VIII. Conclusions

The stability of multiloop systems is investigated here in a fashion which directly incorporates the "structure" of the system in the analysis. This can be accomplished because system stability can often be interpreted in terms of margins within which subloops are stable. The method of analysis is straightforward, and a general analysis procedure is given.

The theory is applied to two specific multiloop systems. A system composed of two single-loop systems, each having a linear and a nonlinear element, in cascade with an outer feedback loop is examined. Conditions are derived which as the outer feedback loop gain approaches zero reduce to the familiar circle criterion for each of the single-loop systems. Also a system is analyzed which consists of an interconnection of three linear time invariant elements, a linear time varying element, a piecewise linear nonlinearity, and a hysterisis nonlinearity.

Appendix
Completion of the Proof of Theorem 1

In order to show that the hypotheses of Theorem 1 are sufficient to guarantee that the matrix $I - [|b_{ij}|g(H_j)]$ has an inverse whose elements are all nonnegative, the following preliminary results (see, e.g., [10] pp. 66 and 71 of vol. II) are employed.

Theorem A: A matrix A having all elements nonnegative has always a nonnegative eigenvalue r such that the moduli of all the eigenvalues of A do not exceed r. To this "maximal" eigenvalue r there corresponds an eigenvector y such that $y \geq 0$ and $y \neq 0$ (refer to Section II for notation). Moreover, the adjoint matrix $B(\lambda) = (\lambda I - A)^{-1}|\lambda I - A|$ has all elements nonnegative for $\lambda \geq r$.

Theorem B: If a matrix G has all off diagonal elements negative or zero and the successive principal minors are positive, then all principal minors are positive.

As in Theorem 1, assume that the successive principal minors of $I - [|b_{ij}|g(H_j)]$ are all positive. Since the last successive principal minor is the determinant of this matrix, the matrix is nonsingular and has an inverse. From Theorem A, it is clear that matrix $[|b_{ij}|g(H_j)]$ has a "maximal" eigenvalue r. Further, the matrix $(I - [|b_{ij}|g(H_j)])^{-1}|I - [|b_{ij}|g(H_j)]|$ has all elements nonnegative if $r \leq 1$. But $|I - [|b_{ij}|g(H_j)]|$ is the last successive principal minor of $I - [|b_{ij}|g(H_j)]$ and is positive. Hence, if $r \leq 1$ then $I - [|b_{ij}|g(H_j)]$ has an inverse with all nonnegative elements.

Now it only need be shown that $r \leq 1$. Since r is an eigenvalue, it is found that

$$0 = |rI - [|b_{ij}|g(H_j)]| = |I - [|b_{ij}|g(H_j)] + (r - 1)I|.$$

This means $(1 - r)$ is an eigenvalue of the matrix $I - [|b_{ij}|g(H_j)]$. Now the characteristic equation for an $n \times n$ matrix B can be written as

$$|B - \lambda I| = (-\lambda)^n + \sum_{k=1}^{n} S_k(-\lambda)^{n-k} = 0$$

where each S_k denotes the sum of all principal minors of order k of the matrix B. Letting $B = I - [|b_{ij}|g(H_j)]$ and $\lambda = 1 - r$ results in

$$(r - 1)^n + \sum_{k=1}^{n} S_k(r - 1)^{n-k} = 0$$

where each S_k represents the sum of all principal minors of order k of $I - [|b_{ij}|g(H_j)]$. But from Theorem B it is clear that all principal minors of $I - [|b_{ij}|g(H_j)]$ are positive. Hence, each $S_k > 0$. Thus, the above characteristic equation cannot be satisfied for $r > 1$. Hence $r, \leq 1$.

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Average Variation L_2-Stability Criteria for Time-Varying Feedback Systems—A Unified Approach

M. K. SUNDARESHAN and M. A. L. THATHACHAR

Abstract—The problem of developing L_2-stability criteria for feedback systems with a single time-varying gain, which impose average variation constraints on the gain, is treated. A unified approach is presented which facilitates the development of such average variation criteria for both linear and nonlinear systems. The stability criteria derived here are shown to be more general than the existing results.

I. Introduction

Following the work of Freedman and Zames [1] in developing L_2-stability criteria for linear systems containing a time-varying gain $k(t)$ in an otherwise time-invariant feedback loop, by imposing averaging constraints on the logarithmic variation of $k(t)$, there has been a considerable activity in obtaining similar results for linear and nonlinear systems. Major results in this direction are the absolute stability criteria of Venkatesh [2] for linear systems and of Narendra and Taylor [3] for nonlinear systems. Venkatesh's criterion imposes a constraint on $k(t)$ more general than [1] in considering only the positive lobes of $d \log k(t)/dt$ for averaging. However, both [2] and [3] make use of Corduneanu's extension of the Lyapunov method for the derivation of results and hence, do not facilitate a straightforward extension to the L_2-stability problem. In a recent paper, the authors [4] have presented L_2-stability criteria for linear systems which impose instantaneous bounds on $1/k(t)$ $dk(t)/dt$, by employing the theory of positivity of compositions of causal operators and time-varying gains. The main aim of the present paper is to emphasize that [4] contains the core of a procedure which in conjunction with a suitable factorization of $k(t)$ yields an extension to average variation stability criteria. This is demonstrated here by deriving within the framework developed in [4], stability criteria which are more general than [2] for linear systems and [3] for nonlinear systems.

II. Formulation of the Problem

A. Notations and Definitions

While the notations employed in [4] will be followed, a few of the important ones are briefly recapitulated. Let, R, R^+, and J^+ denote, respectively, the reals, the nonnegative reals, and the nonnegative integers. An operator H in L_2 (L_{2e}) is a single-valued mapping of L_2

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(L_{2e}) into itself. If H is an operator in L_{2e}, then H is said to be "positive(e)" [strongly positive(e)] if the inequality $\langle x_T(\cdot), (Hx(\cdot))_T \rangle \geqslant \delta \langle x_T(\cdot), x_T(\cdot) \rangle$ holds with $\delta = 0 [\delta > 0]$, $\forall\ x(\cdot) \in L_{2e}$ and $\forall\ T \in R^+$, $x_T(\cdot)$ being the truncation of $x(\cdot)$ defined by, $x_T(t) = x(t)\ \forall\ t \in [0,T]$ and zero otherwise.

Let $\mathcal{B}_c$ denote the Banach algebra with an identity E, of linear time-invariant causal convolution operators in L_2 that are defined by

$$Hx(t) = \sum_{i \in J^+} h_i x(t - \tau_i) + \int_0^\infty h(\tau) x(t - \tau)\, d\tau \quad \forall\ x(\cdot) \in L_2$$

where $\{\tau_i\}$, $i \in J^+$ is a sequence in R^+, $\{h_i\}$, $i \in J^+$ is an l_1-sequence in R and $h(\cdot) \in L_1$. $H(j\omega) = \sum_{i \in J^+} h_i \exp(-j\omega\tau_i) + \int_{-\infty}^{+\infty} h(t) \exp(-j\omega t)\, dt$. An operator $H \in \mathcal{B}_c$ is said to be "regular in $\mathcal{B}_c$" if $H^{-1} \in \mathcal{B}_c$.

$\mathcal{K}$ is the class of time-varying operators K in L_{2e} defined by $Kx(t) = k(t)x(t)\ \forall\ x(\cdot) \in L_{2e}$, $0 < \inf k(t) \leqslant k(t) \leqslant \sup k(t) < \infty\ \forall\ t \in R^+$. Let I denote the identity of $\mathcal{K} \cdot \mathcal{K}^\beta \subset \mathcal{K} \ni K \in \mathcal{K}^\beta \Longrightarrow dk(t)/dt \leqslant 2\beta k(t)$ for some $\beta \in R^+\ \forall\ t \in R^+$. $\mathcal{K}_\alpha \subset \mathcal{K} \ni K \in \mathcal{K}_\alpha \Longrightarrow dk(t)/dt \geqslant -2\alpha k(t)$ for some $\alpha \in R^+\ \forall\ t \in R^+$.

With this notation, it is simple to obtain the following result.

Proposition: If $K \in \mathcal{K}^\beta(\mathcal{K}_\beta)$ has a decomposition $K = \epsilon I + \hat{K}$, $0 < \epsilon < \inf k(t)$, then there exists a scalar $\beta' > \beta$ such that $\hat{K} \in \mathcal{K}^{\beta'}(\mathcal{K}_{\beta'})$ and the difference $(\beta' - \beta)$ can be made arbitrarily small.

B. *System Description*

The system (Fig. 1) is described by the input-output relations, $e_1(\cdot) = u_1(\cdot) - w_2(\cdot)$, $e_2(\cdot) = u_2(\cdot) + w_1(\cdot)$, $w_1(\cdot) = Ge_1(\cdot)$, and $w_2(\cdot) = Ke_2(\cdot)$ where $G \in \mathcal{B}_c$ and $K \in \mathcal{K}$.

C. *The Main Problem*

Given that $u_1(\cdot), u_2(\cdot) \in L_2$ and $e_1(\cdot), e_2(\cdot) \in L_{2e}$, find conditions on G and K which ensure that the system is L_2-stable, i.e., $e_1(\cdot), e_2(\cdot) \in L_2$.

III. Preliminary Results

The following lemmas give conditions for the positivity(e) of compositions of causal operators in L_2 (L_{2e}) and time-varying gains. The proofs of these may be found in [4].

Lemma 1: Let P be a causal operator in L_{2e} and Q be an operator in L_{2e} defined by $Qx(t) = q(t)x(t)\ \forall\ x(\cdot) \in L_{2e}$, $q(\cdot): R^+ \to R$ is absolutely continuous on R^+ and is bounded for all $t \in R^+$. Then QP is positive(e) if 1) P is positive(e) and 2) $q(\cdot)$ is nonnegative and monotone nonincreasing.

Lemma 2: Let P and Q be defined as in lemma 1. Then PQ is positive(e) if 1) P is positive(e) and 2) $q(\cdot)$ is nonnegative and monotone nondecreasing.

Lemma 3: Let $H \in \mathcal{B}_c$ satisfy Re $H(j\omega - \beta) \geqslant 0\ \forall\ \omega \in R$ and some $\beta \in R^+$. Then, KH is positive(e) for all $K \in \mathcal{K}^\beta$ and HK is positive for all $K \in \mathcal{K}_\beta$.

If H further satisfies Re $H(j\omega) \geqslant \epsilon > 0\ \forall\ \omega \in R$, then KH is strongly positive(e) for all $K \in \mathcal{K}^{\beta''}$ and HK is strongly positive(e) for all $K \in \mathcal{K}_{\beta''}$, where $\beta'' < \beta$, the difference $(\beta - \beta'')$ being arbitrarily small.

Note: The first part of Lemma 3, i.e., positivity(e) of KH and HK is proved in [4] (refer to the steps from (5.3) to (5.5) in [4]). The proof of the strong positivity(e) also follows similarly, after making use of the proposition in Section II-A.

IV. Main Results

The lemmas enunciated in the previous section will be crucially used in the derivation of the stability criteria for the system under consideration.

Stability Criteria for Linear Systems

Define the following subclasses of $\mathcal{K}$:

$\mathcal{A}_p\mathcal{K}_{\alpha\beta} \subset \mathcal{K} \ni K \in \mathcal{A}_p\mathcal{K}_{\alpha\beta} \Longrightarrow$

$$\frac{1}{T}\int_t^{t+T} \left[\frac{d \log k(\tau)}{d\tau} - 2(\alpha - \beta)\right]^+ d\tau \leqslant 2\beta \quad (4.1)$$

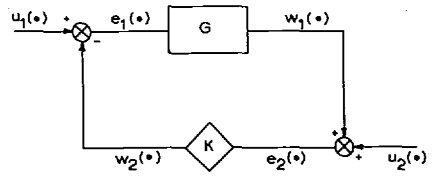

Fig. 1. The linear feedback system under consideration.

for some $T > 0$ and some scalars $\alpha,\beta \in R^+$, $\forall\ t \in R^+$, with $[\cdot]^+$ denoting only the positive lobes of the argument.

$\mathcal{A}_n\mathcal{K}_{\alpha\beta} \subset \mathcal{K} \ni K \in \mathcal{A}_n\mathcal{K}_{\alpha\beta} \Longrightarrow$

$$\frac{1}{T}\int_t^{t+T} \left| \left[\frac{d \log k(\tau)}{d\tau} - 2(\alpha - \beta)\right]^- \right| d\tau \leqslant 2\beta \quad (4.2)$$

for some $T > 0$ and some scalars $\alpha,\beta \in R^+$, $\forall\ t \in R^+$ with $[\cdot]^-$ denoting only the negative lobes of the argument.

The following lemmas depict certain factorization properties of the operators belonging to these classes.

Lemma 4: If $K \in \mathcal{A}_p\mathcal{K}_{\alpha\beta}$, then K admits a factorization $K = K_l K_u$ such that $K_l \in \mathcal{K}_\beta$ and $K_u \in \mathcal{K}^\alpha$.

Proof: The proof follows along the same lines as the proof of a similar factorization lemma in Freedman and Zames [1], by defining $l(t) = \log k(t)$ and obtaining an additive decomposition of $l(t)$. This, however, is contained in Ramarajan and Thathachar.

Lemma 5: If $K \in \mathcal{A}_n\mathcal{K}_{\alpha\beta}$, then K admits a factorization $K = K_l K_u$ such that $K_l \in \mathcal{K}_\alpha$ and $K_u \in \mathcal{K}^\beta$.

Proof: The proof can be obtained from a slight variation of that of Lemma 4, by defining $l(t) = -\log k(t)$.

The following theorem gives a criterion for the L_2-stability of the system under consideration.

Theorem 1: If there exists an operator $M \in \mathcal{B}_c$ and constants $\alpha,\beta^1 \in R^+$ such that,

1) M is regular in β_c (4.3)

2) Re $M(j\omega)\, G(j\omega) \geqslant \delta > 0\ \forall\ \omega \in R$ (4.4)

3) Re $M(j\omega - \beta')\, G(j\omega - \beta') \geqslant 0\ \forall\ \omega \in R$ (4.5)

and

4) Re $M(j\omega - \alpha) \geqslant 0\ \forall\ \omega \in R$, (4.6)

then the system (Fig. 1) is L_2-stable for all $K \in \mathcal{A}_p\mathcal{K}_{\alpha\beta} \cup \mathcal{A}_n\mathcal{K}_{\alpha\beta}$, where $\beta < \beta'$, the difference $(\beta' - \beta)$ being arbitrarily small.

Proof: The proof will be outlined for the case $K \in \mathcal{A}_p\mathcal{K}_{\alpha\beta}$ only, since the other follows from similar arguments. Since $K \in \mathcal{A}_p\mathcal{K}_{\alpha\beta} \Longrightarrow K = K_l K_u$, $K_l \in \mathcal{K}_\beta$ and $K_u \in \mathcal{K}^\alpha$, make the system transformations as shown in Fig. 2. An application of lemma 3 ensures the strong positivity(e) of MGK_l from (4.4) and (4.5) and the positivity(e) of $K_u M^{-1}$ from (4.6). The finiteness of gain of MGK_l (for the definition of the gain of an operator, see [4]) is also ensured since $M,G \in \mathcal{B}_c$ and $K_l \in \mathcal{K}$. Hence, invoking a basic result due to Zames [6], L_2-stability of the system ensues.

Stability Criteria for Nonlinear Systems

In this section, we will consider a system having a different configuration (Fig. 3) from the one described earlier in Section II B, in having a nonlinear operator F in the feedback path. Let $\mathfrak{F}_M$ denote the class of nonlinear operators F in L_{2e} defined by, $Fx(\cdot) = f(x(\cdot))\ \forall\ x(\cdot) \in L_{2e}$, $f(0) = 0$, $xf(x) \geqslant 0\ \forall\ x \in R$ and $[x_1 - x_2][f(x_1) - f(x_2)] \geqslant 0\ \forall\ x_1, x_2 \in R$. Let $\mathfrak{F}_{0M} \subset \mathfrak{F}_M \ni F \in \mathfrak{F}_{0M} \Longrightarrow f(\cdot)$ is odd.

Let us introduce the following subclass of $\mathcal{K}$.

$$\mathcal{A}\mathcal{K}_\beta \subset \mathcal{K} \ni K \in \mathcal{A}\mathcal{K}_\beta \Longrightarrow \left| \frac{1}{t}\int_0^t \left| \frac{d \log k(\tau)}{d\tau} \pm 2\beta \right| d\tau - 2\beta \right| \leqslant \frac{\mu}{t} \quad (4.7)$$

for some scalars $\beta,\mu \in R^+\ \forall\ t \in R^+$.

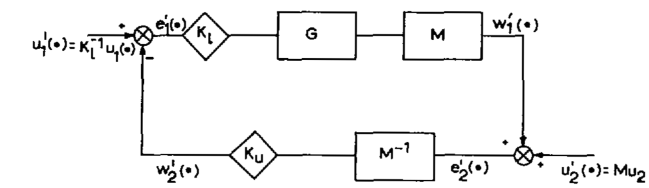

Fig. 2. Transformations of the system for Theorem 1.

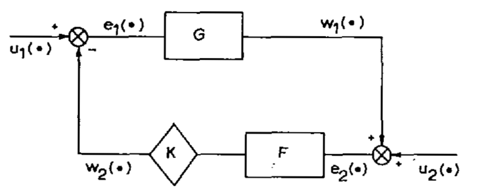

Fig. 3. The nonlinear feedback system under consideration.

It may be shown, following Freedman [7], that this averaging condition is sufficient to ensure a useful factorization of K.

Lemma 6: If $K \in \mathcal{AK}_\beta$, then K admits a factorization $K = K_l K_u$ such that $K_l \in \mathcal{K}_\beta$ and $K_u \in \mathcal{K}^0$, $\mathcal{K}^0$ being the class $[\mathcal{K}^\beta]_{\beta=0}$.

This lemma[1] lends itself useful in proving the following stability criterion.

Theorem 2: If there exists an operator $M \in \mathcal{B}_c$ and a constant $\beta' \in R^+$ such that,

1) M is regular in $\mathcal{B}_c$, (4.8)

2) $$M = E + Z, Z \in \mathcal{B}_c \ni \|Z\|' = \sum_{i \in J^+} |z_i| + \int_0^\infty |z(t)|\, dt < 1 \quad (4.9)$$

3) $\operatorname{Re} M(j\omega)G(j\omega) \geqslant \delta > 0, \quad \forall\, \omega \in R,$ (4.10)

and

4) $\operatorname{Re} M(j\omega - \beta')G(j\omega - \beta') \geqslant 0, \quad \forall\, \omega \in R,$ (4.11)

then the system (Fig. 3) is L_2-stable for all $F \in \mathfrak{F}_{0M}$ and all $K \in \mathcal{AK}_\beta$, where $\beta < \beta'$, the difference $(\beta' - \beta)$ being arbitrarily small.

Corollary: If in addition to the hypotheses of the theorem, Z satisfies $z_i \leqslant 0\ \forall\ i \in J^+$ and $z(t) \leqslant 0\ \forall\ t \in R^+$, then the system is L_2-stable for all $F \in \mathfrak{F}_M$ and $K \in \mathcal{AK}_\beta$.

Proof: Since $K \in \mathcal{AK}_\beta \Longrightarrow K = K_l K_u$, $K_l \in \mathcal{K}_\beta$ and $K_u \in \mathcal{K}^0$ make the system transformations as shown in Fig. 4. It is now sufficient to prove that $K_u F M^{-1}$ is positive(e) and MGK_l is strongly positive(e) with finite gain. The former follows from (4.9) and Lemma 2, while the latter follows from (4.10), (4.11), and Lemma 3. The proof of the corollary may also be obtained from similar arguments.

A Few Remarks

A) In comparison with the criteria of the earlier investigators, it is easy to realize that Theorem 1 is more general than the results of Freedman and Zames [1] and Venkatesh [2], while Theorem 2 is more general than Narendra and Taylor [3]. It should especially be noted that the technique employed in [2] and [3] has been to use Corduneanu's extension of the Lyapunov method, which requires the multiplier $M(s)$ to be a rational function of "s" and the system to be described by a differential equation. Hence the present results are improvements upon [2] and [3] in the following aspects: 1) more general multipliers employed, 2) a broader class of systems considered, 3) less restrictive averaging conditions on the gain, 4) a stronger stability concept employed, and 5) elegance of the method of derivation.

B) An additional novel feature of Theorem 2 is the use of O'Shea type multipliers with integral constraints on the kernel, for time-varying systems with monotone nonlinearities. It is of interest to observe in this context that, in spite of the ample attention this problem has received ever since the appearance in the literature of O'Shea's [9] results for time-invariant systems, it remained unsolved even for the simpler and less general case of the imposition of instantaneous bounds on the logarithmic variation of the gain.

C) Recently, Ramarajan and Thathachar [5] have also obtained results similar to Theorem 1 by following the approach of Freedman and Zames [1]. However, the present method of derivation is different and has followed from an extension of the techniques developed in [4] for the instantaneous bounds case, thus providing a *unified approach for the derivation of average variation criteria with various types of averaging constraints.* In this context, it is interesting to note that the earlier investigators had to resort to entirely different methods for the derivation of stability criteria with bounds of either type on the gain (for example compare the results of Narendra and Taylor in [3] and [8]; while the criteria in [8] which impose instantaneous bounds were derived using the Popov approach, generalisation of these results with averaging bounds in [3] could be obtained only by employing Corduneanu's method).

D) It may be noted that the hypotheses of Theorem 1 differ from Freedman and Zames [1] only in the constraints on $k(t)$ (which are more general). Hence, it is possible to state this result in purely geometrical terms, involving certain "shifted" Nyquist diagrams and dispensing with the explicit use of multipliers, following Freedman [7].

[1] It should be noted that Freedman [7] has obtained a slightly different factorization, $K = K_l K_u \ni K_l \in \mathcal{K}_0$ and $K_u \in \mathcal{K}^\beta$ (where $\mathcal{K}_0 = [\mathcal{K}\beta]_{\beta=0}$). However, the factorization given in lemma 6 also follows from (4.7), the proof being similar to that given in [7], after defining $l(t) = -\log k(t)$.

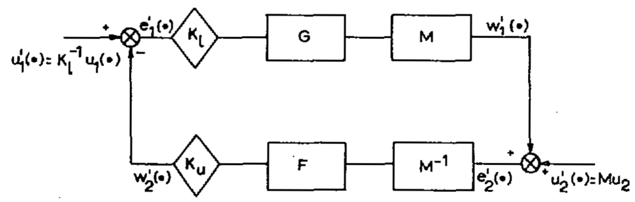

Fig. 4. Transformations of the system for Theorem 2.

V. Conclusions

New criteria for the L_2-stability of systems containing a single time-varying gain $k(t)$ in an otherwise time-invariant negative feedback loop, are developed by imposing averaging constraints on the gain. A unified approach for the derivation of such average variation criteria for both linear and nonlinear systems is delineated. The stability criteria presented here permit a very general class of causal operators in L_2 to be used as multipliers and are more general than the existing results. However, the results of the present findings can be improved by further investigation in developing similar average variation criteria which allow the employment of noncausal multipliers, analogous to the results of [4] for the instantaneous bounds case.

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