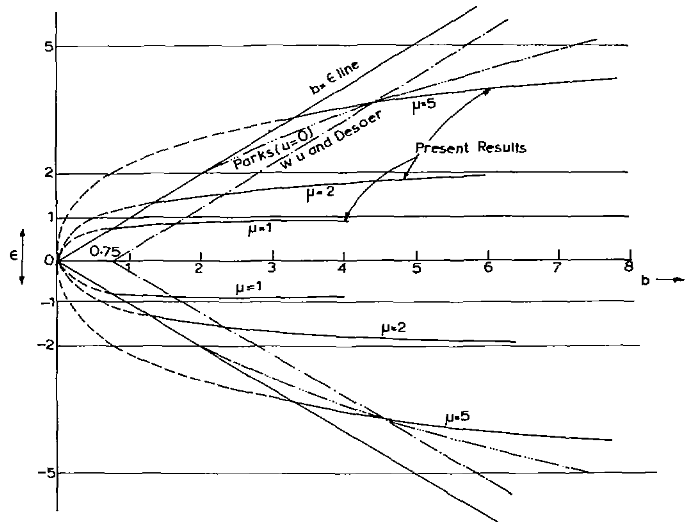

Fig. 4. Stability regions for the Mathieu equation from (4.11).

that the stability region obtained presently is close to Parks' region in the range $-2 < \epsilon < +2$. Further, it may be observed from Fig. 4 that the condition (4.11) gives the same regions as Parks[4] in the parameter ranges $b \leqslant 0.5$ for $\mu = 1$ and $b \leqslant 1.3$ for $\mu = 2$, whereas for $\mu = 5$, the region presently obtained is larger in the range $b \leqslant 4.5$.

2) In comparison with the L_p-stability criterion of Wu and Desoer [17] (for the problem presently considered, i.e., damping coefficient $= 2$, the condition from [17] is $b > |\epsilon| + 0.75$), it may be seen that (4.9) gives a larger region in the parameter ranges, $b \leqslant 2.75$ for $\mu = 1$ and $b \leqslant 2.25$ for $\mu = 2$, while (4.11) gives a larger region in the ranges, $b \leqslant 1.5$ when $\mu = 1$, $b \leqslant 2.25$ when $\mu = 2$ and $b \leqslant 4.75$ when $\mu = 5$. Further, it may be noted that in the latter case, the improvement in the stability region increases with increased μ. Thus one of the strong points of the present criteria is that information on the value of the parameter μ is exploited to obtain enlarged stability regions.

3) It should however be noted that the above comparison is not actually fair to the present criteria; for, the results of Parks [16] and Wu and Desoer [17] were originally derived for the Mathieu equation without any feedback nonlinearity. If the latter results are to remain valid in the nonlinear case, the nonlinearity should be confined to the sector $[0,\epsilon]$ whereas the present criteria permit it to be in the infinite sector. Thus in spite of considering a more general system, the present results give certain stability regions not contained in the earlier ones.

A General Remark: A comparison of the stability regions obtained in this section through the choice of two different periodic functions $q(t)$, shown in Figs. 3 and 4, reveals an interesting point. Note that for the particular case of $\mu = 1$, the choice of $q(t) = (1 + (\epsilon/\mu) \sin \mu t)$ gives a larger region in the range $b \leqslant 0.5$ than the choice of $q(t) = (1 - (\epsilon/2) \cos \mu t)$, while the reverse is true for $b > 0.5$. However, since the stability results obtained through the use of these periodic multipliers are only sufficient conditions, the bounds on the parameters of the Mathieu equation (4.5) which ensure stability of the feedback system may correspond to either of these two regions. In other words, a union of the two stability regions is itself a stability region for the system. Continuing on this theme, one may conclude that the present method of analysis involving the use of periodic multipliers, has opened a new avenue for obtaining enlarged stability regions through a judicious choice of different periodic functions $q(t)$ in the multiplier.

[4] Parks' result is derived by the use of the Circle criterion and hence does not give different conditions for different values of μ.

V. Conclusions

The problem of developing L_2-stability criteria for linear and nonlinear feedback systems containing a time-varying linear operator in L_2, has been treated. A frequency-domain positivity condition for time-varying linear operators is derived and is used to generate L_2-stability criteria, which permit the use of a very general class of multipliers with nonstationary kernels. For the case of feedback systems containing a Popov-type nonlinearity, more explicit conditions in the frequency-domain, involving a first-order time-varying differential multiplier are derived and are shown to improve upon the existing results. The use of periodic multipliers for the L_2-stability of nonlinear damped Mathieu equation with a forcing function, is suggested and is shown to result in stability regions not contained in those given by the existing criteria even for the case of the linear equation. The positivity criterion obtained in this paper, although motivated from a desire to use it in the stability analysis, is of independent interest and has applications in various other areas of Mathematical System Theory.

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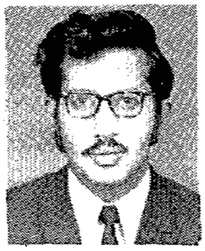

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