Images in linearly conducting dielectrics

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Abstract: Methods of images for electrostatic fields and steady conduction fields are fairly well known. Whenever applicable, they have made the solution of field problems much simpler. However, when dealing with lossy dielectrics both permittivity and nonzero conductivity are to be considered. **A** generalised method of images is developed which can deal with such linearly lossy (conducting) dielectrics. For linearly conducting dielectrics, a point charge is equivalent to a point current source and vice versa. At $t = 0+$, only the dielectric image will be seen. Subsequently, because of the finite nonzero conductivities of the associated media, a current will flow and surface charges will accumulate at the interface due to the mismatch in the material properties. The equation governing this surface-charge accumulatioin is derived. Linearity of the media permits fields in either medium to be obtained by superposing the fields due to the dielectric images and that due to the interfacial charges. The field can be obtained in either medium by replacing these surface charges by an equivalent point charge kept at the appropriate image point. This equivalent point charge satisfies a similar differential equation in time to that of the surface charge. The cases of time-varying chargeicurrent sources and the source-current requirement for keeping any point very close to the source at a specified potential are also discussed.

1 introduction

The method of images for ideal dielectrics and conductors has long been well known [1, 2]. This principle permits simplified analyses and solution of many field problems. However, when an electric field in a linearly conducting dielectric is to be analysed using this method, its permittivity and nonzero conductivity cannot be considered simultaneously. In a large nurnber of cases, these problems need to be studied over time intervals for which neither the ideal dielectric behaviour nor the ideal conductor approximation is strictly valid. For example, in analyses of fields in semiconductors and soil, both permittivity and conductivity need consideration. Thus, a general method of images which permitted consideration of both the nonzero conductivity and the permittivity would be useful. For the steady-state sinusoidal fields, such an image formulation has been realised by simply expressing material properties in the complex material notations. For example, in the calculation of fields around polluted insulators at power frequencies using the charge simulation method (CSM), image charges with complex notation have been used successfully [3, 4]. Similarly, in Sommmerfeld-half-space problems, more general complex images are used [5]. To our knowledge, for the general time-domain case, a method of images is not available in the literature. This work presents such a formulation of a method of images in linearly conducting dielectric media for a point charge and a point current source. Note that the present work is applicable only to static and quasistatic cases wherein the field variations are slow enough to neglect wave-propagation and eddy-current effects.

2 Theory

2.1 Preliminaries

In this work, only images in two semi-infinite media separated by a Cartesian plane are considered. As in the conventional theory of images, only linear homogeneous isotropic materials are considered. To apply this theory, the time variation of charges and currents should be slow enough to permit use of Poisson's equation as the governing equation. This is assured if the electric field produced by the current flow is predominant compared with the electric field produced by the changing magnetic field (produced by the same current). If ω is the dominant angular frequency of the source (charge/current) time variation, the above condition can be written as $1/\sigma >> \omega\mu$, where μ is the magnetic permeability of the media and σ is its electrical conductivity. When this condition is satisfied, only the point form of Faraday's law of Maxwell's equations is relevant, i.e. $\nabla \times \vec{E} = 0$. This is readily satisfied by the electrostatic potential which is the basic field variable of Poisson's equation.

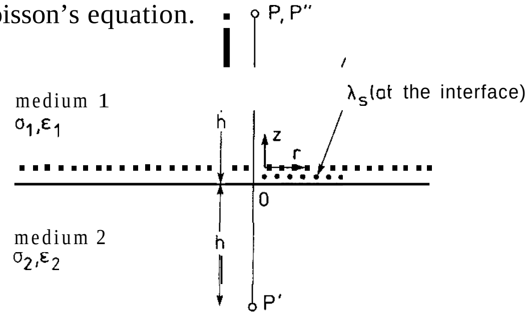

Fig.1
P: source

surface charge distribution q_{s1}, both kept at the image point P'. Similarly, for calculations in mediuni 2, the relevant set of charges are the dielectric image charge q'' and the point charge equivalent of the surface charge distribution q_{s2}, both at P. The whole space is assumed to be of the respective medium. Note that these sets of charges are equivalent to a set of point current sources, as discussed above. Consider a special case of medium 1 with zero conductivity. Then it follows from the above analysis that, initially, only a dielectric image will appear. As the time progresses, due to the current flow, interfacial charge will build up which is governed by

$$\lambda_s = -\left(\frac{q_0}{2\pi}\right)\left\{\frac{h}{(h^2+r^2)^{\frac{3}{2}}}\right\}\left[1-\exp\left\{-\frac{\sigma_2}{(\epsilon_1+\epsilon_2)}t\right\}\right]$$

From eqn. 7 with a value of $\sigma_1 = 0$, the equivalent point charges of the interfacial charge distribution can be obtained. Note that these equivalent point charges are to be kept at the respective image points. As $t \rightarrow \infty$, this equivalent point charge for medium 1 builds up to $-(q + q')$ and that for medium 2 builds up to $-q''$. Therefore, as the time progresses the total image, which is the sum of the dielectric image and the equivalent point charge of the interfacial charges, builds up to $-q$ for the calculation in medium 1 and 0 for that in medium 2. This condition corresponds to medium 2 behaving like a perfect conductor.

2.3 Image of a point current source

Note that the concept of a point current source is rather an unusual concept but has been used as a tool even by Maxwell in his classic work 'A treatise on electricity and magnetism' [2]. Subsequently, Sunde [7] has also utilised it successfully. Such point sources serve as effective basic building blocks to construct all other types of source. Now consider a point current source i_0 kept at P. The actual current source seen by the media is given by eqn. 4. Also, note that these point current sources are equivalent to point charges. Initially, as discussed above, only the permittivities will determine the potential distribution and hence only images of the charges in dielectrics will be seen. Owing to the mismatch of current density at the interface, there will be interfacial charge accumulation λ_s. An expression for λ_s is derived in the Appendix (Section 6.1). For the present situation, $q(t)$ appearing in eqn. 11 is given by

$$q(t) = \left(\frac{\epsilon_1}{\sigma_1}\right) i_0 \left\{1-\exp\left(-\frac{\sigma_1}{\epsilon_1}t\right)\right\}$$

As shown in the Appendix (Section 6.2), this interfacial charge can be replaced by a set of equivalent point charges $q_{s_{1,2}}$ kept at the respective image points. Since any point charge is equivalent to a point current source, point current sources $i_{s_{1,2}}$ can be used instead of equivalent point charges $q_{s_{1,2}}$. These equivalent point current sources follow

$$\frac{\partial i_{s_{1,2}}}{\partial t} + \left(\frac{\sigma_1+\sigma_2}{\epsilon_1+\epsilon_2}\right) i_{s_{1,2}} = \left(\frac{\sigma_{1,2}}{\epsilon_{1,2}}\right)\left\{\frac{\sigma_1\epsilon_2-\sigma_2\epsilon_1}{\epsilon_1(\epsilon_1+\epsilon_2)}\right\} \times \left(\frac{2\epsilon_{1,2}}{\epsilon_1+\epsilon_2}\right)\left(\frac{\epsilon_1}{\sigma_1}\right) i(t) \quad (8)$$

For the present case, these equivalent current sources are given by

$$i_{s_{1,2}} = \left(\frac{\sigma_{1,2}}{\epsilon_{1,2}}\right)\left(\frac{2\epsilon_{1,2}}{\epsilon_1+\epsilon_2} i_0\right)\left(\frac{\epsilon_1}{\sigma_1}\right) \times \left[\left\{\frac{\sigma_1\epsilon_2-\sigma_2\epsilon_1}{\epsilon_1(\sigma_1+\sigma_2)}\right\} + \exp\left(-\frac{\sigma_1}{\epsilon_1}t\right) - \frac{\sigma_1}{\epsilon_1}\left(\frac{\epsilon_1+\epsilon_2}{\sigma_1+\sigma_2}\right)\exp\left\{-\frac{(\sigma_1+\sigma_2)t}{(\epsilon_1+\epsilon_2)}\right\}\right] \quad (9)$$

Therefore, the set of image current sources for the present case is as follows: For medium 1, point current source $i(t)$ at P and the dielectric image source i', the equivalent point current source i_{s_1} for the interfacial charges, both kept at the image point P', are relevant. Here,

$$i(t) = i_0 \left\{1-\exp\left(-\frac{\sigma_1}{\epsilon_1}t\right)\right\}$$

and

$$i' = \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right) i(t)$$

For the medium 2, the dielectric image source

$$i'' = \left(\frac{\sigma_2\epsilon_1}{\sigma_1\epsilon_2}\right)\left(\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\right) i(t)$$

and the equivalent point current source i_{s_2} for the interfacial charges, both kept at P, are relevant. As above, during calculations, the whole space is assumed to be of the respective medium. Further, as time progresses, these set of sources will converge into a set of images in conductors. For the above two cases, i.e. a point charge and a constant point current source, the condition on conductivity and magnetic permeability (stated in the beginning of Section 2.1) can be further quantified as follows: $1/\sigma >> \sqrt{(\mu_0\mu_r/\varepsilon_0\varepsilon_r)}$ or $1/\sigma >> 376.63/\sqrt{\varepsilon_r}$, a condition which is readily satisfied by all practical insulating materials, many types of soils and semiconductors.

3 Generalisation

If the point charges or current sources placed at P are time varying, then a generalisation of the above analysis is possible under the condition that the time variation of the associated charges and currents is slow enough to still have Poisson's equation as the governing equation.

In the following, during the calculations using images, the whole of the space is assumed to be of the respective medium, as the conventional image theory. Consider first the time-varying charge. If at P the charge is $q_0(t)$, then, for the calculations in medium 1 the relevant set of charges is $q(t)$ at P, its dielectric image charge

$$q'(t) = \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right) q(t)$$

and the point-charge equivalent of the interfacial charge q_{s_1} both located at image point P'. Similarly, for medium 2, the dielectric image charge

$$q''(t) = \left(\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\right) q(t)$$

and q_{s_2} the point-charge equivalent of interfacial charge distribution are relevant. $q_{s_{1,2}}$ can be obtained by solving eqn. 13. As mentioned above, for the steady-state sinusoidal fields, complex images have been used in the literature [3, 4]. If the point charge at P is varying sinusoidally in time, it will set up a sinusoidally varying field everywhere. Computing the equivalent point charge of interfacial charge distribution and taking only the steady-state part, it can be verified that, for the medium 1,

$$q_s = q_0 \frac{2(\sigma_1\epsilon_2 - \sigma_2\epsilon_1)}{(\epsilon_1+\epsilon_2)^2} \frac{\exp(j\omega t)}{\frac{\sigma_1+\sigma_2}{\epsilon_1+\epsilon_2} + j\omega}$$

Adding this to the dielectric image for medium 1, and rearranging the terms, the total image seen is

$$\left\{ \frac{(\epsilon_1 - \frac{j\sigma_1}{\omega}) - (\epsilon_2 - \frac{j\sigma_2}{\omega})}{(\epsilon_1 - \frac{j\sigma_1}{\omega}) + (\epsilon_2 - \frac{j\sigma_2}{\omega})} \right\} q_0$$

which is exactly the complex image formulation employed for the steady-state sinusoidal case [3, 4]. This would suggest an alternative method for the deduction of the generalised images through the frequency-domain analysis. The Fourier transform of the source (charge or current) can be convoluted with the complex images for obtaining the time-domain solution. However, such an approach does not portray the true physical picture of the phenomena. Also, in many practical applications of the method of images, as in CSM, spatial and temporal distributions of the source are not known *a priori*. This complicates the problem further and can make it very difficult to solve. Assuming that the source characteristics are fixed *a priori*, the frequency-domain technique is an indirect method and cannot be simpler than the direct time-domain method, except probably in specific cases.

The time-varying point-current source is more general than the time-varying point-charge case and is now considered. If a point current source $i_0(t)$ is kept at P, then the actual current source $i(t)$ seen by the medium can be obtained by solving eqn. 3. The relevant current sources for the calculation in medium 1 are $i(t)$ at P, the dielectric image source $i'(t)$ and equivalent surface charge source i_{s1}, both located at the image point P'. For that in medium 2, the relevant set of sources is the dielectric-image source $i''(t)$ and equivalent surface-charge source i_{s2}, both located at P, where

$$i'(t) = \left(\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\right) i(t)$$

$$i''(t) = \left(\frac{\sigma_2\epsilon_1}{\sigma_1\epsilon_2}\right)\left(\frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\right) i(t)$$

The equivalent surface-charge sources are obtained by solving eqn. 8.

In many practical situations, the voltages are fixed rather than the charges or currents. It will thus be relevant to study the source-current requirement for such cases. In the following, the source current $i_0(t)$ required for keeping the potential of the spherical surface of radius a constructed around P is considered. The radius a is chosen to be very small compared with the the smallest dimension in the problem geometry. Using eqn. 12 of the Appendix (Section 6.2) and eqn. 3, the expression for the source current is

$$i_0(t) = \left(\frac{\sigma_1}{\epsilon_1} + \frac{\partial}{\partial t}\right)\left(\frac{4\pi\epsilon_1 a}{k_1} v(t) - \frac{k_2}{k_1^2} 4\pi\epsilon_1 a \right.$$
$$\left. \times \int_0^t \left[v(t-\tau) \exp\left\{-\tau\left(\frac{\sigma_1+\sigma_2}{\epsilon_1+\epsilon_2} + \frac{k_2}{k_1}\right)\right\}\right] d\tau \right) \quad (10)$$

where

$$k_1 = \left\{1 + \frac{a}{2h}\left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\right)\right\}$$

$$k_2 = \left\{\frac{a}{2h}\frac{\sigma_1\epsilon_2 - \sigma_2\epsilon_1}{(\epsilon_1+\epsilon_2)^2}\right\}$$

Note that if $v(t = 0^+)$ is not zero, then there will be a current surge at $t = 0^+$ to build the capacitive-voltage distribution.

The important feature here is that the field problems involve permittivities and conductivities participating simultaneously in determining the equivalent charges and/or current sources, and hence the potential distribution. The governing equation still remains the same: Poisson's equation. Thus there is no direct dependency on time and these time-independent equations are linked by a time-dependent material boundary condition yielding a time-dependent field distribution. Such field problems are termed 'capacitive-resistive' fields and are discussed in [8] for the Laplacian fields.

The generalised method of images developed above has been used successfully for modelling of the stepped leader of a lightning stroke. For modelling the charge in the leader sheath, cylindrical and spherical charge distributions have been used. The magnitude of the charge has been determined by field conditions and varies dynamically. The full modelling details which include transient-field calculation, conduction through a nonlinear leader and, importantly, the air-breakdown process will form the subject of a future paper. Also, this example portrays the simplicity of the direct time-domain technique for the generalised images over the frequency-domain approach which seems to be impractical for this problem.

4 Conclusions

Problems involving images in linearly conducting dielectrics require simultaneous consideration of permittivities and conductivities. Here, owing to the mismatch (at the interface) of the material properties, interfacial charges build up with time. The general equations governing the build up for a time-varying point charge and a point current source have been derived. It is also shown that, for the field calculation in either medium, this interfacial charge can be replaced by point sources kept at the respective image points. Expressions for the magnitude of the images of a point charge and a point current source (kept in a system of two isotropic linear semi-infinite media) have also been derived. It is shown that the general images reduce to the complex images employed for the steady-state sinusoidal excitation.

The source-current requirement for satisfying the specified potential in the vicinity of the source has also been deduced.

In all the analyses, only the geometric properties are used and hence general images for other geometries can be deduced on similar lines.

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6 Appendixes

6.1 Derivation of equation for charge build-up

In this Appendix, the differential equation governing the accumulation of the interfacial charges is derived. Let medium 1 with conductivity σ_1 and permittivity ε_1 and medium 2 with conductivity σ_2 and permittivity ε_2 meet in a Cartesian plane. Let P be a point in medium 1 and P' be its image point in the conventional sense (see Fig. 1). The origin is the intersection of the line joining PP' and the interface. If a point charge $q(t)$ is present at P, then it will set up a field. This field will cause a current flow in both the media. Therefore, dissipation of these charges gives rise to point-current sources i_q at P and i'_q at P' for calculations in medium 1 and i''_q at P for calculations in medium 2. The subscript q is used to distinguish these sources from the set of image sources in conductors. If $\sigma_1/\varepsilon_1 \neq \sigma_2/\varepsilon_2$, then there will be a mismatch of the current densities at the interface resulting in accumulation of an interfacial charge λ_s. This interfacial charge density λ_s is seen as a sheet of charge of density $2\{\varepsilon_1/(\varepsilon_1 + \varepsilon_2)\}\lambda_s$ from medium 1 and density $2\{\varepsilon_2/(\varepsilon_1 + \varepsilon_2)\}\lambda_s$ from medium 2. Equivalent current sheets follow similarly. If, at any instant., $q(t)$ is the charge seen at P and if λ_s is the interfacial charge density then the field at any point can be calculated by considering them independently. First, considering the point charge alone, dielectric images can be used continuously. The field due to the interfacial charge distribution can then be added to obtain the total field. At the interface, therefore, the current densities are

$$J_{1n} = \sigma_1 E_{1n}$$

$$= -\{q(t) - q'(t)\}\frac{\sigma_1 h}{4\pi\epsilon_1(r^2+h^2)^{\frac{3}{2}}} + \frac{\sigma_1\lambda_s}{\epsilon_1+\epsilon_2}$$

$$J_{2n} = \sigma_2 E_{2n}$$

$$= -q''(t)\frac{\sigma_2 h}{4\pi\epsilon_2(r^2+h^2)^{\frac{3}{2}}} + \frac{\sigma_2\lambda_s}{\epsilon_1+\epsilon_2}$$

Substituting these values of current densities in eqn. 5 and then replacing the dielectric-image charges in terms of $q(t)$, the governing equation for the interfacial charge can be obtained as

$$\frac{\partial\lambda_s}{\partial t} + \left(\frac{\sigma_1+\sigma_2}{\epsilon_1+\epsilon_2}\right)\lambda_s = \frac{2h}{4\pi(r^2+h^2)^{\frac{3}{2}}}\left\{\frac{\sigma_1\epsilon_2 - \sigma_2\epsilon_1}{\epsilon_1(\epsilon_1+\epsilon_2)}\right\}q(t) \tag{11}$$

with $\lambda_s(t = 0^+) = 0$. Note that the charge $q(t)$ at P may be due to either a point charge or a point current source.

6.2 Equivalent point charge

The basic advantage of the method of images is simplification of the analysis and computation of fields. With the charge build-up at the interface, it may be thought that the original simplicity will be lost. It is now shown that the interfacial charge distribution can be replaced by a simple equivalent charge kept at the respective image point.

The equation for the charge build-up has been derived in Section 6.1. Applying Laplace transformation to eqn. 11, and then solving for λ_s, the following expression results:

$$\lambda_s[s] = \frac{2h}{4\pi(r^2+h^2)^{\frac{3}{2}}}AF[s]q[s] \tag{12}$$

where

$$A = \frac{\sigma_1\epsilon_2 - \sigma_2\epsilon_1}{\epsilon_1(\epsilon_1+\epsilon_2)}$$

$$F[s] = \frac{1}{s + \frac{\sigma_1+\sigma_2}{\epsilon_1+\epsilon_2}}$$

and $q[s]$ is the Laplace transform of $q(t)$, the charge seen at P. From eqn. 12 it is clear that the spatial pattern of the interfacial charge distribution will depend only on the geometrical parameters and time dependency will only affect its amplitude. In the following, a sample calculation is shown for medium 1.

The normal field at the interface due to λ_s is

$$E_{1n} = \frac{1}{2\epsilon_1}\frac{2\epsilon_1}{\epsilon_1+\epsilon_2}\lambda_s$$

Substituting for λ_s,

$$E_{1n} = \frac{q[s]}{4\pi\epsilon_1}\frac{h}{(r^2+h^2)^{\frac{3}{2}}}\frac{2\epsilon_1}{\epsilon_1+\epsilon_2}AF[s]$$

The potential along the axis of symmetry can be computed from the expression for potential of a ring charge along its axis:

$$v(z) = \int_0^\infty \frac{\lambda_s\frac{2\epsilon_1}{\epsilon_1+\epsilon_2}}{4\pi\epsilon_1(r^2+h^2)^{\frac{3}{2}}}2\pi r dr = \frac{q[s]\frac{2\epsilon_1}{\epsilon_1+\epsilon_2}}{4\pi\epsilon_1(z+h)}AF[s]$$

At the boundary at infinity, it can seen that the potential due to λ_s vanishes. Note that the normal field at the interface and the potential along the axis both satisfy the same differential equation in time as that governing the interfacial charge distribution. From the expression for the field at the interface (boundary), the potential along the axis and the vanishing potential at the far boundary, it can be identified that all these are as same as that produced by a point charge of magnitude:

$$q_{s_1}[s] = q[s]\frac{2\epsilon_1}{\epsilon_1+\epsilon_2}AF[s]$$

located at the image point P'. As both the field due to the interfacial charge distribution and this equivalent charge satisfy Laplace's equation in medium 1 and both of them satisfy the same boundary conditions, they produce the same field distribution in medium 1. Therefore, one can be replaced by the other for all field calculations in medium 1. **A** similar analysis can be carried out for medium 2.

Thus, the field of the interfacial surface-charge distribution in media 1 and 2 can be obtained from these equivalent point charges $q_{s1,2}$ kept at the respective image points. The equation governing these charges is then

$$\frac{\partial q_{s_{1,2}}}{\partial t} + \left(\frac{\sigma_1+\sigma_2}{\epsilon_1+\epsilon_2}\right)q_{s_{1,2}} = \frac{2\epsilon_1}{(\epsilon_1+\epsilon_2)}\left\{\frac{\sigma_1\epsilon_2-\sigma_2\epsilon_1}{\epsilon_1(\epsilon_1+\epsilon_2)}\right\}q(t) \tag{13}$$