Noncausal Multipliers for Nonlinear System Stability

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Abstract—Using the Popov approach, new absolute stability conditions in multiplier form are derived for a single-loop system with a time-invariant stable linear element G in the forward path and a nonlinear time-varying gain $k(t)\phi(\cdot)$ in the feedback path. The classes of nonlinearities considered are the monotonic, odd monotonic, and power law. The stability multiplier contains causal and noncausal terms; for absolute stability, the latter give rise to a lower bound (which is believed to be new) on dk/dt and the former, as in earlier investigations, to an upper bound on dk/dt . Asymptotic stability conditions for a linear system are realized as a limiting case of the absolute stability conditions derived for the power law nonlinearity.

I. INTRODUCTION

THE FEEDBACK system of Fig. 1 contains a stable time-invariant block G and a nonlinear time-varying feedback gain $k(t)\phi(\cdot)$. The system is governed by the set of equations

$$\begin{aligned} dx/dt &= Ax - b k(t) \phi(\sigma) \\ \sigma &= c'x \end{aligned} \quad (1)$$

where A is an $n \times n$ (constant) stable matrix; b, c, x (state) are $n \times 1$ vectors; σ the output of the system is a scalar. The time-varying gain $k(t)$ is assumed to be continuous and bounded; for convenience, it is allowed to belong to the infinite sector or $0 < \epsilon \leq k(t) < \infty$. The transfer function of the system is $G(s) = c'(sI - A)^{-1}b$. $\phi(\sigma)$ is a memoryless continuous nonlinearity satisfying in general the conditions

$$\phi(0) = 0, \quad \sigma\phi(\sigma) > 0, \quad \text{for all } \sigma \neq 0.$$

This class of functions is denoted by P , i.e., $\phi(\sigma) \in P$. For simplicity in the proofs of the theorems it is assumed that there exist positive constants h_1 (however small) and h_2 such that

$$h_1\sigma^2 \leq \sigma\phi(\sigma) < h_2\sigma^2.$$

If $\phi(\sigma)$ satisfies additional conditions, then it belongs to other classes. For example,

1) $\phi(\sigma) \in P_M$, the class of monotonically nondecreasing functions

$$[\phi(\sigma_1) - \phi(\sigma_2)](\sigma_1 - \sigma_2) \geq 0, \quad \text{for all } \sigma_1 \text{ and } \sigma_2 \quad (2)$$

Fig. 1. Nonlinear time-varying feedback system.

2) $\phi(\sigma) \in P_{MO}$, the class of odd monotonically nondecreasing functions (2) and

$$\phi(\sigma_1)(\sigma_1 + \sigma_2) - \phi(\sigma_2)(\sigma_1 - \sigma_2) \geq 0, \quad \text{for all } \sigma_1 \text{ and } \sigma_2 \quad (3)$$

3) $\phi(\sigma) \in P_{P^{m_0}}$, the class of power law functions

$$\phi(\sigma) = -\phi(-\sigma)$$

$$|\sigma_1/\sigma_2|^{1/m_0} \leq |\phi(\sigma_1)/\phi(\sigma_2)| \leq |\sigma_1/\sigma_2|^{m_0},$$

$$\text{for } |\sigma_1| \geq |\sigma_2| \text{ and } m_0 \geq 1.$$

Or, equivalently, (2) and

$$\phi(\sigma_1)(c\sigma_1 + \sigma_2) + \phi(\sigma_2)(c\sigma_2 - \sigma_1) \geq 0, \quad \text{for all } \sigma_1 \text{ and } \sigma_2 \quad (4)$$

where

$$c = \max_{0 < \sigma < \infty} \left| \frac{\sigma^{m_0} - \sigma}{1 + \sigma^{m_0+1}} \right|.$$

As $m_0 \rightarrow \infty$, $c \rightarrow 1$ so that $\phi(\sigma) \in P_{MO}$ and (4) reduces to (3); as $m_0 \rightarrow 1$, $c \rightarrow 0$ giving a linear feedback; $c = 0.3536$ corresponds to $m_0 = 3$. (See Thathachar *et al.* [10].)

Assuming that the closed-loop system is asymptotically stable for all constant linear feedback with $k(t) \equiv K \in (0, \infty)$, the problem is to find conditions to guarantee the absolute stability of the null solution of (1) when $k(t) \in (0, \infty)$ and $\phi(\sigma)$ belongs to any one of the preceding classes.

Definitions

1) A causal (or nonanticipative) system is one whose response to an impulse applied at $t = 0$ is nonzero for

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$t \geq 0$ and zero for $t < 0$; the complex-frequency function of such a system is said to be causal.

It is to be noted that passive systems (or systems with a positive real impedance function) are necessarily causal.

2) A noncausal (or anticipative) system is one whose response to an impulse applied at $t = 0$ is nonzero for $t \leq 0$ and zero for $t > 0$; the complex-frequency function of such a system is said to be noncausal.

Historical Credits

It appears from a survey of literature that Popov's approach [1] has not been applied to the present problem. Using Zames' positive operator theory [2], Cho and Narendra [3], and in a Lyapunov framework, Narendra and Taylor [4] derived absolute stability conditions in terms of positive real (or causal) multipliers and a local bound on dk/dt (which depends upon $\phi(\sigma)$ and the multiplier employed). But they do not consider a noncausal multiplier and hence their results are less general than those presented in Section III.

Major contents of this paper are the following.

1) A new method is presented for establishing the non-negativeness of the integral

$$\int_0^T k(t)\phi(\sigma)[Z\sigma(t)] dt$$

with Z the operator representation of the multiplier chosen. This method easily accommodates noncausal operators in contrast with Zames' [2] and Narendra and Taylor's [4] approaches.

2) Theorems 3 and 4 are presented (in Section III), which contain in part a special noncausal multiplier. The resulting additional lower bound on dk/dt appears to be the first of its kind.

The paper is divided into two main parts: the first part introduces the Popov approach and deals with causal multipliers; Theorem 1, containing an RL-RC multiplier, is proved in detail to provide motivation for the second part, which deals with causal and noncausal multipliers. The proofs of the main results (Theorems 3 and 4) do not differ very much from the proof of Theorem 1; therefore, only the necessary changes are indicated.

II. SOLUTION OF THE MAIN PROBLEM—CAUSAL MULTIPLIER

A. Introduction to Popov Approach

The integral equation representation of the system is

$$x(t) = \exp(At)x(0) - \int_0^t \exp[A(t-\tau)]bk(\tau)\phi(\sigma(\tau)) d\tau \quad (5a)$$

$$\sigma(t) = c' \exp(At)x(0) - \int_0^t c' \exp[A(t-\tau)]bk(\tau)\phi(\sigma(\tau)) d\tau. \quad (5b)$$

The absolute stability of the null solution (NS) of (5) implies the absolute stability of the NS of (1). It is obvious that a knowledge of the behavior of $\sigma(t)$ enables one to deduce the behavior of $x(t)$ from (5a); for example, if $\sigma(t)$ is bounded, so is $x(t)$, that is, all the components of $x(t)$ are bounded; if $\sigma(t)$ tends to zero, so do the components of $x(t)$.

The method used by Popov [1] is to obtain an integral inequality of the form

$$a(|\sigma|) + \int_0^t b(|\sigma|) dt < d(|\sigma(0)|) \quad (6)$$

where $a(r)$, $b(r)$, and $d(r)$ are continuous functions zero in the origin, the first two functions being monotonically increasing. From inequality (6), the absolute stability of the NS of (1) can be proved [1].

In order to realize the integral inequality of the desired form (6), a quadratic functional $\rho(T)$ is considered in σ for which one seeks, firstly, a lower bound of the form

$$a(|\sigma|) + \int_0^t b(|\sigma|) dt + d_1(|\sigma(0)|) \quad (7)$$

(where $d_1(|\sigma(0)|)$ is a function of the initial condition $\sigma(0)$) and, secondly, an upper bound of the form $d_2(|\sigma(0)|)$. The Fourier transformation is used to obtain this upper bound after a choice (following Popov [1]) of a convenient functional $\rho(T)$.

Let $g(t) = c' \exp(At)b$ be the Laplace inverse of $G(s)$, and $f(t) = c' \exp(At)x(0)$. The assumption of a stable matrix A implies the existence of constants $r_1, r_0 > 0$, such that

$$|g(t)| \leq r_1 \exp(-r_0 t), \quad \text{for all } t \geq 0.$$

Also, for the same reason, there exists a constant $r_2 > 0$ such that

$$|f(t)| \leq r_2 \exp(-r_0 t), \quad t \geq 0.$$

Consequently, the Fourier transforms $G(j\omega)$ and $F(j\omega)$ of $g(t)$ and $f(t)$, respectively, exist.

Notation:

$$\Phi(\sigma) = \int_0^\sigma \phi(\sigma) d\sigma > 0, \quad \text{for } \sigma \neq 0$$

$$\delta_\sigma = \Phi(\sigma)/\phi(\sigma)\sigma$$

$$\delta_{\max} = \max_\sigma \Phi(\sigma)/\phi(\sigma)\sigma$$

$$\psi(\sigma, t) = k(t)\phi(\sigma).$$

$\delta_D(t-T)$ denotes a unit impulse function occurring at $t = T$. Z is the operator representation of an impedance function $z(s)$. The small Greek letters $\alpha, \beta, \gamma, \mu, \nu, \lambda, \theta, \zeta, \xi$, and η are constants. The subscript i when applied to different letters need not have the same range of values, for example,

$$\sum_{i=1}^{m_1} \gamma_i; \quad \sum_{i=1}^{m_1} \sum_{i=1}^{m_2} \gamma_i'.$$

B. Mathematical Preliminaries

Lemma 1

The integral

$$I_0 = \int_0^T \psi(\sigma, t) [\alpha \sigma(t) + \beta (d\sigma/dt)] dt, \quad \alpha > 0, \quad \beta \geq 0$$

is positive (except for a constant term of the form $d_1(|\sigma(0)|)$ appearing in (7)) if, for some $\epsilon_0 > 0$,

$$(dk/dt)_{\delta_{\max}} \leq (\alpha/\beta)k(t) - \epsilon_0, \quad \text{for all } t. \quad (8)$$

Proof:

$$I_0 = \int_0^T \alpha k(t) \phi(\sigma) \sigma dt + \int_0^T \beta k(t) \phi(\sigma) (d\sigma/dt) dt.$$

Integrate the second term by parts to get

$$I_0 = \int_0^T [\alpha k(t) - \beta \delta_\sigma (dk/dt)] \phi(\sigma) \sigma dt + \beta k(T) \Phi(\sigma(T)) - \beta k(0) \Phi(\sigma(0))$$

which is positive (except for the last constant term) if (8) is satisfied. Q.E.D.

Consider

$$z_1(s) = \sum_{i=1}^{m_1} \gamma_i (s + \nu_i \mu_i) / (s + \mu_i),$$

$$\gamma_i, \mu_i > 0, \quad \nu_i \geq 0, \quad \text{for all } i. \quad (9)$$

Lemma 2

For $\phi(\sigma) \subset P_M$, the integral

$$I_1 = \int_0^T \psi(\sigma, t) [Z_1 \sigma(t)] dt$$

is nonnegative (except for a constant term of the form $d_1(|\sigma(0)|)$ in (7)) if

$$0 \leq \nu_i < 1, \quad \text{for all } i$$

and

$$(dk/dt)_{\delta_{\max}} \leq \min_i \nu_i \mu_i k(t). \quad (10)$$

Proof: Let

$$z_{1i}(s) = (s + \nu_i \mu_i) / (s + \mu_i) = 1 + (\nu_i - 1) \mu_i / (s + \mu_i).$$

Its inverse transform

$$z_{1i}(t) = \delta_D(t) + (\nu_i - 1) \mu_i \exp(-\mu_i t).$$

Let

$$\begin{aligned} I_{1i} &= \int_0^T \psi(\sigma, t) [Z_{1i} \sigma(t)] dt \\ &= \int_0^T k(t) \phi(\sigma) \cdot \left\{ \sigma(t) + (\nu_i - 1) \mu_i \int_0^t \exp[-\mu_i(t - \tau)] \sigma(\tau) d\tau \right\} dt \end{aligned} \quad (11)$$

so that

$$I_1 = \sum_i \gamma_i I_{1i}.$$

When $\phi(\sigma) \subset P_M$, (2) is satisfied. In (11), let

$$\sigma_1 = \sigma$$

$$\sigma_1 - \sigma_2 = \sigma(t) + (\nu_i - 1) \mu_i \int_0^t \exp[-\mu_i(t - \tau)] \sigma(\tau) d\tau$$

from which

$$\sigma_2 = -\mu_i (\nu_i - 1) \int_0^t \exp[-\mu_i(t - \tau)] \sigma(\tau) d\tau$$

and

$$d\sigma_2/dt = -\mu_i \sigma_2 + \mu_i (1 - \nu_i) \sigma(t)$$

or

$$\sigma(t) = \sigma_1(t) = \frac{d\sigma_2/dt + \mu_i \sigma_2}{\mu_i (1 - \nu_i)}. \quad (12)$$

Adding to and subtracting from (11) the expression $\phi(\sigma_2)(\sigma_1 - \sigma_2)$, one has

$$\begin{aligned} I_{1i} &= \int_0^T k(t) [\phi(\sigma_1) - \phi(\sigma_2)] (\sigma_1 - \sigma_2) dt \\ &\quad + \int_0^T k(t) \phi(\sigma_2) (\sigma_1 - \sigma_2) dt. \end{aligned} \quad (13)$$

The first integral of (13) is nonnegative in virtue of (2); the second integral on using (12) becomes

$$\begin{aligned} &\int_0^T k(t) \phi(\sigma_2) (\sigma_1 - \sigma_2) dt \\ &= \frac{1}{(1 - \nu_i) \mu_i} \int_0^T k(t) \phi(\sigma_2) [d\sigma_2/dt + \nu_i \mu_i \sigma_2] dt. \end{aligned} \quad (14)$$

When $k(t)$ is a constant, integral (14) (and hence I_{1i}) is nonnegative if $0 \leq \nu_i < 1$ (for all i), thereby verifying earlier results [5], [6]. However, when $k(t)$ is time-varying, use Lemma 1 to get (10) for the nonnegativity of I_{1i} . Q.E.D.

Consider

$$z_2(s) = \sum_{i=1}^{m_2} \gamma'_i (s + \nu'_i \mu'_i) / (s + \mu'_i),$$

$$\gamma'_i, \mu'_i > 0, \quad \nu'_i \geq 0, \quad \text{for all } i. \quad (15)$$

The proof of the following lemma is based on (3) and is similar to the proof of Lemma 2.

Lemma 3

For $\phi(\sigma) \subset P_{MO}$, the integral

$$I_2 = \int_0^T \psi(\sigma, t) [Z_2 \sigma(t)] dt$$

is nonnegative (except for a constant term of the form $d_1(|\sigma(0)|)$ in (7)) if

$$1 < \mu'_i < 2, \quad \text{for all } i$$

and

$$(dk/dt)\delta_{\max} \leq \min_i (2 - \nu_i') \mu_i' k(t). \quad (16)$$

Corollary: For $\phi(\sigma) \subset P_{Mo}$, the integral

$$I_{12} = \int_0^T \psi(\sigma, t) [(Z_1 + Z_2)\sigma(t)] dt$$

is nonnegative (except for a constant term of the form $d_1(|\sigma(0)|)$ in (7)) if (10) and (16) are simultaneously satisfied.

The following two lemmas are extensions of the results found in Popov [1]; the proofs are omitted here.

Lemma 4

If $|f(t)| \leq r_2 \exp(-r_0 t)$, with $r_2, r_0 > 0$, then there exists a constant R_1 independent of T such that

$$\left| \int_0^T \{\alpha f(t) + \beta(df/dt) + (Z_1 + Z_2)f(t)\} \psi(\sigma, t) dt \right| \leq R_1 \sup_{0 \leq t \leq T} |\sigma(t)|. \quad (17)$$

Lemma 5 (Popov-Barbalat)

Let $y(t)$ map $[0, \infty)$ into the real line, and be differentiable. If y and dy/dt are bounded on $[0, \infty)$, $J(y(t)) = 0$, for $y(t) = 0$ continuous, and $J(y(t)) > 0$, for $y(t) \neq 0$, $0 < \epsilon_0 \leq \epsilon(t)$, for all t , and

$$\int_0^\infty \epsilon(t) J(y(t)) dt < \infty$$

then

$$\lim_{t \rightarrow \infty} y(t) = 0.$$

C. Absolute Stability

Based on the preceding preliminaries, the absolute stability conditions for (5) are derived assuming that $\phi(\sigma) \subset P_{Mo}$. The stability multiplier, being then an RC-RL impedance, contains as special cases the multipliers for $\phi(\sigma) \subset P$ and $\phi(\sigma) \subset P_M$. Therefore, when $\phi(\sigma) \subset P$ or P_M , Theorem 1 holds after casting out the inadmissible terms from the multiplier. The functions $z_1(s)$ and $z_2(s)$ are as defined in (9) and (15), respectively.

Theorem 1

The system governed by (5) is absolutely stable for $\phi(\sigma) \subset P_{Mo}$ if there exist constant $\alpha, \beta > 0$; $\gamma_i, \gamma_i', \mu_i, \mu_i' \geq 0$; $0 \leq \nu_i < 1$, $1 < \nu_i' \leq 2$, for all i ; and a multiplier

$$z(s) = \alpha + \beta s + z_1(s) + z_2(s)$$

such that

a) $\operatorname{Re} Z(j\omega)G(j\omega) \geq 0$, for all real ω , and, for some $\epsilon_0 > 0$, as small as desired,

b) $(dk/dt)\delta_{\max} \leq \min_i (\alpha/\beta, \nu_i \mu_i, (2 - \nu_i') \mu_i') k(t) - \epsilon_0$.

(18)

Proof: Let

$$\psi_T = \psi(\sigma, t), \quad 0 \leq t \leq T$$

$$= 0, \quad t > T$$

$$\sigma_T = \sigma(t), \quad 0 \leq t \leq T$$

$$= 0, \quad t > T$$

$$f_T = f(t), \quad 0 \leq t \leq T$$

$$= \int_0^T g(t - \tau) \psi(\sigma(\tau), \tau) d\tau, \quad t > T.$$

Define

$$\rho_1(T) = \int_0^T \{\alpha[\sigma(t) - f(t)] + \beta(d/dt)[\sigma(t) - f(t)]\} \psi(\sigma, t) dt + \int_0^T \psi(\sigma, t) [(Z_1 + Z_2)(\sigma(t) - f(t))] dt \quad (19)$$

which can be written as

$$\rho_1(T) = \int_0^\infty \{\alpha(\sigma_T - f_T) + \beta(d/dt)(\sigma_T - f_T)\} \psi_T dt + \int_0^\infty \psi_T [(Z_1 + Z_2)(\sigma_T - f_T)] dt. \quad (20)$$

Let the Fourier transforms of σ_T, ψ_T be, respectively, Σ_T, Ψ_T . By virtue of the truncation and the assumption on G , Parseval's theorem is applicable to (20), and on application gives

$$\rho_1(T) = -\frac{1}{2\pi} \int_{-\infty}^\infty \operatorname{Re} z(j\omega) G(j\omega) |\Psi_T|^2 d\omega. \quad (21)$$

Since $|\Psi_T|^2$ is real, $\rho_1(T) \leq 0$ according to hypothesis a) of Theorem 1, implying thereby that

$$\int_0^T [\alpha\sigma(t) + \beta(d\sigma/dt)] \psi(\sigma, t) dt + \int_0^T \psi(\sigma, t) [(Z_1 + Z_2)\sigma(t)] dt \leq \int_0^T \{\alpha f + \beta(df/dt)\} + [(Z_1 + Z_2)f(t)] dt. \quad (22)$$

Consider the left-hand side of the inequality (22): from Lemma 1, satisfaction of (8) guarantees the positiveness of the first integral. Let

$$\alpha k(t) - \beta(dk/dt)\delta_{\max} = \epsilon_1(t) \geq \epsilon_0 > 0, \quad \text{for all } t.$$

The second integral of (22) is nonnegative (from Corollary of Lemma 3) if (10) and (16) are simultaneously satisfied. As for the right-hand side of (22), Lemma 4 gives its upper bound in the inequality (17).

Consequently, (22) becomes

$$\begin{aligned} & \int_0^T \epsilon_1(t) \phi(\sigma) \sigma dt + \beta k(T) \Phi(\sigma(T)) \\ & + \int_0^T \psi(\sigma, t) [(Z_1 + Z_2) \sigma(t)] dt \\ & \leq \beta k(0) \Phi(\sigma(0)) + R_1 \sup_{0 \leq t \leq T} |\sigma(t)|. \end{aligned} \quad (23)$$

By assumption, there exists a positive constant h_1 such that $\phi(\sigma)\sigma \geq h_1\sigma^2$, from which $\Phi(\sigma) \geq h_1\sigma^2/2$. Let $\beta_1 = \beta h_1 k(T)/2$; $R_0 = \beta k(0) \Phi(\sigma(0))$. Then the crucial inequality (23) takes the form

$$\begin{aligned} & \int_0^T \epsilon_1(t) \phi(\sigma) \sigma dt + \beta_1 \sigma^2(T) + \int_0^T \psi(\sigma, t) [(Z_1 + Z_2) \sigma(t)] dt \\ & \leq R_0 + R_1 \sup_{0 \leq t \leq T} |\sigma(t)| \end{aligned} \quad (24)$$

where constants R_0, R_1 are independent of T . Boundedness of $\sigma(t)$ and asymptotic stability follow from (24) as in Popov's proof [1]. Q.E.D.

Corollary 1: $\phi(\sigma) \subset P$, $z_1(s)$ and $z_2(s)$ are inadmissible. Hypothesis b) reads

$$(dk/dt) \delta_{\max} \leq (\alpha/\beta) k(t) - \epsilon_0.$$

Note that, for a general $\phi(\sigma) \subset P$, $0 < \delta_{\max} \leq \infty$.

Corollary 2: $\phi(\sigma) \subset P_{M, z_2(s)}$ is inadmissible. Hypothesis b) accordingly reads

$$(dk/dt) \delta_{\max} < \min_i (\alpha/\beta, \nu_i \mu_i) k(t) - \epsilon_0.$$

Note that, for $\phi(\sigma) \subset P_M$, $0 < \delta_{\max} \leq 1$.

D. Power Law Nonlinearity (Class $P_P^{m_0}$)

Preliminaries: In view of the inequality (2) characterizing $\phi(\sigma) \subset P_P^{m_0}$, Lemma 2 holds here, but Lemma 3 needs modification, as it is based on (3) instead of (4). The proof is similar to that of Lemma 3.

Lemma 6

For $\phi(\sigma) \subset P_P^{m_0}$, the integral

$$I_3 = \int_0^T \psi(\sigma, t) [Z_2 \sigma(t)] dt$$

is nonnegative (except for a constant term of the form $d_1(|\sigma(0)|)$ in (7)) if

$$1 < \nu_i' \leq (1 + 1/c^2), \quad \text{for all } i$$

$$(dk/dt) \delta_{\max} \leq \min_i (1 + c^2 - \nu_i' c^2) \mu_i' k(t). \quad (25)$$

Corollary: When $\phi(\sigma) \subset P_P^{m_0}$, the integral

$$I_{13} = \int_0^T \psi(\sigma, t) [(Z_1 + Z_2) \sigma(t)] dt$$

is nonnegative (except for a constant term of the form $d_1(|\sigma(0)|)$ in (7)) if (10) and (25) are simultaneously obeyed.

Based on the preceding results, the following theorem generalizes Theorem 1. Its proof, being analogous to the proof of Theorem 1, is omitted.

Theorem 2

The system governed by (5) is absolutely stable for $\phi(\sigma) \subset P_P^{m_0}$, if there exist constants $\alpha, \beta > 0$; $\gamma_i, \gamma_i', \mu_i, \mu_i' \geq 0$; $0 \leq \nu_i < 1$, $1 < \nu_i' \leq (1 + 1/c^2)$, for all i , and a multiplier $z(s) = \alpha + \beta s + z_1(s) + z_2(s)$ such that

- $\operatorname{Re} z(j\omega) G(j\omega) \geq 0$, for all real ω , and, for some $\epsilon_0 > 0$, as small as desired,
- $(dk/dt) \delta_{\max} \leq \min_i (\alpha/\beta, \nu_i \mu_i, (1 + c^2 - \nu_i' c^2) \mu_i')$

$$k(t) - \epsilon_0.$$

Corollary 1: If $\phi(\sigma) \subset P_{Mo}$, replace c by 1 to get Theorem 1.

Corollary 2: If $\phi(\sigma) = \sigma$, $c = 0$, and $\delta_{\max} = 1/2$, Theorem 2 then gives conditions for the asymptotic stability of the linear system. In this case, $\nu_i' > 1$, for all i , and hypothesis b) reads

$$dk/dt \leq 2 \min_i (\alpha/\beta, \nu_i \mu_i, \mu_i') k(t) - \epsilon_0.$$

Remark: It is to be observed that the multiplier of Corollary 2 is not a general positive real function (which is in fact expected in view of Gruber and Willems' result [8]). The asymptotic stability conditions for the linear system obtained by a limit process are consequently not the best possible.

III. SOLUTION OF THE MAIN PROBLEM—CAUSAL AND NONCAUSAL MULTIPLIERS

A. Preliminaries

- Consider the impulse response function

$$\begin{aligned} & \sum_{i=1}^{m_3} -\xi_i \exp(\zeta_i t), \quad t \leq 0, \quad \xi_i, \zeta_i > 0, \quad \text{for all } i \\ & 0, \quad t > 0 \end{aligned}$$

which describes a noncausal system. The Fourier transform exists and is equal to $\sum_i -\xi_i/(\zeta_i - j\omega)$.

- Let $\eta_i \geq 0$, for all i ,

$$\begin{aligned} & \lambda_i = \zeta_i - \xi_i, \quad \theta_i = \lambda_i/\zeta_i \\ & z_1^a(s) = \sum_i \eta_i (\zeta_i \theta_i - s) / (\zeta_i - s) \\ & = \sum_i \eta_i [1 - \xi_i/(\zeta_i - s)]. \end{aligned} \quad (26)$$

Define

$$\begin{aligned} I_1^a &= \int_0^T k(t) \phi(\sigma) [Z_1^a \sigma(t)] dt \\ &= \int_0^T k(t) \phi(\sigma) \left(\int_{-\infty}^0 z_1^a(\tau) \sigma(t - \tau) d\tau \right) dt \\ &= \int_0^T k(t) \phi(\sigma) \left(\int_t^\infty z_1^a(t - \tau) \sigma(\tau) d\tau \right) dt. \end{aligned} \quad (27)$$

Let

$$I_{1i}^a = \int_0^T k(t) \phi(\sigma) \left(\sigma(t) - \xi_i \int_t^\infty \exp[\zeta_i(t-\tau)] \sigma(\tau) d\tau \right) dt \quad (28)$$

so that $I_1^a = \sum_i \eta_i I_{1i}^a$.

When $\phi(\sigma) \subset P_M$, (2) is satisfied. In (28), let $\sigma_1 = \sigma(t)$, and

$$\sigma_1 - \sigma_2 = \sigma(t) - \xi_i \int_t^\infty \exp[\zeta_i(t-\tau)] \sigma(\tau) d\tau$$

from which

$$\sigma_2 = \xi_i \int_t^\infty \exp[\zeta_i(t-\tau)] \sigma(\tau) d\tau \quad (29)$$

and

$$d\sigma_2/dt = \zeta_i \sigma_2 - \xi_i \sigma(t)$$

or

$$\sigma(t) = \sigma_1 = (\zeta_i \sigma_2 - d\sigma_2/dt)/\xi_i. \quad (30)$$

Add to and subtract from (28) the expression $\phi(\sigma_2)(\sigma_1 - \sigma_2)$ to get

$$I_{1i}^a = \int_0^T k(t) [\phi(\sigma_1) - \phi(\sigma_2)] (\sigma_1 - \sigma_2) dt + \int_0^T k(t) \phi(\sigma_2) (\sigma_1 - \sigma_2) dt. \quad (31)$$

If $\phi(\sigma) \subset P_M$, the first integral of (31) is nonnegative in virtue of (2). The second integral of (31), on using (30) and integrating by parts, gives

$$\begin{aligned} & \int_0^T k(t) \phi(\sigma_2) (\sigma_1 - \sigma_2) dt \\ &= -[k(T) \Phi(\sigma_2(T))/\xi_i] + k(0) \Phi(\sigma_2(0))/\xi_i \\ &+ \frac{1}{\xi_i} \int_0^T [(dk/dt) \delta_{\sigma_2} + \zeta_i \theta_i k(t)] \phi(\sigma_2) \sigma_2 dt. \end{aligned} \quad (32)$$

Since $0 < \delta_{\max} \leq 1$ for $\phi(\sigma) \subset P_M$ and $k(t) \subset (0, \infty)$, the right-hand side integral of (32) is nonnegative if

$$(dk/dt) \delta_{\max} \geq -\zeta_i \theta_i k(t), \quad \text{for all } t. \quad (33)$$

Consequently, if $\phi(\sigma) \subset P_M$ and (33) is satisfied, I_{1i}^a is nonnegative except for the term $-k(T) \Phi(\sigma_2(T))/\xi_i$ in (32). This in turn implies that the inequality

$$(dk/dt) \delta_{\max} \geq -\min_i \zeta_i \theta_i k(t), \quad \text{for all } t \quad (34)$$

guarantees the nonnegativity of I_1^a given in (27), save for the term

$$\sum_i -\eta_i k(T) \Phi(\sigma_2(T))/\xi_i. \quad (35)$$

3) It is now intended to make I_1^a nonnegative by adding a positive term to (35). To this end, add $\alpha + \beta s$ (with $\alpha, \beta > 0$) to $z_1^a(s)$. Let

$$z_{01}^a(s) = \alpha + \beta s + z_1^a(s)$$

and define

$$\begin{aligned} I_{01}^a &= \int_0^T k(t) \phi(\sigma) [Z_{01}^a \sigma(t)] dt \\ &= \int_0^T [\alpha k(t) - \beta (dk/dt) \delta_\sigma] \phi(\sigma) \sigma dt \\ &+ \beta k(T) \Phi(\sigma(T)) - \beta k(0) \Phi(\sigma(0)) + I_1^a. \end{aligned} \quad (36)$$

Let it be assumed that (for some $\epsilon_0 > 0$)

$$-\min_i \zeta_i \theta_i k(t) \leq (dk/dt) \delta_{\max} \leq (\alpha/\beta) k(t) - \epsilon_0. \quad (37)$$

Then, except for a constant term of the form $d_1(|\sigma(0)|)$ in (7), (36) is positive if

$$\beta k(T) \Phi(\sigma(T)) - \sum_i \eta_i k(T) \Phi(\sigma_2(T))/\xi_i > 0$$

or, adding and subtracting $\sum_i \eta_i k(T) \Phi(\sigma(T))/\xi_i$,

$$\begin{aligned} & [\beta - \sum_i (\eta_i/\xi_i)] k(T) \Phi(\sigma(T)) \\ &+ \sum_i \eta_i \frac{k(T)}{\xi_i} \int_{\sigma_2(T)}^{\sigma(T)} \phi(\sigma) d\sigma > 0. \end{aligned} \quad (38)$$

The first term of (38) is positive if $\beta > \sum_i \eta_i/\xi_i$. As regards the second term of (38), let T be so chosen that $\sigma(t)$ attains its extremum value σ_{ext} at $t = T$ (i.e., either its positive maximum $\sigma_{\max}^+$ or its negative maximum $\sigma_{\max}^-$). Recalling that

$$\sigma_2(T) = \xi_i \int_T^\infty \exp[\zeta_i(T-\tau)] \sigma(\tau) d\tau$$

one has

$$\begin{aligned} \sigma_2(T) &\geq \xi_i (\sigma_{\max}^-) \int_i^\infty \exp[\zeta_i(T-\tau)] d\tau \\ &= \xi_i \sigma_{\max}^- / \zeta_i \end{aligned} \quad (39)$$

$$\begin{aligned} |\sigma_2(T)| &\leq \xi_i (\sigma_{\max}^+) \int_i^\infty \exp[\zeta_i(T-\tau)] d\tau \\ &= \xi_i \sigma_{\max}^+ / \zeta_i. \end{aligned} \quad (40)$$

It is to be noted that $\sigma_2(T) \geq \eta \sigma_{\max}^-$ implies $\sigma_2(T) > \sigma_{\max}^-$ if $\eta < 1$, and $|\sigma_2(T)| \leq \eta \sigma_{\max}^+$ implies $|\sigma_2(T)| < \sigma_{\max}^+$ if $\eta < 1$. Then, for $0 < \theta_i < 1$ or $\zeta_i > \xi_i$, one concludes from (39) and (40) that $\sigma_2(T) > \sigma_{\max}^-$ when $\sigma_2(T)$ is negative, and $\sigma_2(T) < \sigma_{\max}^+$ when $\sigma_2(T)$ is positive. Consequently, the integral

$$\int_{\sigma_2(T)}^{\sigma_{\text{ext}}} \phi(\sigma) d\sigma$$

is nonnegative by virtue of the fact that $\phi(\sigma) \subset P$. The following is a summary of the preceding findings.

Result 1

The integral (36) (with $\alpha, \beta > 0$, and T so chosen that σ attains its extremum value at $t = T$) is positive (except for a constant term of the form $d_1(|\sigma(0)|)$ in (7)) if

(for some $\epsilon_0 > 0$)

$$-\min_i \zeta_i \theta_i k(t) \leq (dk/dt) \delta_{\max} \leq (\alpha/\beta) k(t) - \epsilon_0$$

$$\beta > \sum_i \eta_i / \zeta_i (1 - \theta_i), \quad 0 < \theta_i < 1, \quad \text{for all } i.$$

4) Consider the noncausal impulse response function

$$\sum_{i=1}^{m_4} \xi_i' \exp(\zeta_i' t), \quad t \leq 0, \quad \xi_i', \zeta_i' > 0, \quad \text{for all } i$$

$$0, \quad t > 0.$$

The Fourier transform exists and is given by

$$\sum_i \xi_i' / (\zeta_i' - j\omega).$$

Let $\eta_i' \geq 0$, for all i ; $\lambda_i' = \zeta_i' + \xi_i'$, $\theta_i' = \lambda_i' / \zeta_i'$; and

$$z_2^a(s) = \sum_i \eta_i' (\theta_i' \zeta_i' - s) / (\zeta_i' - s)$$

$$= \sum_i \eta_i' [1 + \xi_i' / (\zeta_i' - s)]. \quad (41)$$

The proof of the following result is similar to that of Result 1 and is hence omitted.

Result 2

When $\phi(\sigma) \subset P_{MO}$, the integral (with $\alpha, \beta > 0$, and T so chosen that σ attains its maximum value $\sigma_{\max}$ at $t = T$)

$$I_{02}^a = \int_0^T k(t) \phi(\sigma) [\alpha \sigma + \beta (d\sigma/dt) + Z_2^a \sigma(t)] dt$$

is positive (except for a constant term of the form $d_1(|\sigma(0)|)$ in (7)) if (for some $\epsilon_0 > 0$)

$$-\min_i (2 - \theta_i') \zeta_i' k(t) \leq (dk/dt) \delta_{\max} \leq \alpha k(t) / \beta - \epsilon_0$$

$$\beta > \sum_i \eta_i' / \zeta_i' (\theta_i' - 1), \quad 1 < \theta_i' < 2, \quad \text{for all } i.$$

A combination of Lemmas 1–3 and Results 1 and 2 yields the following.

Result 3

When $\phi(\sigma) \subset P_{MO}$, the integral (with $\alpha, \beta > 0$, and T so chosen that $\sigma(t)$ attains its maximum value at $t = T$)

$$I^a = \int_0^T k(t) \phi(\sigma)$$

$$\cdot [\alpha \sigma + \beta (d\sigma/dt) + (Z_1 + Z_2 + Z_1^a + Z_2^a) \sigma(t)] dt$$

is positive (except for a constant term of the form $d_1(|\sigma(0)|)$ in (7)) if (for some $\epsilon_0 > 0$)

$$-\min_i (\zeta_i \theta_i (2 - \theta_i') \zeta_i') k(t)$$

$$\leq (dk/dt) \delta_{\max}$$

$$\leq \min_i (\alpha / \beta, \nu_i \mu_i, (2 - \nu_i') \mu_i') k(t) - \epsilon_0 \quad (42)$$

$$\beta > \sum_i [\eta_i / (1 - \theta_i) \zeta_i] + \sum_i [\eta_i' / (\theta_i' - 1) \zeta_i'],$$

$$0 \leq \nu_i < 1, \quad 1 < \nu_i' \leq 2,$$

$$0 < \theta_i < 1, \quad 1 < \theta_i' < 2, \quad \text{for all } i.$$

Lemma 7

If $|f(t)| \leq r_2 \exp(-r_0 t)$ with $r_2, r_0 > 0$, then there exists a constant R_1' independent of T such that

$$\left| \int_0^T k(t) \phi(\sigma) [(Z_1^a + Z_2^a) f(t)] dt \right| \leq R_1' \sup_{0 \leq t \leq T} |\sigma(t)|. \quad (43)$$

Proof: The proof is similar to that found in Popov [1].

B. Main Results

Having settled the preliminaries, a major result (Theorem 3) of this section can be stated. This theorem, believed to be new, shows that the price to be paid for the introduction of a noncausal function into the stability multiplier is a lower bound on dk/dt . Theorem 3 includes Theorem 1 as a special case. When $\phi(\sigma) \subset P_M$, the theorem still holds after casting out inadmissible terms from the multiplier.

Theorem 3

The system (5) is absolutely stable for $\phi(\sigma) \subset P_{MO}$ if there exist constant $\alpha, \beta > 0$; $\gamma_i, \gamma_i', \eta_i, \eta_i', \mu_i, \mu_i' \geq 0$; $\zeta_i, \zeta_i' > 0$; $0 \leq \nu_i < 1$, $1 < \nu_i' \leq 2$, $0 < \theta_i < 1$, $1 < \theta_i' < 2$, for all i ,

$$\beta > \sum_i [\eta_i / (1 - \theta_i) \zeta_i] + \sum_i [\eta_i' / (\theta_i' - 1) \zeta_i']$$

and a multiplier

$$z(s) = \alpha + \beta s + z_1(s) + z_2(s) + z_1^a(s) + z_2^a(s)$$

such that

$$\text{a) } \operatorname{Re} z(j\omega) G(j\omega) \geq 0, \quad \text{for all real } \omega, \quad \text{and for some } \epsilon_0 > 0, \text{ as small as desired,}$$

$$\text{b) } \min_i (\zeta_i \theta_i (2 - \theta_i') \zeta_i') k(t)$$

$$\leq (dk/dt) \delta_{\max}$$

$$\leq \min_i (\alpha / \beta, \nu_i \mu_i, (2 - \nu_i') \mu_i') k(t) - \epsilon_0.$$

Proof: It is similar to the proof of Theorem 1. The points of departure are 1) the negative terms due to Z_1^a and Z_2^a are to be dominated by the positive contribution of βs (see preliminary results); 2) Lemma 7 is to be taken into account.

As in the proof of Theorem 1, hypothesis a) leads to the inequality

$$\int_0^T k(t) \phi(\sigma) [\alpha \sigma + \beta (d\sigma/dt) + (Z_1 + Z_2 + Z_1^a + Z_2^a) \sigma(t)] dt$$

$$\leq \int_0^T k(t) \phi(\sigma) [\alpha f + \beta (df/dt)$$

$$+ (Z_1 + Z_2 + Z_1^a + Z_2^a) f(t)] dt$$

$$\leq R_1 \sup_{0 \leq t \leq T} |\sigma(t)| \quad (44)$$

the constant R_1 being independent of T .

Concerning Result 2, let

$$\sigma_3(T) = \xi_i' \int_T^\infty \exp[\xi_i'(T - \tau)] \sigma(\tau) d\tau.$$

For the definition of $\sigma_2(T)$, see (29). Hypothesis b) on application to the left-hand side of (44) gives, as in the proof of Theorem 1, for an $\epsilon_2(t) \geq \epsilon_0 > 0$, for all t , and a constant R_0 independent of T ,

$$\begin{aligned} & \int_0^T \epsilon_2(t) \phi(\sigma) \sigma dt + k(T) \{ \beta \Phi(\sigma(T)) - \sum_i \eta_i \Phi(\sigma_2(T)) / \xi_i \\ & - \sum_i \eta_i' \Phi(\sigma_3(T)) / \xi_i' \} \\ & + (\text{nonnegative terms due to } Z_1, Z_2, Z_1^a, Z_2^a) \\ & \leq R_0 + R_1 \sup_{0 \leq t \leq T} |\sigma(t)|. \end{aligned} \quad (45)$$

Since the inequality (45) has been stated for every positive T , it holds in particular for T so chosen as to permit $\sigma(t)$ to attain its maximum value at $t = T$. Therefore, $\sigma(t)$ in $\Phi(\sigma(T))$ of the left-hand side of (45) can be replaced by $\sigma_{\max}$ without affecting the inequality. This enables one to use Result 3, which guarantees that the expression inside the braces of (45) is positive and greater than or equal to $\beta_0 k(T) \Phi(\sigma(T))$ for some $\beta_0 > 0$. Consequently,

$$\begin{aligned} & \int_0^T \epsilon_2(t) \phi(\sigma) \sigma dt + \beta_0 k(T) \Phi(\sigma(T)) + (\text{nonnegative terms}) \\ & \leq R_0 + R_1 \sup_{0 \leq t \leq T} |\sigma(t)| \end{aligned} \quad (46)$$

from which stability and asymptotic stability of $\sigma(t)$ ensue as in Popov's proof [1].

Corollary: If $\phi(\sigma) \subset P_M$, the terms $z_2(s)$ and $z_2^a(s)$ are inadmissible in the multiplier. Hypothesis b) accordingly reads

$$- \min_i \xi_i \theta_i k(t) \leq (dk/dt) \delta_{\max} \leq \min_i (\alpha/\beta, \nu_i \mu_i) k(t) - \epsilon_0.$$

C. Power Law Nonlinearity

The present aim is to include $z_1^a(s)$ and $z_2^a(s)$ in the multiplier of Theorem 2. By doing so, Theorem 3 is generalized.

Because the property (2) of $\phi(\sigma)$ is common for both $\phi(\sigma) \subset P_{P^{m_0}}$ and $\phi(\sigma) \subset P_M$, Result 1 concerning $z_1^a(s)$ holds for $\phi(\sigma) \subset P_{P^{m_0}}$. As regards $z_2^a(s)$, an analysis similar to the one leading to Result 2 gives the following.

Result 4

When $\phi(\sigma) \subset P_{P^{m_0}}$ and $c \neq 0$, the integral (with $\alpha, \beta > 0$ and T chosen so as to make $\sigma(t)$ attain its maximum at $t = T$)

$$I_{0p}^a = \int_0^T k(t) \phi(\sigma) [\alpha \sigma + \beta (d\sigma/dt) + (Z_2^a \sigma(t))] dt$$

is positive (except for a constant term of the form $d_1(|\sigma(0)|)$ in (7)) if (for some $\epsilon_0 > 0$)

$$- \min_i (1 + c^2 - c^2 \theta_i') \xi_i' k(t) \leq (dk/dt) \delta_{\max}$$

$$\leq (\alpha/\beta) k(t) - \epsilon_0$$

$$\beta > \sum_i \eta_i' / c^2 (\theta_i' - 1) \xi_i', \quad 1 < \theta_i' < 1 + 1/c^2, \quad \text{for all } i.$$

A combination of the Corollary to Lemma 6, Result 1, and Result 4 gives the following.

Result 5

When $\phi(\sigma) \subset P_{P^{m_0}}$ and $c \neq 0$, the integral (with $\alpha, \beta > 0$ and T chosen so as to allow $\sigma(t)$ to reach its maximum value at $t = T$)

$$\begin{aligned} I_p^a &= \int_0^T k(t) \phi(\sigma) \\ & \cdot [\alpha \sigma + \beta (d\sigma/dt) + (Z_1 + Z_2 + Z_1^a + Z_2^a) \sigma(t)] dt \end{aligned}$$

is positive (except for a constant term of the form $d_1(|\sigma(0)|)$ in (7)) if (for some $\epsilon_0 > 0$)

$$\begin{aligned} & - \min_i (\theta_i \xi_i (1 + c^2 - c^2 \theta_i') \xi_i') k(t) \\ & \leq (dk/dt) \delta_{\max} \\ & \leq \min_i (\alpha/\beta, \nu_i \mu_i (1 + c^2 - c^2 \nu_i') \mu_i') k(t) - \epsilon_0 \end{aligned}$$

$$\beta > \sum_i \eta_i / (1 - \theta_i) \xi_i + \sum_i \eta_i' / c^2 (\theta_i' - 1) \xi_i',$$

$$0 \leq \nu_i < 1, \quad 1 < \nu_i' \leq 1 + 1/c^2,$$

$$0 < \theta_i < 1, \quad 1 < \theta_i' < 1 + 1/c^2, \quad \text{for all } i.$$

A limiting case of Result 5 is the following.

Result 6

When $\phi(\sigma) = \sigma$, the integral (with $\alpha, \beta > 0$ and T chosen so as to allow $\sigma(t)$ attain its maximum value at $t = T$)

$$\begin{aligned} I_L &= \int_0^T k(t) \phi(\sigma) \\ & \cdot [\alpha \sigma + \beta (d\sigma/dt) + (Z_1 + Z_2 + Z_1^a + Z_2^a) \sigma(t)] dt \end{aligned}$$

is positive (except for a constant term of the form $d_1(|\sigma(0)|)$ in (7)) if (for some $\epsilon_0 > 0$)

$$\begin{aligned} & -2 \min_i (\theta_i \xi_i \xi_i') k(t) \leq dk/dt \\ & \leq 2 \min_i (\alpha/\beta, \nu_i \mu_i \mu_i') k(t) - \epsilon_0 \end{aligned}$$

$$\beta > \sum_i \eta_i / (1 - \theta_i) \xi_i + \sum_i \eta_i' (\theta_i' - 1) / \xi_i',$$

$$0 \leq \nu_i < 1, \quad \nu_i' > 1, \quad 0 < \theta_i < 1, \quad \theta_i' > 1, \quad \text{for all } i.$$

Based on the preceding results, the following theorem generalizes Theorem 3. Its proof resembles the proof of Theorem 3 and is hence not given.

Theorem 4

The system governed by (5) is absolutely stable for $\phi(\sigma) \subset P_{P^m0}$, $c \neq 0$, if there exist constants $\alpha, \beta > 0$; $\eta_i, \eta'_i, \gamma_i, \gamma'_i, \mu_i, \mu'_i \geq 0$; $0 \leq \nu_i < 1$, $1 < \nu'_i \leq (1 + 1/c^2)$, $0 < \theta_i < 1$, $1 < \theta'_i < (1 + 1/c^2)$, for all i ,

$$\beta > \sum_i \eta_i / (1 - \theta_i) \zeta_i + \sum_i [\eta'_i / c^2 (\theta'_i - 1) \zeta'_i]$$

and a multiplier

$$z(s) = \alpha + \beta s + z_1(s) + z_2(s) + z_1^a(s) + z_2^a(s)$$

such that

- a) $\operatorname{Re} z(j\omega)G(j\omega) \geq 0$, for all real ω , and, for some $\epsilon_0 > 0$, as small as desired,
- b) $-\min_i (\theta_i \zeta_i (1 + c^2 - c^2 \theta'_i) \zeta'_i) k(t)$
 $\leq (dk/dt) \delta_{\max}$
 $\leq \min_i (\alpha / \beta, \nu_i \mu_i (1 + c^2 - c^2 \nu'_i) \mu'_i) k(t) - \epsilon_0$.

Corollary 1: When $c = 1$ (or $\phi(\sigma) \subset P_{MO}$), Theorem 3 is obtained.

Corollary 2: (Based on Result 6) When $\phi(\sigma) = \sigma$, the linear system governed by (5) is asymptotically stable if there exist constants $\alpha, \beta > 0$; $\eta_i, \eta'_i, \gamma_i, \gamma'_i, \mu_i, \mu'_i \geq 0$; $0 \leq \nu_i < 1$, $\nu'_i > 1$, $0 < \theta_i < 1$, $\theta'_i > 1$, for all i ,

$$\beta > \sum_i [\eta_i / (1 - \theta_i) \zeta_i] + \sum_i [\eta'_i (\theta'_i - 1) / \zeta'_i]$$

and a multiplier

$$z(s) = \alpha + \beta s + z_1(s) + z_2(s) + z_1^a(s) + z_2^a(s)$$

such that

- a) $\operatorname{Re} z(j\omega)G(j\omega) \geq 0$ for all real ω , and, for some $\epsilon_0 > 0$, as small as desired,
- b) $-2 \min_i (\theta_i \zeta_i \zeta'_i) k(t) \leq dk/dt$
 $\leq 2 \min_i (\alpha / \beta, \nu_i \mu_i \mu'_i) k(t) - \epsilon_0$.

D. Remarks

1) The present method of introducing a noncausal function into the stability multiplier is different from O'Shea's [7] correlation method developed explicitly for a time-invariant system. (See Zames and Falb [9] for a rigorous treatment of the correlation method).

2) The multiplier introduced by O'Shea [7] is very general, but the accompanying time domain restriction on the multiplier is not always easy to verify. The stability multiplier of the paper contains an RC-RL reflected impedance, and there is no explicit time domain restriction on it. It is possible to extend the class of multipliers (for $\phi(\sigma) \subset P_{MO}, P_{P^m0}$) by considering biquadratic functions.

3) It has been found that the correlation technique, on extension to the time-varying feedback problem, leads to

an inconsequential bound on dk/dt ; namely, $k(t)$ is either a constant or a nonincreasing function of time. However, if the time-varying gain $k(t)$ is periodic, one can arrive at useful conditions for absolute stability of the system in terms of a special multiplier; a detailed correlation-bound analysis is not essential. This contribution will appear elsewhere.

E. Examples

1) Let

$$G(s) = s/(s+1)^3, \quad \phi(\sigma) \subset P_M.$$

With constant linear feedback, the system is asymptotically stable for all gains $\subset [0, \infty)$. Choose

$$z(s) = 2 + s + [1 - 2/(6-s)] \\ = 2 + s + (4-s)/(6-s).$$

$\xi = 2$, $\zeta = 6$, $\alpha = 2$, $\beta = 1$. Note that $\zeta > \xi$, $\beta > 1/\xi$. Also $0 < \delta_{\max} \leq 1$. Corollary 1 of Theorem 3 is applicable. It can be verified that $\operatorname{Re} z(j\omega)G(j\omega) \geq 0$, for all real ω . The system is, therefore, absolutely stable for $\phi(\sigma) \subset P_M$ if, for some $\epsilon_0 > 0$, as small as desired

$$-4k(t) \leq (dk/dt) \delta_{\max} \leq 2k(t) - \epsilon_0.$$

When the system is linear, $\delta_{\max} = 1/2$ and the preceding bound on dk/dt becomes

$$-8k(t) \leq dk/dt \leq 4k(t) - \epsilon_0.$$

Observe that for the linear system, using positive real multipliers alone, it is unlikely that a result better than $dk/dt \leq 2k(t) - \epsilon_0$ (for asymptotic stability) can be established.

2) Let

$$G(s) = s^3/(s^5 + 5s^4 + 4s^3 + 3s^2 + 2s + 5), \quad \phi(\sigma) \subset P_{MO}.$$

With linear constant feedback, the system is asymptotically stable for all gains $\subset [0, \infty)$. Choose

$$z(s) = 2s + 4 + [1 + 1/(6-s)].$$

Here $\xi' = 1$, $\zeta' = 6$, $\alpha = 4$, $\beta = 2$, $\zeta' - \xi' > 0$, $\beta > 1/\xi'$, $0 < \delta_{\max} \leq 1$. Theorem 3 is applicable. It is easy to verify that $\operatorname{Re} z(j\omega)G(j\omega) \geq 0$, for all real ω .

Therefore, the nonlinear time-varying feedback system is absolutely stable for $\phi(\sigma) \subset P_{MO}$ if, for some $\epsilon_0 > 0$, as small as desired,

$$-5k(t) \leq (dk/dt) \delta_{\max} \leq 2k(t) - \epsilon_0.$$

When the system is linear, $\delta_{\max} = 1/2$. The preceding constraint on dk/dt becomes

$$-10k(t) \leq dk/dt \leq 4k(t) - \epsilon_0.$$

Observe that in the linear time-varying case, because of the third-order zero in $G(s)$, it is not possible to find a positive real $z(s)$ such that $z(s)G(s)$ is strictly positive real. Consequently Gruber and Willems' [8] criterion for the linear time-varying system is not applicable here.

IV. CONCLUSIONS

New absolute stability criteria are derived for a nonlinear time-varying feedback system illustrated in Fig. 1. The classes of the nonlinearities considered are P_M , P_{MO} , and $P_{P^{m_0}}$ (with $m_0 \geq 1$). The criteria are expressed in terms of a multiplier containing causal and noncausal functions. A significant outcome of the presence of noncausal functions in the multiplier is that absolute stability can be established in cases where a purely causal multiplier is ineffective because the phase angle of $G(s)$ is outside the $\pm 90^\circ$ band in many intervals along the $j\omega$ axis; in return, dk/dt is to be bounded from below also. Such a lower bound on dk/dt appears to be the first of its kind. As in earlier investigations, the causal part of the multiplier gives rise to an upper bound on dk/dt . These bounds on dk/dt are dependent on the form of the nonlinearity and the multiplier chosen.

The search for a $z(s)$ to satisfy hypothesis a) of the theorems is rather cumbersome. It would be very useful in practice to have a direct method by which a candidate for $z(s)$ is obtained directly from the phase angle characteristic of $G(s)$. This is an area deserving study.

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