

Extended Kac–Akhiezer formulae and the Fredholm determinant of finite section Hilbert–Schmidt kernels

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Abstract. This paper deals with some results (known as Kac–Akhiezer formulae) on generalized Fredholm determinants for Hilbert–Schmidt operators on L_2 -spaces, available in the literature for convolution kernels on intervals. The Kac–Akhiezer formulae have been obtained for kernels which are not necessarily of convolution nature and for domains in \mathbb{R}^n .

Keywords. Hilbert–Schmidt integral operator; Fredholm determinant; Kac–Akhiezer formula.

1. Introduction

The classical Fredholm determinant [2] of a given symmetric Hilbert–Schmidt integral (briefly H-S) operator T is an analytic function $D_T(\lambda)$ with the property that $\lambda (\neq 0)$ is a zero of $D_T(\lambda)$ if and only if $1/\lambda$ is an eigenvalue of T . We shall call any analytic function $f_T(\lambda)$, as a Fredholm determinant of T if its zeros are precisely reciprocals of the non-zero eigenvalues of T .

For convolution kernels such a Fredholm determinant was obtained by Kac [3], [4] and Akhiezer [1]. Their result is briefly summarized below:

Let $k(x)$ satisfy the following conditions:

1. $k(x)$ is a bounded, continuous, real valued function on $(-\infty, \infty)$,
 2. $k(x) = k(-x)$,
 3. $\hat{k} \in L_1(\mathbb{R}^1)$, where \hat{k} is the Fourier transform of k ,
 4. $\int_{-\infty}^{\infty} |xk(x)| dx < \infty$,
- and
5. $\int_{-\infty}^{\infty} |xk(x)| dx < 1/2$.

Then the Fredholm determinant $D_T(\lambda)$ of the symmetric H-S operator

$$T_\tau f = \int_0^\tau k(x-y)f(y)dy$$

on $L_2[0, \tau]$ is given by

$$D_\tau(\lambda) = \prod_{j=1}^{\infty} [1 - \lambda \lambda_j(\tau)] = \exp \left[- \int_0^\tau \alpha(0, s, \lambda) ds \right], \quad (1.1)$$

where $\alpha(x, \tau, \lambda)$ is the solution of the integral equation:

$$\alpha(x, \tau, \lambda) = \lambda k(x) + \lambda \int_0^\tau k(x-y)\alpha(y, \tau, \lambda)dy.$$

We shall refer to (1.1) as the Kac–Akhiezer formula.

In the above result we observe that the main assumptions on the kernel are:

- (i). $k(x)$ is a continuous function on $(-\infty, \infty)$;
- (ii). The domain of the integral is an interval in \mathbb{R}^1 ;
- (iii). The kernel is of convolution type.

This result has been extended for the case when $k(x)$ is only continuous in a neighbourhood of the origin. Further, when $k(x)$ is not continuous near the origin, a meromorphic function whose poles and zeros correspond to the eigenvalues has been obtained in [5], [7]. Subsequently Rao and Sukavanam [9] have extended these results for normal integral operators. But all these results are with the assumptions (ii) and (iii).

The aim of this paper is to obtain these results by replacing (ii) and (iii) by (ii)' and (iii)' below, where (ii)' is essentially replacing the interval by a suitable smooth domain in \mathbb{R}^n and (iii)' is essentially replacing the convolution nature of the kernel by an assumption that the map $x \mapsto k(x, \cdot)$ is continuous.

- (ii)': \mathcal{C}_1 : For every $\tau \geq 0$, Ω_τ is a bounded subset of \mathbb{R}^n .
- \mathcal{C}_2 : $\{\mathbf{0}\} = \Omega_0 \subseteq \Omega_{\tau_1} \subseteq \Omega_{\tau_2}$ if $0 \leq \tau_1 \leq \tau_2$, where $\mathbf{0}$ is the zero vector in \mathbb{R}^n .
- \mathcal{C}_3 : The map $\tau \mapsto \mu(\bar{\Omega}_\tau)$ from the interval $[0, \infty)$ to \mathbb{R}^1 is absolutely continuous in every bounded interval in $[0, \infty)$, where μ is the Lebesgue measure on \mathbb{R}^n and $\bar{\Omega}_\tau$ is the closure of Ω_τ with respect to the standard topology of \mathbb{R}^n .
- \mathcal{C}_4 : For every $\tau \geq 0$ there exists a surface measure σ_τ on $\partial\bar{\Omega}_\tau$, such that

- (i). $g(t) \stackrel{\text{def}}{=} \int_{\partial\bar{\Omega}_\tau} f(x) d\sigma_\tau(x)$ is an integrable function on $[0, \tau]$ for $f \in C^0(\bar{\Omega}_\tau)$, where $C^0(\bar{\Omega}_\tau)$ is the space of all continuous functions on $\bar{\Omega}_\tau$, and
- (ii). $\int_{\bar{\Omega}_\tau} f(x) dx = \int_{t=0}^\tau \left[\int_{\partial\bar{\Omega}_t} f(x) d\sigma_t(x) \right] dt, \quad \forall f \in C^0(\bar{\Omega}_\tau).$

(A simple motivation for such a family $\{\Omega_\tau\}_{\tau \geq 0}$ in \mathbb{R}^n is the ball in \mathbb{R}^n of radius τ and centre at $\mathbf{0}$.)

(iii)': The kernel is such that

- \mathcal{C}_5 : $k(x, y)$ is defined in $\mathbb{R}^n \times \mathbb{R}^n$, and is symmetric.
- \mathcal{C}_6 : k is locally square integrable.
- \mathcal{C}_7 : For every $x \in \mathbb{R}^n$, $k(x, y)$ is well defined and is in $L_2(\mathbb{R}^n)$, and for any two compact subsets E, F of \mathbb{R}^n , the mapping $x \mapsto k_x$ from E into $L_2(F)$ is continuous; where $k_x(\cdot) = k(x, \cdot)$.

By \mathcal{C}_5 and \mathcal{C}_6 it follows that for every $\tau > 0$.

$$(K_\tau f)x = \int_{\bar{\Omega}_\tau} k(x, y)f(y)dy \quad (f \in L_2(\bar{\Omega}_\tau))$$

defines a compact self adjoint operator on $L_2(\bar{\Omega}_\tau)$. Further we assume

\mathcal{C}_8 : For all $\tau > 0, 1$ is not in the spectrum of K_τ .

Then $(I - K_\tau)^{-1}$ exists for every $\tau > 0$ where I is the identity operator on $L_2(\bar{\Omega}_\tau)$.

Let $\alpha_1(x, y, \tau)$ denote the unique solution of the equation

$$\alpha(x, y, \tau) = k(y, x) + \int_{\bar{\Omega}_\tau} k(y, z)\alpha(x, z, \tau)dz \tag{1.2}$$

and $\alpha_2(x, y, \tau)$ be that of

$$\alpha(x, y, \tau) = k(y, -x) + \int_{\bar{\Omega}_\tau} k(y, z)\alpha(x, z, \tau)dz. \tag{1.3}$$

Since K_τ is a compact self adjoint operator on $L_2(\bar{\Omega}_\tau)$ it follows from the general theory of such operators that K_τ has a discrete set of real eigenvalues $\{\lambda_i(\tau)\}_{i=1}^\infty$ with the origin as the only possible limit point.

The main results we prove are Theorems 1.1 and 1.2 below:

Theorem 1.1. *Suppose the domain satisfies conditions $\mathcal{C}_1 - \mathcal{C}_4$ and the kernel, conditions $\mathcal{C}_5 - \mathcal{C}_7$, and suppose that \mathcal{C}_8 holds. Then there exists a function $u(x, y, \tau)$ such that for each fixed $x \in \bar{\Omega}_\tau, u$ is continuous in the variable y on $\bar{\Omega}_\tau$ and*

$$u(x, y, \tau) = \alpha_1(x, y, \tau) - k(y, x) \tag{1.4}$$

in the L_2 sense as a function in the variable y on $\bar{\Omega}_\tau$ and the corresponding Kac–Akhiezer formula is given by

$$\exp\left\{-\int_{t=0}^\tau dt \int_{\partial\bar{\Omega}_\tau} u(x, x, t)d\sigma_t(x)\right\} = \prod_{i=1}^\infty [\exp(\lambda_i(\tau))][1 - \lambda_i(\tau)]. \tag{1.5}$$

If k satisfies the conditions $\mathcal{C}_1 - \mathcal{C}_7$, we can apply the Theorem 1.1 to the kernel

$$g_\lambda(x, y) \stackrel{\text{def}}{=} \lambda k(x, y) \quad (\lambda \in \mathbb{R}), \tag{1.6}$$

if $1/\lambda$ is not in the spectrum of K_τ . For the kernel g_λ the eigenvalues will be $\{\lambda\lambda_n(\tau)\}_{n=1}^\infty$ and the corresponding Kac–Akhiezer formula is given by

$$\begin{aligned} \delta_1(\lambda, \tau) &\stackrel{\text{def}}{=} \exp\left\{-\int_{t=0}^\tau dt \int_{\partial\bar{\Omega}_\tau} u(x, x, t; \lambda)d\sigma_t(x)\right\} \\ &= \prod_{i=1}^\infty [\exp(\lambda\lambda_i(\tau))][1 - \lambda\lambda_i(\tau)], \end{aligned} \tag{1.7}$$

where $u(\cdot, \cdot, t; \lambda)$ is as in (1.4) with k replaced by g_λ and α_1 is the solution of (1.2) with k replaced by g_λ .

It is to be noted that $\delta_1(\lambda, \tau)$ has an analytic extension in the complex λ -plane. It follows from (1.7) that $\lambda_0 \in \mathbb{R}^1 \setminus \{0\}$ is an eigenvalue of K_τ if and only if $1/\lambda_0$ is a zero of this extended function.

Apart from the conditions $\mathcal{C}_1 - \mathcal{C}_8$ if the Ω_τ 's are symmetric for $\tau > 0$ and the kernel k is real valued. we prove

Theorem 1.2. Suppose as in Theorem 1.1, conditions $\mathcal{C}_1 - \mathcal{C}_8$ hold and in addition the domains Ω_τ (for $\tau > 0$) are symmetric and k is real valued. Then there exists a function $v(x, y, \tau)$ such that for each fixed $x \in \bar{\Omega}_\tau$, v is continuous in the variable y on $\bar{\Omega}_\tau$ and

$$v(x, y, \tau) = \alpha_2(x, y, \tau) - k(y, -x) \quad (1.8)$$

in the L_2 sense as a function in the variable y on $\bar{\Omega}_\tau$ and the corresponding Kac–Akhiezer formula is given by

$$\exp \left\{ - \int_{t=0}^{\tau} dt \int_{\partial \bar{\Omega}_t} v(x, x, t) d\sigma_t(x) \right\} = \frac{\Pi_i^+ [\exp(\lambda_i(\tau))] [1 - \lambda_i(\tau)]}{\Pi_j^- [\exp(\lambda_j(\tau))] [1 - \lambda_j(\tau)]}, \quad (1.9)$$

where Π_i^+ and Π_j^- are products over all eigenvalues that have symmetric and anti-symmetric eigenfunctions respectively.

Consider the kernel k and the family $\{\bar{\Omega}_\tau\}_{\tau \geq 0}$ as in Theorem 1.2. We can apply Theorem 1.2 to $g_\lambda(x, y)$. For this kernel the eigenvalues will be $\{\lambda_n(\tau)\}_{n=1}^\infty$ and the corresponding Kac–Akhiezer formula is given by

$$\begin{aligned} \delta_2(\lambda, \tau) &\stackrel{\text{def}}{=} \exp \left\{ - \int_{t=0}^{\tau} dt \int_{\partial \bar{\Omega}_t} v(x, x, t; \lambda) d\sigma_t(x) \right\} \\ &= \frac{\Pi_i^+ [\exp(\lambda \lambda_i(\tau))] [1 - \lambda \lambda_i(\tau)]}{\Pi_j^- [\exp(\lambda \lambda_j(\tau))] [1 - \lambda \lambda_j(\tau)]}, \end{aligned} \quad (1.10)$$

where $v(\cdot, \cdot; t; \lambda)$ is as in (1.8) with k replaced by g_λ and α_2 is the solution of (1.3) with k replaced by g_λ .

It is to be noted that $\delta_2(\lambda, \tau)$ has a meromorphic extension in the complex λ -plane. It follows from (1.10) that $\lambda_0(\neq 0) \in \mathbb{R}^1$ is an eigenvalue of K_τ if and only if $1/\lambda_0$ is a zero or a pole of this function.

We observe that Theorem 1.1 is the generalization of the classical Kac–Akhiezer formula (1.1) as extended by Vittal Rao [8] for kernels continuous near the origin and Theorem 1.2 is the generalization of the meromorphic function extension obtained by Vittal Rao [8].

2. Some properties of the eigenvalues

We now obtain some properties of the eigenvalues and eigenfunctions similar to those obtained by Vittal Rao [7]. The ideas behind the proofs are similar to those in [7].

Let $\lambda_i^+(\tau)$ and $\lambda_i^-(\tau)$ be respectively the i th positive and negative eigenvalues of K_τ . By the minimax characterization [6], [10] of the eigenvalues we have

Lemma 2.1. For each $i \in \mathbb{N}$, $\lambda_i^+(\tau)$ and $\lambda_i^-(\tau)$ are respectively nondecreasing and non-increasing functions for $\tau \geq 0$ if the Ω_τ 's satisfy the conditions $\mathcal{C}_1 - \mathcal{C}_2$ and the kernel k satisfies the conditions $\mathcal{C}_5 - \mathcal{C}_6$.

Lemma 2.2. $\lambda_i(\tau)$ is an absolutely continuous function in the variable τ in every bounded

interval of $[0, \infty)$ for each $i \in \mathbb{N}$, if k is a locally integrable symmetric kernel on $\mathbb{R}^n \times \mathbb{R}^n$ and the Ω_τ 's satisfy the conditions $\mathcal{C}_1 - \mathcal{C}_3$.

Proof. For $\tau \geq 0$ we define

$$F(\tau) = \int_{\bar{\Omega}_\tau} \int_{\bar{\Omega}_\tau} |k(x, y)|^2 dx dy. \tag{2.1}$$

For a $\tau_0 \in (0, \infty)$ and a finite sequence of disjoint open intervals $\{(\tau_j, \eta_j)\}_{j=1}^m$ in $[0, \tau_0]$, we have

$$\begin{aligned} & \sum_{j=1}^m F(\eta_j) - F(\tau_j) \\ &= \sum_{j=1}^m \left\{ \int_{\bar{\Omega}_{\eta_j}} dx \int_{\bar{\Omega}_{\eta_j} \setminus \bar{\Omega}_{\tau_j}} |k(x, y)|^2 dy + \int_{\bar{\Omega}_{\eta_j} \setminus \bar{\Omega}_{\tau_j}} dx \int_{\bar{\Omega}_{\tau_j}} |k(x, y)|^2 dy \right\} \\ &\leq \sum_{j=1}^m \left\{ \int_{\bar{\Omega}_{\tau_0}} dx \int_{\bar{\Omega}_{\eta_j} \setminus \bar{\Omega}_{\tau_j}} |k(x, y)|^2 dy + \int_{\bar{\Omega}_{\eta_j} \setminus \bar{\Omega}_{\tau_j}} dx \int_{\bar{\Omega}_{\tau_0}} |k(x, y)|^2 dy \right\} \\ &= 2 \int_{\bar{\Omega}_{\tau_0}} \int_{\cup_{j=1}^m (\bar{\Omega}_{\eta_j} \setminus \bar{\Omega}_{\tau_j})} |k(x, y)|^2 dx dy \end{aligned} \tag{2.2}$$

and

$$(\mu \times \mu)[\bar{\Omega}_{\tau_0} \times \cup_{j=1}^m (\bar{\Omega}_{\eta_j} \setminus \bar{\Omega}_{\tau_j})] = \mu(\bar{\Omega}_{\tau_0}) \sum_{j=1}^m [\mu(\bar{\Omega}_{\eta_j}) - \mu(\bar{\Omega}_{\tau_j})]. \tag{2.3}$$

Let ν be the measure defined on the elementary sets as

$$\nu(E \times F) = \iint_{E \times F} |k(x, y)|^2 dx dy.$$

Since the map $\tau \mapsto \mu(\bar{\Omega}_{\tau_0})\mu(\bar{\Omega}_\tau)$ is absolutely continuous in $[0, \tau_0]$ and $\nu \ll \mu \times \mu$, using (2.2) and (2.3) one has $F(\tau)$ is absolutely continuous in the interval $[0, \tau_0]$. From (2.1) and the classical theory of H-S operators [6], we have

$$F(\tau) = \sum_{i=1}^{\infty} \lambda_i^2(\tau);$$

hence by Lemma 2.1, we have for $i \in \mathbb{N}$, and $j = 1, 2, 3, \dots, m$,

$$F(\eta_j) - F(\tau_j) \geq \lambda_i^2(\eta_j) - \lambda_i^2(\tau_j) \geq 0,$$

from which it follows that $\lambda_i^2(\tau)$ is an absolutely continuous function in the variable τ in the interval $[0, \tau_0]$. Hence by Lemma 2.1 we have $\lambda_i^+(\tau)$ is absolutely continuous in the interval $[0, \tau_0]$. By considering $-K_\tau$ instead of K_τ one can conclude the same for the negative eigenvalues. This completes the proof of the lemma. ■

Lemma 2.1 implies that for each $i \in \mathbb{N}$, $\lambda_i(\tau)$ is differentiable almost everywhere. Let us denote an eigenfunction corresponding to an eigenvalue $\lambda_i(\tau)$ of K_τ by $\phi_i(\cdot, \tau)$.

Lemma 2.3 Let $i \in \mathbb{N}$ be fixed and $\lambda_i(\tau)$ be of multiplicity $m_i(\tau)$. Then there exist $m_i(\tau)$ orthonormal eigenfunctions $\{\phi_{ij}(\cdot, \tau)\}_{j=1}^{m_i(\tau)}$ such that

$$\frac{d\lambda_i(\tau)}{d\tau} = \lambda_i(\tau) \int_{\partial\bar{\Omega}_\tau} |\phi_{ij}(x, \tau)|^2 d\sigma_\tau(x); \quad j = 1, 2, \dots, m_i(\tau), \quad (2.4)$$

for almost every τ , if the Ω_τ and k satisfy the conditions $\mathcal{C}_1 - \mathcal{C}_7$.

Proof. It is enough if we prove the result for the positive eigenvalues of K_τ . Let $\tilde{\tau} \geq 0$ be such that $\lambda_i^+(\tau) = 0$ for all $\tau \leq \tilde{\tau}$ and $\lambda_i^+(\tau) > 0$ for $\tau > \tilde{\tau}$. Hence (2.4) is true for almost every $\tau \leq \tilde{\tau}$ and for any choice of eigenfunctions. Now $\lambda_i^+(\tau)$ and $\mu(\bar{\Omega}_\tau)$ are differentiable almost everywhere by Lemma 2.1 and assumption \mathcal{C}_3 respectively. Let $\tau_0 > \tilde{\tau}$ be such that both $d\lambda_i^+(\tau)/d\tau$ and $d\mu(\bar{\Omega}_\tau)/d\tau$ exist at $\tau = \tau_0$. Let $\{\tau_p\}_{p=1}^\infty$ be an increasing sequence in the interval $(\tilde{\tau}, \tau_0)$ such that

$$\tau_p \rightarrow \tau_0 \text{ as } p \rightarrow \infty.$$

First let us consider the case when the multiplicity of $\lambda_i^+(\tau_0)$ namely $m_i(\tau_0)$ is one. Consider the sequence of eigenvalues $\{\lambda_i(\tau_p)\}_{p=1}^\infty$ and a corresponding sequence $\{\phi_i(\cdot, \tau_p)\}_{p=1}^\infty$ of normalized eigenfunctions of K_{τ_p} .

We have

$$\lambda_i^+(\tau_p)\phi_i(x, \tau_p) = \int_{\bar{\Omega}_{\tau_p}} k(x, y)\phi_i(y, \tau_p)dy \quad (2.5)$$

and

$$\int_{\bar{\Omega}_{\tau_p}} |\phi_i(x, \tau_p)|^2 dx = 1. \quad (2.6)$$

Though initially the eigenfunctions $\phi_i(x, \tau_p)$ are defined only in $\bar{\Omega}_{\tau_p}$, the right hand side of (2.5) allows us to extend $\phi_i(x, \tau_p)$ for all $x \in \mathbb{R}^n$. Hence without loss of generality we can take each one of the $\phi_i(\cdot, \tau_p)$ to be defined on $\bar{\Omega}_{\tau_0}$.

For each $x \in \bar{\Omega}_\tau$, we define a linear functional $F_{x,\tau}$ on $L_2(\bar{\Omega}_\tau)$ by

$$F_{x,\tau}(f) = \int_{\bar{\Omega}_\tau} k(x, y)f(y)dy.$$

Condition \mathcal{C}_7 implies that the map $x \mapsto k_x$ from $\bar{\Omega}_\tau$ to $L_2(\bar{\Omega}_\tau)$ is weakly continuous; and hence $\{k_x\}_{x \in \bar{\Omega}_\tau}$ is a weakly compact subset of $L_2(\bar{\Omega}_\tau)$. Hence for $f \in L_2(\bar{\Omega}_\tau)$, there exists a constant $c_{f,\tau} \in (0, \infty)$ such that

$$\sup_{x \in \bar{\Omega}_\tau} \left| \int_{\bar{\Omega}_\tau} k_x(y)f(y)dy \right| \leq c_{f,\tau},$$

which implies that

$$\sup_{x \in \bar{\Omega}} |F_{x,\tau}(f)| \leq c_{f,\tau}.$$

Hence by uniform boundedness principle one can choose a constant $M_\tau \in (0, \infty)$ such that

$$\|F_{x,\tau}\| \leq M_\tau, \quad \forall x \in \bar{\Omega}_\tau. \quad (2.7)$$

Since

$$\|F_{x,\tau}\|^2 = \int_{\bar{\Omega}_\tau} |k(x,y)|^2 dy,$$

it now follows from (2.7) that,

$\forall \tau > 0$, there exists a constant $M_\tau \in (0, \infty)$ such that

$$\int_{\bar{\Omega}_\tau} |k(x,y)|^2 dy \leq M_\tau^2, \quad \forall x \in \bar{\Omega}_\tau. \tag{2.8}$$

We now prove the following properties of the eigenvalue sequence $\{\phi_i(\cdot, \tau_p)\}_{p=1}^\infty$:

(a) $\phi_i(\cdot, \tau_p) \in C^0(\bar{\Omega}_{\tau_0}), \forall p. \tag{2.9}$

(b) There exists a $J_i \in [0, \infty)$ such that $|\phi_i(x, \tau_p)| \leq J_i, \forall x \in \bar{\Omega}_{\tau_0}$ and $\forall p. \tag{2.10}$

(c) There exists a subsequence $\{\phi_i(\cdot, \tau_q)\}_{q=1}^\infty$ which converges uniformly on $\bar{\Omega}_{\tau_0}. \tag{2.11}$

Proof of (a): For each p , by \mathcal{C}_7 , the map

$$x \mapsto \int_{\bar{\Omega}_{\tau_p}} k(x,y)\phi_i(y, \tau_p) dy,$$

from the compact set $\bar{\Omega}_{\tau_0}$ into \mathbb{C} is continuous; hence by (2.5) the map

$$x \mapsto \lambda_i^+(\tau_p)\phi_i(x, \tau_p)$$

from $\bar{\Omega}_{\tau_0}$ into \mathbb{C} is continuous. Now (a) follows as $\lambda_i^+(\tau_p) > 0$.

Proof of (b): Applying Schwartz inequality to the right hand side of (2.5) and using (2.6) we get

$$|\lambda_i^+(\tau_p)\phi_i(x, \tau_p)|^2 \leq \int_{\bar{\Omega}_{\tau_p}} |k(x,y)|^2 dy \quad \forall x \in \bar{\Omega}_{\tau_0} \text{ and } \forall p. \tag{2.12}$$

Now using \mathcal{C}_2 and (2.12) we get

$$|\lambda_i^+(\tau_p)\phi_i(x, \tau_p)| \leq M_{\tau_0}, \quad \forall x \in \bar{\Omega}_{\tau_0} \text{ and } \forall p$$

where M_{τ_0} is as given in (2.8). Since $\lambda_i^+(\tau_1) > 0$, (b) follows by Lemma 2.1.

Proof of (c): We shall do this in two steps.

Step 1. By (b) and Banach–Alaoglu theorem one can find a sub sequence $\{\phi_i(\cdot, \tau_q)\}_{q=1}^\infty$ of $\{\phi_i(\cdot, \tau_p)\}_{p=1}^\infty$ and a $\psi_i \in L_2(\bar{\Omega}_{\tau_0})$ such that $\phi_i(\cdot, \tau_q) \rightarrow \psi_i$ weakly in $L_2(\bar{\Omega}_{\tau_0})$ as $q \rightarrow \infty$. Since the operator $K_{\tau_0}: L_2(\bar{\Omega}_{\tau_0}) \rightarrow L_2(\bar{\Omega}_{\tau_0})$ is compact, it now follows that

$$\|K_{\tau_0}\phi_i(\cdot, \tau_q) - K_{\tau_0}\psi_i\|_{L_2(\bar{\Omega}_{\tau_0})} \rightarrow 0 \text{ as } q \rightarrow \infty.$$

We see that

$$\begin{aligned} & \left\| \int_{\bar{\Omega}_{\tau_q}} k(\cdot, y)\phi_i(y, \tau_q)dy - \int_{\bar{\Omega}_{\tau_0}} k(\cdot, y)\psi_i(y)dy \right\|_{L_2(\bar{\Omega}_{\tau_0})} \\ & \leq \left\| \int_{\bar{\Omega}_{\tau_0} \setminus \bar{\Omega}_{\tau_q}} k(\cdot, y)\phi_i(y, \tau_q)dy \right\|_{L_2(\bar{\Omega}_{\tau_0})} + \left\| \int_{\bar{\Omega}_{\tau_0}} k(\cdot, y)[\phi_i(y, \tau_q) - \psi_i(y)]dy \right\|_{L_2(\bar{\Omega}_{\tau_0})} \\ & \leq \left(\int_{\bar{\Omega}_{\tau_0}} \int_{\bar{\Omega}_{\tau_0} \setminus \bar{\Omega}_{\tau_q}} |k(x, y)|^2 dx dy \right)^{1/2} [\mu(\bar{\Omega}_{\tau_0} \setminus \bar{\Omega}_{\tau_q})]^{1/2} J_i + \|K_{\tau_0}\phi_i(\cdot, \tau_q) - K_{\tau_0}\psi_i\|_{L_2(\bar{\Omega}_{\tau_0})}, \end{aligned}$$

where J_i is as given in (2.10). Hence we have

$$\left\| \frac{1}{\lambda_i^+(\tau_q)} \int_{\bar{\Omega}_{\tau_q}} k(\cdot, y)\phi_i(y, \tau_q)dy - \frac{1}{\lambda_i^+(\tau_0)} \int_{\bar{\Omega}_{\tau_0}} k(\cdot, y)\psi_i(y)dy \right\|_{L_2(\bar{\Omega}_{\tau_0})} \rightarrow 0 \text{ as } q \rightarrow \infty;$$

from which it follows that

$$\psi_i = \frac{1}{\lambda_i^+(\tau_0)} K_{\tau_0}\psi_i \tag{2.13}$$

and

$$\|\phi_i(\cdot, \tau_q) - \psi_i\|_{L_2(\bar{\Omega}_{\tau_0})} \rightarrow 0 \text{ as } q \rightarrow \infty. \tag{2.14}$$

From (2.6) it follows that

$$1 = \int_{\bar{\Omega}_{\tau_0}} |\phi_i(x, \tau_q)|^2 dx + \int_{\bar{\Omega}_{\tau_0} \setminus \bar{\Omega}_{\tau_q}} |\phi_i(x, \tau_q)|^2 dx \quad \forall q. \tag{2.15}$$

As

$$\mu(\bar{\Omega}_{\tau_0} \setminus \bar{\Omega}_{\tau_q}) \rightarrow 0 \text{ when } q \rightarrow \infty, \tag{2.16}$$

using (2.10) in (2.15) we get

$$\lim_{q \rightarrow \infty} \int_{\bar{\Omega}_{\tau_0}} |\phi_i(x, \tau_q)|^2 dx = 1. \tag{2.17}$$

Since the map $f \mapsto \|f\|$ from $L_2(\bar{\Omega}_{\tau_0})$ to \mathbb{R}^1 is continuous, from (2.14) and (2.17) it follows that

$$\int_{\bar{\Omega}_{\tau_0}} |\psi_i(x)|^2 dx = 1. \tag{2.18}$$

From (2.13) and (2.18) we have ψ_i is a normalized eigenfunction of K_{τ_0} corresponding to the eigenvalue $\lambda_i^+(\tau_0)$.

Step 2. Let us denote the ψ_i by $\phi_i(\cdot, \tau_0)$. Then (2.13) can be written as

$$\lambda_i^+(\tau_0)\phi_i(x, \tau_0) = \int_{\bar{\Omega}_{\tau_0}} k(x, y)\phi_i(y, \tau_0)dy. \tag{2.19}$$

From (2.5) and (2.19) it follows that for any $x \in \bar{\Omega}_{\tau_0}$,

$$\begin{aligned} & |\lambda_i^+(\tau_q)\phi_i(x, \tau_q) - \lambda_i^+(\tau_0)\phi_i(x, \tau_0)| \\ & \leq \left(\int_{\bar{\Omega}_0 \setminus \bar{\Omega}_{\tau_q}} |k(x, y)|^2 dy \right)^{1/2} \left(\int_{\bar{\Omega}_0 \setminus \bar{\Omega}_{\tau_q}} |\phi_i(y, \tau_0)|^2 dy \right)^{1/2} \\ & \quad + \left(\int_{\bar{\Omega}_{\tau_q}} |k(x, y)|^2 dy \right)^{1/2} \left(\int_{\bar{\Omega}_{\tau_q}} |\phi_i(y, \tau_q) - \phi_i(y, \tau_0)|^2 dy \right)^{1/2} \\ & \leq M_{\tau_0} J_i [\mu(\bar{\Omega}_{\tau_0} \setminus \bar{\Omega}_{\tau_q})]^{1/2} + M_{\tau_0}^{1/2} \|\phi_i(\cdot, \tau_q) - \phi_i(\cdot, \tau_0)\|_{L_2(\bar{\Omega}_{\tau_0})}. \end{aligned}$$

Now (2.14) and (2.16) imply that

$$\phi_i(x, \tau_q) \rightarrow \phi_i(x, \tau_0) \text{ uniformly on } \bar{\Omega}_{\tau_0} \text{ as } q \rightarrow \infty.$$

This completes the proof of (c).

From (2.5) and (\mathcal{C}_4 ii) it can be easily seen that

$$\begin{aligned} & \frac{\lambda_i^+(\tau_0) - \lambda_i^+(\tau_q)}{\tau_0 - \tau_q} \int_{\bar{\Omega}_0} \phi_i(x, \tau_q) \overline{\phi_i(x, \tau_0)} dx \\ & = \frac{\lambda_i^+(\tau_0)}{\tau_0 - \tau_q} \int_{\bar{\Omega}_0 \setminus \bar{\Omega}_{\tau_q}} \{\phi_i(y, \tau_q) - \phi_i(x, \tau_0)\} \overline{\phi_i(y, \tau_0)} dy \\ & \quad + \frac{\lambda_i^+(\tau_0)}{\tau_0 - \tau_q} \int_{\bar{\Omega}_0 \setminus \bar{\Omega}_{\tau_q}} |\phi_i(y, \tau_0)|^2 dy \\ & = \frac{\lambda_i^+(\tau_0)}{\tau_0 - \tau_q} \left[\int_{t=\tau_q}^{\tau_0} \int_{\partial \bar{\Omega}_t} \{\phi_i(y, \tau_q) - \phi_i(y, \tau_0)\} \overline{\phi_i(y, \tau_0)} d\sigma_t(y) \right. \\ & \quad \left. + \int_{\bar{\Omega}_0 \setminus \bar{\Omega}_{\tau_q}} |\phi_i(y, \tau_0)|^2 dy \right]. \end{aligned}$$

Letting $q \rightarrow \infty$ and using (c) and (\mathcal{C}_4 i), we get (2.4). This proof can be easily modified for the case $m_i(\tau_0) > 1$ along similar lines as in [7]. ■

3. Proofs of the main results

Proof of Theorem 1.1: (2.8) implies that the series $\sum_{i=1}^{\infty} \lambda_i(\tau) \langle \alpha_1, \phi_i \rangle \phi_i(y, \tau)$ converges uniformly and absolutely [6] to $\int_{\bar{\Omega}_1} k(y, z) \alpha_1(x, z, \tau) dz$ where

$$\langle \alpha_1, \phi_i \rangle = \int_{\bar{\Omega}_1} \alpha_1(x, z, \tau) \overline{\phi_i(z, \tau)} dz.$$

Now using (1.2) we get

$$\alpha_1(x, y, \tau) - k(y, x) = \sum_{i=1}^{\infty} \lambda_i(\tau) \langle \alpha_1, \phi_i \rangle \phi_i(y, \tau). \tag{3.1}$$

From the above equation one easily gets

$$\langle \alpha_1, \phi_i \rangle = \frac{\lambda_i(\tau)}{1 - \lambda_i(\tau)} \overline{\phi_i(x, \tau)}. \quad (3.2)$$

Let $u(x, y, \tau) = \alpha_1(x, y, \tau) - k(y, x)$. We have from (3.1) and (3.2)

$$u(x, y, \tau) = \sum_{i=1}^{\infty} \frac{(\lambda_i(\tau))^2}{1 - \lambda_i(\tau)} \phi_i(y, \tau) \overline{\phi_i(x, \tau)}. \quad (3.3)$$

For each fixed $x \in \bar{\Omega}_\tau$, the series in the right hand side of (3.3) converges uniformly and absolutely in the variable y and hence the sum is a continuous function in y . It therefore follows that $u(x, y, \tau)$ has a continuous representation. Letting $y \rightarrow x$ in (3.3) and using monotone convergence theorem and Lemma 2.3 we get

$$\int_{\partial \bar{\Omega}_\tau} u(x, x, \tau) d\sigma_\tau(x) = \sum_{i=1}^{\infty} \frac{\lambda_i(\tau)}{1 - \lambda_i(\tau)} \frac{d\lambda_i(\tau)}{d\tau}.$$

By Lemma 2.2 as in [8] we get

$$-\int_{t=0}^{\tau} dt \int_{\partial \bar{\Omega}_\tau} u(x, x, t) d\sigma_\tau(x) = \sum_{i=1}^{\infty} \{\lambda_i(\tau) + \log[1 - \lambda_i(\tau)]\}.$$

Now (1.5) is easily obtained by exponentiating both sides. ■

Proof of Theorem 1.2: Since the kernel is real valued, one can choose ϕ_i 's such that they are real and

$$\phi_i(-x, \tau) = \pm \phi_i(x, \tau), \quad \forall x \in \bar{\Omega}_\tau, \quad (3.4)$$

where \pm correspond respectively to those eigenvalues having symmetric and skew symmetric eigenfunction. By using arguments as in the proof of Theorem 1.1, one can prove that

$$v(x, y, \tau) = \sum_{i=1}^{\infty} \frac{(\lambda_i(\tau))^2}{1 - \lambda_i(\tau)} \phi_i(y, \tau) \overline{\phi_i(-x, \tau)}, \quad (3.5)$$

where $v(x, y, \tau)$ is a continuous function on $\bar{\Omega}_\tau$ in the variable y for each fixed $x \in \bar{\Omega}_\tau$. Letting $y \rightarrow x$ in (3.5) and using (3.4) we get

$$v(x, x, \tau) = \pm \sum_{i=1}^{\infty} \frac{(\lambda_i(\tau))^2}{1 - \lambda_i(\tau)} |\phi_i(x, \tau)|^2.$$

Now by arguments as in the proof of Theorem 1.1 one can arrive at (1.9). ■

4. Some remarks

Remark 1. (Non symmetric case): In all the above lemmas if the kernel is not symmetric then the analogous results hold for the singular numbers, $s_i(\tau)$. That is, we get for

each $i = 1, 2, 3, \dots$,

- (1). $s_i(\tau)$ is a nondecreasing function in τ ,
- (2). $s_i(\tau)$ is absolutely continuous in each bounded interval of $[0, \infty)$,
- (3). There exist $m_i(\tau)$ orthonormal eigenfunctions $\phi_{ij}(\cdot, \tau) \in L_2(\Omega_\tau)$ corresponding to the eigenvalue $s_i^2(\tau)$ of $K_\tau^* K_\tau$ such that

$$\frac{ds_i(\tau)}{d\tau} = \frac{1}{2} s_i(\tau) \int_{\partial\Omega_\tau} |\phi_{ij}(x, \tau)|^2 d\sigma_\tau(x) \quad \text{for } j = 1, 2, 3, \dots, m_i(\tau),$$

and for almost every $\tau \geq 0$; where $m_i(\tau)$ is the multiplicity of the eigenvalue $s_i^2(\tau)$.

Remark 2. Lemma 2.3, Theorems 1.1 and 1.2 as well as their proofs remain valid even if we replace the condition \mathcal{C}_7 by a weaker one given below:

Given two compact subsets E and F of \mathbb{R}^n the map $x \mapsto k_x$ from E to $L_2(F)$ is weakly continuous, meaning that,

“for each $f \in L_2(F)$, the map $x \mapsto \int_F k_x(y) \overline{f(y)} dy$ from E to \mathbb{C} is continuous.”

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References

- [1] Akhiezer N I, A continual analogue of some theorems on Toeplitz matrices, *AMS Transl.*, **50** (1966) 295–316
- [2] Dunford N and Schwartz J T, *Linear operators*, Vol II, (Interscience) 1963
- [3] Kac M, Toeplitz-matrices, translation kernels and a related problem in probability theory, *Duke Math. J.*, **21** (1954) 501–509
- [4] Kac M, Theory and application of Toeplitz forms, in *Summer institute on spectral theory and statistical mechanics*, (Brookhaven National Laboratory), (1965) pp. 1–56
- [5] Mullikin T W and Vittal Rao R, Extended Kac–Akhiezer formula for the Fredholm determinant of integral operators, *J. Math. Anal. Appl.*, **61** (1977) 409–415
- [6] Riesz F and Nagy Sz, *Functional analysis*, (Unger, New York) (1955)
- [7] Vittal Rao R, On the eigenvalues of the integral operators with difference kernels, *J. Math. Anal. Appl.* **53** (1976) 554–566
- [8] Vittal Rao R, Extended Akhiezer formula for the Fredholm determinant of difference kernels, *J. Math. Anal. Appl.* **54** (1976) 79–88
- [9] Vittal Rao R and Sukavanam N, Kac–Akhiezer formula for normal integral operators *J. Math. Anal. Appl.*, **114** (1986) 458–467
- [10] Zabraiko P P *et al*, *Integral equations – a reference text*. (Noordhoff international publishing, Leyden) 1975