equation shows the system to be stable for all positive values of τ, K, and the feedback parameters.

This generalized error coefficient approach can be used to analyze the effects of any type of nonunity feedback, whether inherent or deliberate, upon the performance of a linear system. The simple examples of this paper indicate the need for caution in using nonunity feedback since the effects can be significant and are not always apparent. The examples further illustrate why the outer (position) loop of so many control systems is indeed unity and suggest the possible use of the generalized error coefficients in the synthesis of feedback transfer functions that will improve the accuracy as well as the transient response of a system. Finally, state variable feedback can be handled by the reduction of the block diagram or signal flow graph to obtain the H_{eq} of Melsa and Schultz [5], which can then be substituted for H into the expressions for the generalized error coefficients.

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On the Positivity of Certain Nonlinear Time-Varying Operators

Y. V. VENKATESH

Abstract—**Sufficient conditions are given for the positivity of a composition of two positive operators, one of which is nonlinear and time varying.**

We consider the following question, which is of some interest in stability. Given two positive operators, one of which is nonlinear and time varying, under what conditions is their composition positive?

This correspondence gives the main lemma (Lemma 2), which forms the pivotal result in the author's report [1] establishing sufficient conditions (more general than those of [2]) for the stability of nonlinear time-varying feedback systems.

Let $N(\cdot)$ be a real-valued function on $(-\infty, \infty)$ with the following properties: 1) $N(0) = 0$; 2) N is monotone nondecreasing, i.e., $(r-s)(N(r)-N(s)) \geqslant 0$; 3) there is a constant $C > 0$ such that $|N(r)| \leq C|r|$ for all r. N will be called odd if $N(-v) = -N(v)$ for all real v. Let $L_2(-\infty, \infty)$ be the space of real-valued square integrable functions on $(-\infty, \infty)$.

The derivation of conditions for the composition of operators to be positive is based on the following area inequality (1).

Lemma 1: If $N(\cdot)$ is monotone nondecreasing, then

$$xN(x) - yN(x) \geqslant P(x) - P(y) \tag{1}$$

for all x and y, where $P(x) = \int_0^x N(s)\,ds$.

Proof: See the proof of Lemma 7 in Zames and Falb [3].

Define

$$\delta_s = \sup_x \,[P(x)/N(x)x]$$

$$\delta_i = \inf_x \,(P(x)/N(x)x].$$

Lemma 2: Let μ be a constant equal to $\xi > 0$ for $\tau \geqslant 0$, and to $\zeta < 0$ for $\tau < 0$. If $N(\cdot)$ is monotone nondecreasing, then

$$\int_{-\infty}^{\infty} e^{\mu(t-\tau)}x(t-\tau)N(x(t))\,dt \leq (1+\delta_s-\delta_i)\int_{-\infty}^{\infty} e^{\mu t}x(t)N(x(t))\,dt \tag{2}$$

for all τ and any $x(\cdot)$ in $L_2(-\infty, \infty)$. If, in addition, $N(\cdot)$ is odd, then

$$\left|\int_{-\infty}^{\infty} e^{\mu(t-\tau)}x(t-\tau)N(x(t))\,dt\right| \leq (1+\delta_s-\delta_i)\int_{-\infty}^{\infty} e^{\mu t}x(t)N(x(t))\,dt \tag{3}$$

for all τ and any $x(\cdot)$ in $L_2(-\infty, \infty)$.

Proof: Since $|N(s)| \leqslant C|s|$ and $x(\cdot)$ is in $L_2(-\infty, \infty)$, then $N(x(\cdot))$ is in $L_2(-\infty, \infty)$ and $P(x(\cdot))$ is in $L_1(-\infty, \infty)$. Thus,

$$\int_{-\infty}^{\infty} e^{\mu t}x(t)N(x(t))\,dt - \int_{-\infty}^{\infty} e^{\mu(t-\tau)}x(t-\tau)N(x(t))\,dt \geqslant \int_{-\infty}^{\infty} e^{\mu t}P(x(t))\,dt - \int_{-\infty}^{\infty} e^{\mu t}P(x(t-\tau)e^{-\mu\tau})\,dt. \tag{4}$$

Consider the last integral of inequality (4). By a change of the variable of integration to $t_1 = t - \tau$, we get

$$\int_{-\infty}^{\infty} e^{\mu t}P(x(t-\tau)e^{-\mu\tau})\,dt = \int_{-\infty}^{\infty} e^{\mu(t_1+\tau)}P(x(t_1)e^{-\mu\tau})\,dt_1.$$

Now,

$$P(x(t)) \geqslant \delta_i N(x(t))x(t)$$

and

$$P(x(t_1)e^{-\mu\tau}) \leqslant \delta_s N(x(t_1)e^{-\mu\tau})x(t_1)e^{-\mu\tau}.$$

Hence, from (4),

$$\int_{-\infty}^{\infty} e^{\mu t}x(t)N(x(t))\,dt - \int_{-\infty}^{\infty} e^{\mu(t-\tau)}x(t-\tau)N(x(t))\,dt \geqslant \delta_i\int_{-\infty}^{\infty} e^{\mu t}N(x(t))x(t)\,dt - \delta_s\int_{-\infty}^{\infty} e^{\mu t_1}N(x(t_1)e^{-\mu\tau})x(t_1)\,dt_1. \tag{5}$$

$N(\cdot)$ being monotonic, we have

$$N(x(t_1)e^{-\mu\tau})x(t_1) \leqslant N(x(t_1))x(t_1), \qquad \forall\,\tau$$

because $\mu = \xi > 0$ for $\tau \geqslant 0$ and $\mu = \zeta < 0$ for $\tau < 0$. Use this in (5) to get (2) by a mere transposition.

If $N(\cdot)$ is odd, then $P(\cdot)$ is even, and so

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{\mu t}x(t)N(x(t))\,dt + \int_{-\infty}^{\infty} e^{\mu(t-\tau)}x(t-\tau)N(x(t))\,dt \\
&= \int_{-\infty}^{\infty} e^{\mu t}x(t)N(x(t))\,dt - \int_{-\infty}^{\infty} e^{\mu(t-\tau)}(-x(t-\tau))N(x(t))\,dt \\
&\geqslant \int_{-\infty}^{\infty} e^{\mu t}P(x(t))\,dt - \int_{-\infty}^{\infty} e^{\mu t}P(-x(t-\tau)e^{-\mu\tau})\,dt \\
&\geqslant \delta_i\int_{-\infty}^{\infty} e^{\mu t}x(t)N(x(t))\,dt - \delta_s\int_{-\infty}^{\infty} e^{\mu t_1}x(t_1)N(x(t_1))\,dt_1.
\end{aligned}$$

Thus, (3) holds.

Inequalities (2) and (3) can be used to derive positivity conditions for certain nonlinear time-varying operators.

Let $\mathfrak{N}$, $\mathcal{K}$ be the mapping of $L_2(-\infty, \infty)$ into $L_2(-\infty, \infty)$ given by

$$\mathfrak{N}x(t) = \mathfrak{N}(x(t)) \qquad \mathcal{K}x(t) = k(t)x(t)$$

where $\mathfrak{N}(\cdot)$ is as defined above; and $k(\cdot)$ is a real-valued function, absolutely continuous on $[0, \infty)$.

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Let L_{2e} be the space of real-valued functions $x(\cdot)$ on the interval $(-\infty, \infty)$, for which for every finite $t \geqslant 0$, $\int_0^t x^2(\tau)\, d\tau < \infty$.

Let $\mathcal{L}$ denote the class of operators $\mathcal{Z}: L_{2e} \to L_{2e}$ satisfying an equation of the type

$$\mathcal{Z}x(t) = \sum_{i=1}^{\infty} z_i x(t - \tau_i) + \sum_{j=1}^{\infty} z_j' x(t + \tau_j') + \int_{-\infty}^{\infty} z(\tau) x(t - \tau)\, d\tau + x(t) \quad (6)$$

where z_i, z_j' (for all i, j) are real constants; $\sum_{i=1}^{\infty}(|z_i| + |z_i'|) < \infty$; $z(\cdot)$ is a real-valued function on $(-\infty, \infty)$, and $\int_{-\infty}^{\infty} |z(\tau)|\, d\tau < \infty$; $0 < \tau_1, \tau_1' < \tau_2, \tau_2' < \cdots$. It is shown in [3] that any convolution of form (6) is a bounded linear transformation of $L_2(-\infty, \infty)$ into itself and that the set of all such convolutions can be viewed as a commutative Banach algebra with an identity.

Finally, let $\mathcal{K}$ be a given operator in $\mathcal{L}$. $\mathcal{K}$ is called positive if $\int_0^T x(t) \mathcal{K} x(t)\, dt \geqslant 0$ for all x in the domain of $\mathcal{K}$ and all $T \geqslant 0$.

The lemma given below is proved (under somewhat less general conditions) in [1]; its proof is based on inequalities (2) and (3).

Lemma 3: The inequality

$$\int_0^T \mathcal{Z}x(t) \mathcal{K} \mathcal{N} x(t)\, dt \geqslant 0, \qquad \forall x \in \text{do}\mathcal{Z}; \quad \forall T \geqslant 0 \qquad (7)$$

where do$\mathcal{Z}$ denotes the domain of $\mathcal{Z}$, holds for all monotonely nondecreasing $\mathcal{N}$ if the following conditions are satisfied:
1) $z_i, z_j' \leqslant 0$ for $i,j = 0,1,2,\cdots$; 2) $z(\tau) \leqslant 0$ almost everywhere;

$$3) \sum_{i=1}^{\infty} |z_i| e^{\gamma \tau_i} + \sum_{j=1}^{\infty} |z_j'| e^{\nu \tau_j'} + \int_0^{\infty} |z(\tau)| e^{\gamma \tau}\, d\tau + \int_{-\infty}^{0} |z(\tau)| e^{-\nu \tau}\, d\tau \leqslant \frac{1}{(1 + \delta_s - \delta_i)}$$

where γ, ν are nonnegative constants; 4) with $\xi = \max \gamma$ and $\zeta = \max \nu$ for which 3) is verified, $k(t)e^{-\xi t}$ is nonincreasing, and $k(t)e^{\zeta t}$ is nondecreasing.

The inequality (7) holds for all monotonically nondecreasing *odd* $\mathcal{N}$ if conditions 3) and 4) are satisfied.

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A Stackelberg Solution for Games with Many Players

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***Abstract*—The concept of Stackelberg solution is widened to include games with many leaders and many followers. Necessary conditions for the existence of an open-loop Stackelberg solution in differential games where each player is using a Nash strategy within his group are also derived.**

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I. Introduction

The Stackelberg solution [1] for a two-player nonzero-sum differential game where, because of a bias in the prior information sets, one player assumes the role of a leader and the other player assumes the role of a follower has been investigated in [2] and [3]. In this note the Stackelberg concept is widened to include games where the number of players are more than two. The players are assumed to be divided into two groups, a group of leaders and a group of followers. The group of leaders announces its strategies before the group of followers, and every player may or may not be cooperating within his own group. Naturally, in order to be able to define a Stackelberg strategy for this game, the leaders as a group are assumed to know the rationale according to which the followers are playing, and the followers are assumed to be rational in the sense that they will play according to this rationale. Several interesting games can be formulated using the above model such as, for example, oligopoly markets where there are several large firms as price leaders and several other smaller firms as price followers.

In this note, preliminary results as a basis for a more general theory of a two-group Stackelberg game are obtained. Because of its analytical simplicity, only the case where Nash strategies are the rationales used within each group is treated. Other cases where minimax, Pareto, and Stackelberg strategies are used within each group can be similarly studied; however, this will not be done in this note.

II. Definition and Properties

Let N and M denote the number of players in groups 1 and 2, respectively, and let U_i, $i = 1,\cdots,N$, and V_i, $i = 1,\cdots,M$, be their sets of admissible controls. Let $U = \pi_{i=1}^{N} U_i$ and $V = \pi_{i=1}^{M} V_i$ be the sets of admissible controls for groups 1 and 2, respectively. Let $J_{ji}(u_1,\cdots,u_N; v_1,\cdots,v_M) \triangleq J_{ji}(u;v)$ (where $u = (u_1,\cdots,u_N) \in U$ and $v = (v_1,\cdots,v_M) \in V$) for $j = 1$; $i = 1,\cdots,N$ and $j = 2$; $i = 1,\cdots,M$, be the cost functions for the $N + M$ players, i.e., $J_{ji}: U \times V \to R$ for all j and i where j refers to the group, and i refers to the player in this group.

Definition 1: If there exist mappings $T_i: V \to U_i$, $i = 1,\cdots,N$, such that for any $v \in V$

$$J_{1i}(Tv;v) \leq J_{1i}(T_1 v,\cdots,u_i,\cdots,T_N v;v); \qquad \forall u_i \in U_i, \quad i = 1,\cdots,N$$

where $Tv = (T_1 v,\cdots,T_N v)$ and if there exists a $v_{s2} \in V$ such that

$$J_{2i}(Tv_{s2};v_{s2}) \leq J_{2i}(Tv_{s2}^{(i)};v_{s2}^{(i)}); \qquad i = 1,\cdots,M$$

where $v_{s2}^{(i)} = (v_{1s2},\cdots,v_{(i-1)s2},v_i,v_{(i+1)s2},\cdots,v_{Ms2})$, then the strategies $(u_{s2},v_{s2}) \in U \times V$ where $u_{1s2} = Tv_{2s2}$ are called *Stackelberg strategies with group 2 as Nash leaders and group 1 as Nash followers.*

In other words, this Stackelberg strategy is the optimal (in the sense of Nash) strategy for the leaders when the followers react by playing according to a Nash optimal solution. It protects every player in the leader group from attempts by any other one player in the leader group to deviate from his Stackelberg strategy, causing the followers to deviate also, in order to further reduce their costs. In this sense this Stackelberg strategy is safer for the leaders than their corresponding Nash strategies, which in effect safeguards every player from attempts by any one other player only to further reduce his cost.

Following the same terminology as in [3], the Nash rational reaction set of group 1 when group 2 is the leader group is defined by

$$D_1 = \{(u,v) \in U \times V: u = Tv, \qquad \forall v \in V\}.$$

A similar definition can be made for the Nash rational reaction set D_2 of group 2 when group 1 is the leader group. It follows from the definitions of D_1 and D_2 that the Nash solution for the given game with $N + M$ players is obtained by taking the intersection of D_1 and D_2.

When $N = 1$ the game reduces to a many-leader one-follower