it follows that

$$\int_0^{t_m} |w_1(t)|\,dt \le \left[t_m \int_0^{t_m} |w_1(t)|^2 dt\right]^{1/2} = \left[\frac{t_m}{t_m}\int_0^{t_m} \left|\sum_{k=-\infty}^{+\infty} c_k e^{jk(2\pi/t_m)t}\right|^2 dt\right]^{1/2} = \left[\sum_{k=-\infty}^{+\infty} |c_k|^2\right]^{1/2}. \tag{24}$$

Recalling (23), if $w_1(t)\simeq 0$ for $t>t_m$, the inequality (24) can be written in the form

$$\int_0^{t_m} |w_1(t)|\,dt \le \left[\sum_{k=-\infty}^{+\infty} |W(jk(2\pi/t_m))|^2\right]^{1/2}. \tag{25}$$

The evaluation of the term in the right-hand side of (25) is very easy if $W_1(s)$ is of the form

$$W_1(s) = \sum_{i=0}^{m} a_i s^i \Big/ \sum_{i=0}^{n} b_i s^i$$

with $n>m$, $\sum_i b_i s^2$ strictly Hurwitzian, and all the poles p_i simple. Then

$$\left[\sum_{-\infty}^{+\infty} |W_1(jk\omega_0)|^2\right]^{1/2} = \left[j\frac{2\pi B_1}{\omega_0}\cot j\pi\frac{p_1}{\omega_0} + \cdots + \frac{2\pi B_n}{\omega_0}\cot j\pi\frac{p_n}{\omega_0}\right]^{1/2} \tag{26}$$

where $\omega_0=2\pi/t_m$, $B_i=B_i'W_i(-p_i)$, and B_i' are the residues of $W_1(s)$ at the poles p_i.

It is also possible to find a relation similar to (25) if the poles of $W_1(s)$ are not simple.

The accuracy of the approximation of the integral (17) by the right-hand side of (25) depends on t_m since the L_2 norm approaches the L_1 norm as t_m goes to zero.

Conclusions

A sufficient condition is derived here for the boundedness of the responses of a class of nonlinear feedback systems to amplitude limited signals. This condition depends on the linear part of the system and on the range of the derivative of the nonlinear characteristic. An upper bound on the responses is then determined. On the other hand, it is possible to find an interval, if it exists, in which the amplitude of the responses is within a bound fixed a priori.

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Stability of a System of Linear Differential Equations

***Abstract*—Conditions for asymptotic stability of an *n*th order linear time-invariant differential equation of a particular type with matrix coefficients are shown to be similar to those obtained through the Routh-Hurwitz inequalities, but wherein real numbers and positivity are replaced by the matrix coefficients and positive definiteness.**

Introduction

Seshu and Reed have mentioned the conditions under which a second-order linear time-invariant differential equation with real constant matrix coefficients is stable.[1] Ezeilo established stability conditions for a system of n third-order nonlinear differential equations of a particular class.[2]

In this correspondence, conditions for asymptotic stability in the large of nth-order ordinary differential equations with time-invariant matrix coefficients are derived.

Notation

Let R^m and T_+ denote the real Euclidean m-dimensional space and the real interval $0\le t<\infty$, respectively. Consider a system of m nth-order differential equations

$$X^{(n)} + B_1X^{(n-1)} + B_2X^{(n-2)} + \cdots + B_{n-1}X^{(1)} + B_nX = 0 \tag{1}$$

where $X\in R^m$, X is an m vector, and B_i, $i=1, 2, \cdots, n$, are $m\times m$ real constant matrices. It is assumed that (1) satisfies all the requirements for uniqueness and existence of solution $\phi(t;\ X_1^0, \cdots, X_n^0;\ t_0)$ which corresponds to the initial state $(t_0;\ X_1^0, \cdots, X_n^0)$ in the motion space $R^m\times T_+$, where the vectors $X_2, X_3, \cdots, X_n$ represent the successive time derivatives of the vector $X_1=X$. Equation (1) is a vector version for the system of real linear time-invariant m nth-order differential equations

$$x_j^{(n)} + \sum_{k=1}^{m} 1_{\beta\cdot k}x_k^{(n-1)} + \cdots + \sum_{k=1}^{m} n_{\beta\cdot k}x_k = 0, \qquad j = 1, \cdots, m. \tag{2}$$

Definition 1: The system of differential equations (1) is said to be asymptotically stable if the vectors $X_i(i=1, \cdots, n)\to 0$ as time $t\to\infty$. The vectors $X_i\to 0$ imply that every component of X_i, the $x_j^{(i-1)}$ $(j=1, \cdots, m)\to 0$ as $t\to\infty$.

Definition 2: The determinant formed, similar to that of the Routh–Hurwitz determinant but with matrix coefficients $[B_i]$ of (1) treated in place of real numbers, will be called the matrix Routh–Hurwitz determinant.

Theorem 1

If the matrix coefficients $[B_i]$ of (1) can be simultaneously diagonalized by a nonsingular transformation, then positive definiteness of all minors Δ_i $(i=1, \cdots, n)$ of the matrix Routh–Hurwitz determinant are necessary and sufficient criteria for asymptotic stability of the system of equations (1).

Proof: Let the matrix $[L]$ diagonalize the matrices $[B_i]$ such that matrices $[\Lambda_i]$ are diagonal with the transformation,

$$[L]^{-1}[B_i][L] = [\Lambda_i], \qquad i = 1, \cdots, n.$$

Then the positive definiteness of the product of matrices

$$\{[B_1][B_2], \cdots, [B_l]\}, \qquad l \le i = 1, \cdots, n$$

is equivalent to the positive definiteness of the product of matrices

$$\{[\Lambda_1][\Lambda_2], \cdots, [\Lambda_l]\}$$

since

$$\{[\Lambda_1], \cdots, [\Lambda_l]\} = \{[L]^{-1}[B_1][L]\cdot[L]^{-1}[B_2][L], \cdots, \cdot[L]^{-1}[B_l][L]\} = [L]^{-1}\{[B_1, \cdots, [B_l]\}[L].$$

Consider the equation with diagonal matrix coefficients $[\Lambda_i]$ obtained by a transformation $[X]=[L]\ [Y]$ which is equivalent, in

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the sense of Liapunov transformation, to the given system of equations (1). The modified equation is

$$Y^{(n)} + \Lambda_1 Y^{(n-1)} + \cdots + \Lambda_{n-1} Y^{(1)} + \Lambda_n Y = 0 \tag{3}$$

which is a set of differential equations of the form

$$y_j^{(n)} + {}^1\lambda_j y_j^{(n-1)} + \cdots + {}^{(n-1)}\lambda_j y_j^{(1)} + {}^n\lambda_j y_j = 0, \quad j = 1, \cdots, m. \tag{4}$$

Let ${}^i\delta_i$, $i=1, \cdots, n$, be the ith-order Routh–Hurwitz determinant minor corresponding to the single differential equation in the y_jth component of the m vector $[Y]$. Then the determinants ${}^i\delta_i$, $j=1, \cdots, m$, are the jjth diagonal elements of the matrix Routh–Hurwitz determinant Δ_i. For the determinants ${}^i\delta_i$ to be positive it is necessary and sufficient that the determinants Δ_i be positive definite Q.E.D.

It may be noted that the matrix coefficients $[B_i]$ may be symmetric or nonsymmetric. It is known that any two symmetric matrices can be simultaneously diagonalized if one of them is positive definite. It then follows that the other coefficient matrices of (1) must be expressible as functions involving positive integer powers of these two matrices.

The conditions under which a set of nonsymmetric matrices can be simultaneously diagonalized are not known at present.

Example

Consider the following system of differential equations for stability:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 7 & 4 \\ -6 & 18 \end{bmatrix} \begin{bmatrix} \dddot{x} \\ \dddot{y} \end{bmatrix} + \begin{bmatrix} 19 & 8 \\ -12 & 41 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} -4 & 12 \\ -18 & 29 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It is known that the preceding coefficient matrices can be diagonalized simultaneously. The matrix Routh–Hurwitz test shows that the given system of equations satisfy the conditions for asymptotic stability.

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Minimum-State Realizations of Linear Time-Varying Systems

Abstract—**An algorithm for removing uncontrollable (unobservable) state variables from time-varying systems is presented. The algorithm makes use of the time-variable controllability (observability) matrix which is augmented with the identity matrix and transformed into Hermite normal form. In this new set of coordinates, the uncontrollable (unobservable) state variables are easily identified and removed. An example is given.**

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It is well known[1],[2] that a system which is uncontrollable and/or unobservable can be reduced to a zero-state equivalent system of lower dynamic order by removing the uncontrollable and unobservable modes. This problem of reducing the order of a system has been considered by Silverman and Meadows,[1],[3] Glass and D'Angelo,[4] and others. It is the purpose of this correspondence to present a very simple procedure for removing uncontrollable and unobservable state variables of time-variable linear systems to yield a zero-state equivalent system with the minimum number of state variables. In earlier papers,[5],[6] algorithms for constructing minimum-state realizations of time-invariant systems were presented. Some of these techniques will be applied to the time-varying system problem.

The system is represented by the state variable equations

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t). \end{aligned} \tag{1}$$

Use will be made of the controllability and observability matrices for the time-varying system given previously.[3],[7] These are as follows:

$$Q_c(t) = [P_0(t) \vdots P_1(t) \vdots \cdots \vdots P_{n-1}(t)] \tag{2}$$

where

$$P_{k+1}(t) = -A(t)P_k(t) + \dot{P}_k(t), \quad P_0(t) = B(t)$$

and

$$Q_0(t) = [R_0(t) \vdots R_1(t) \vdots \cdots \vdots R_{n-1}(t)] \tag{3}$$

where

$$R_{k+1}(t) = A'(t)R_k(t) + \dot{R}_k(t), \quad R_0(t) = C'(t)$$

and where the prime indicates transpose and the dot indicates time derivative.

An algorithm can now be stated.

Algorithm I

Given system (1) where $A(t)$, $B(t)$, and $C(t)$ are $n \times n$, $n \times m$, and $p \times n$ matrices, respectively, with possibly time-varying elements, and where matrices $A(t)$, $B(t)$, and $C(t)$ together with their first $n-2$, $n-1$, and $n-1$ derivatives, respectively, are continuous functions, then any uncontrollable state variables in the system may be removed as follows:

1) Form the controllability matrix $Q_c(t)$ as given in (2).

2) Augment $Q_c(t)$ with an $n \times n$ identity matrix, i.e.,

$$[Q_c(t) \vdots \mathrm{I}].$$

3) Transform $Q_c(t)$ into Hermite normal (row-echelon) form $(Q_{c_h}(t))$ using elementary row operations¹ and obtain the transformation matrix T_c, which must be nonsingular and differentiable, where

$$[Q_c(t) \vdots \mathrm{I}] \Rightarrow [Q_{c_h}(t) \vdots T_c^{-1}]. \tag{4}$$

4) Identify the uncontrollable state variables of the system, in this new set of coordinates, as those state variables corresponding to the all-zero rows of $Q_{c_h}(t)$ where row i corresponds to state variable x_i. Transform the system $\mathcal{S} = \{A, B, C\}$ into the new coordinate system $\hat{\mathcal{S}} = \{\hat{A}, \hat{B}, \hat{C}\}$ where

$$\begin{aligned} \hat{A} &= (T_c^{-1}AT_c - T_c^{-1}\dot{T}_c) \\ \hat{B} &= T_c^{-1}B \\ \hat{C} &= CT_c. \end{aligned} \tag{5}$$

5) For each uncontrollable state variable x_i as identified in step 4), delete row i and column i from $\hat{A}$, row i from $\hat{B}$, and column i from $\hat{C}$ to obtain the reduced system

$$\hat{\mathcal{S}}_r = \{A_r, \hat{B}_r, \hat{C}_r\}.$$

6) Subsystem $\hat{\mathcal{S}}_r$ is totally controllable² and is zero-state equivalent to $\mathcal{S}$.

Proof of this algorithm follows directly from Theorem 5.1 of Nering,[8] Lemma 1 of Silverman and Meadows,[1] and Corollary 11.3.15 of Zadah and Desoer,[9] and is presented in Albertson[10], but it will not be given here because of space limitations.

The algorithm for removing unobservable modes is the dual of Algorithm I, where $Q_0(t)$ given in (3) is utilized and where the transformation

$$[Q_0(t) \vdots I] \Rightarrow [Q_{0_h}(t) \vdots T_0'] \tag{6}$$

is carried out using elementary row operations. The matrix T_0' is the transpose of the desired transformation matrix.

Example

The following example, considered previously,[1],[4] illustrates the use of the algorithm. Consider system (1) where

$$A(t) = \begin{bmatrix} t-1 & 0 & -t+2 \\ -t-2 & 1 & t+2 \\ t & 0 & -t+1 \end{bmatrix}$$

$$B(t) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{7}$$

$$C(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The controllability matrix using (2) is

$$Q_c(t) = \begin{bmatrix} 1 & 1-t & 0 \\ 1 & 1+t & 2 \\ 0 & -t & -1 \end{bmatrix} \tag{8}$$

Now, using (4),

$$\begin{bmatrix} 1 & 1-t & 0 & \cdot & 1 & 0 & 0 \\ 1 & 1+t & 2 & \cdot & 0 & 1 & 0 \\ 0 & -t & -1 & \cdot & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1-t & 0 & \cdot & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdot & -1 & 1 & 2 \\ & 0-t & -1 & \cdot & 0 & 0 & 1 \end{bmatrix}. \tag{9}$$

It is noted that $Q_{c_h}(t)$ has rank 2 everywhere and that state variable x_2, in the new coordinate system, is uncontrollable. Also note that

$$T_c^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

¹ It may not always be necessary to reduce $Q_c(t)$ completely to Hermite form, but only to the point where the all-zero rows of Q_{c_h} are identified (see example).

² See page 689 of Silverman and Meadows[1] for the definition of total controllability.