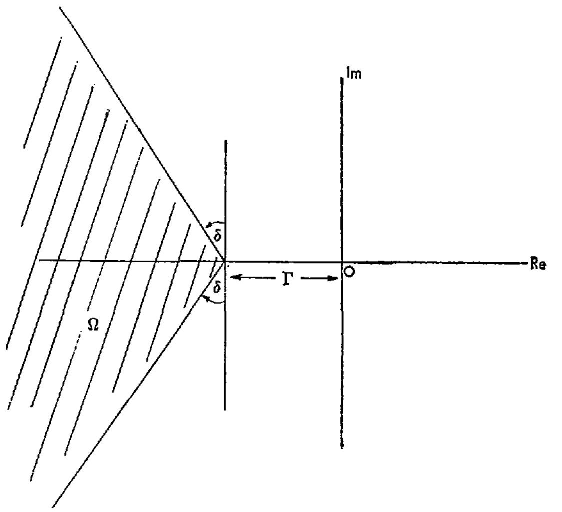


Fig. 1. Region for determining if the $\{\lambda_i\}$ are contained.

*Proof:* Let the eigenvalues of A be denoted by $\lambda_i$, $i = 1,2,\cdots,n$, and the eigenvalues of $A^*$ by $\lambda_i^*$, $i = 1,2,\cdots,2n$. Then

$$\{\lambda_i^*\} = \{\hat{\lambda}_i, \tilde{\hat{\lambda}}_i\} \tag{1}$$

where the tilde denotes complex conjugate and

$$\hat{\lambda}_i = (\lambda_i + \Gamma)(\cos\delta + j\sin\delta), \quad i = 1,2,\cdots,n. \tag{2}$$

Now Re $(\lambda_i^*) \leq 0$, $i = 1,2,\cdots,2n$, iff Re $(\hat{\lambda}_i) \leq 0$, Re $(\tilde{\hat{\lambda}}_i) \leq 0$, $i = 1,2,\cdots,n$, or iff

$$\text{Re}\,[(\lambda_i + \Gamma)(\cos\delta \pm j\sin\delta)] \leq 0, \quad i = 1,2,\cdots,n \tag{3}$$

or iff

$$\{\lambda_i\} \in \Omega. \tag{4}$$

Q.E.D.

This theorem gives a simple condition for determining if all the eigenvalues of a real matrix A are real by letting $\delta \to \pi/2$ and letting $\Gamma \to -\infty$ (see the following corollary).

*Corollary*

The real matrix A has all real eigenvalues occurring in it if and only if Re $[\lambda_i(A^*)] \leq 0$, $i = 1,2,\cdots,2n$, where $\delta \to \pi/2$ and $\Gamma \to -\infty$.

Edward J. Davison
N. Ramesh
Dept. of Elec. Engrg.
University of Toronto
Toronto, Ont., Canada

## Comparison of Open- and Closed-Loop Sensitivities for Systems with Stochastic Inputs

*Abstract*—**For a stable linear system with a random input, expressions are derived for open- and closed-loop trajectory sensitivity. A condition is derived under which a closed-loop system is superior to the open-loop system.**

Manuscript received September 8, 1969.

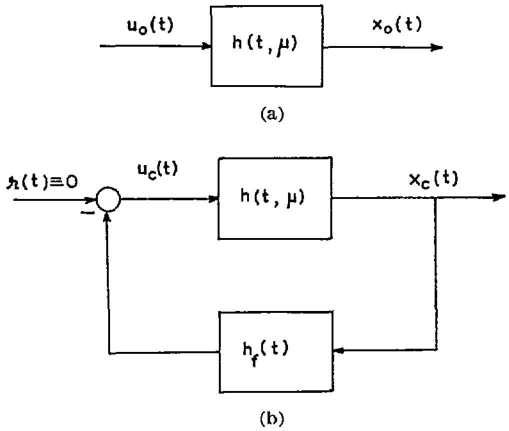


Fig. 1. (a) Open-loop system. (b) Nominally equivalent closed-loop system.

### I. Introduction

It is well known that feedback can provide a reduction of sensitivity to deviations of plant parameters from their nominal values. Kreindler [1] has demonstrated the closed-loop sensitivity reduction of linear optimal control systems. This applies for the deterministic case. The purpose of this correspondence is to present a parallel development for linear systems with stochastic inputs.

### II. Problem Development

Fig. 1 shows an open-loop system and a nominally equivalent closed-loop system. The system is linear and stable. The input to the system is a random process $u(t)$ and the corresponding output is the random process $x(t)$; open- and closed-loop are indicated by the subscripts o and c, respectively. Then

$$x_o(t) = \int_{-\infty}^{\infty} h(\tau,\mu) u_o(t-\tau)\, d\tau \tag{1}$$

where $\mu$ is an uncertain parameter of the system. If the open-loop sensitivity $\partial x_o(t)/\partial\mu$ is denoted by $\sigma_o(t)$, the expression for $\sigma_o(t)$ can be obtained by partial differentiation of (1) with respect to $\mu$ and is given by

$$\sigma_o(t) = \int_{-\infty}^{\infty} h_\mu(\tau,\mu) u_o(t-\tau)\, d\tau \tag{2}$$

where

$$h_\mu(\tau,\mu) = \partial h(\tau,\mu)/\partial\mu\,|_{\mu=\mu_o}$$

where $\mu_o$ is the nominal value of parameter $\mu$.

For the closed-loop system,

$$u_c(t) = -\int_{-\infty}^{\infty} h_f(\tau) x_c(t-\tau)\, d\tau \tag{3}$$

and

$$x_c(t) = \int_{-\infty}^{\infty} h(\alpha,\mu) u_c(t-\alpha)\, d\alpha. \tag{4}$$

Substituting the value of $u_c(t)$ from (3) in (4), we obtain

$$x_c(t) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha,\mu) h_f(\beta) x_c(t-\alpha-\beta)\, d\alpha\, d\beta. \tag{5}$$

The expression for closed-loop trajectory sensitivity $\partial x_c(t)/\partial\mu$ denoted by $\sigma_c(t)$ may be written as

$$\sigma_c(t) = \int_{-\infty}^{\infty} h_\mu(\alpha,\mu) u_c(t-\alpha)\, d\alpha - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha,\mu) h_f(\beta) \sigma_c(t-\alpha-\beta)\, d\alpha\, d\beta. \tag{6}$$

Since $u_c = u_o$, (6) gives, in view of (2),

$$\sigma_c(t) = \sigma_o(t) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha,\mu)h_f(\beta)\sigma_c(t-\alpha-\beta)\,d\alpha\,d\beta. \quad (7)$$

Equation (7) is rewritten as

$$\sigma_o(t) = \sigma_c(t) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha,\mu)h_f(\beta)\sigma_c(t-\alpha-\beta)\,d\alpha\,d\beta. \quad (8)$$

The autocorrelation function of the open-loop sensitivity is defined by $R_o(t_1,t_2)$ and is given by [2]

$$\begin{aligned} R_o(t_1,t_2) &= E(\sigma_o(t_1)\sigma_o(t_2)) \\ &= E\left[\left\{\sigma_c(t_1) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha,\mu)h_f(\beta)\sigma_c(t_1-\alpha-\beta)\,d\alpha\,d\beta\right\}\right. \\ &\quad \left.\cdot\left\{\sigma_c(t_2) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\gamma,\mu)h_f(\delta)\sigma_c(t_2-\gamma-\delta)\,d\gamma\,d\delta\right\}\right]. \end{aligned} \quad (9)$$

If the input random process is stationary in the wide sense, we have

$$R_o(t_1,t_2) = R_o(\tau) \quad (10)$$

where $\tau = t_2 - t_1$.

From (9) and (10) we get

$$\begin{aligned} R_o(\tau) &= R_c(\tau) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha,\mu)h_f(\beta)R_c(\tau+\alpha+\beta)\,d\alpha\,d\beta \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\gamma,\mu)h_f(\delta)R_c(\tau-\gamma-\delta)\,d\gamma\,d\delta \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha,\mu)h(\gamma,\mu)h_f(\beta)h_f(\delta) \\ &\quad \cdot R_c(\tau+\alpha+\beta-\gamma-\delta)\,d\alpha\,d\beta\,d\gamma\,d\delta. \end{aligned} \quad (11)$$

The spectral density $S_o(f)$ is given by

$$S_o(f) = \int_{-\infty}^{\infty} R_o(\lambda)\exp(-j\omega\lambda)\,d\lambda \quad (12)$$

From (11) we get

$$\begin{aligned} S_o(f) &= S_c(f) + H^*H_f^*S_c(f) + HH_fS_c(f) \\ &\quad + H^*HH_f^*H_fS_c(f) \\ &= |1+HH_f|^2S_c(f) \end{aligned} \quad (13)$$

where $H$ and $H_f$ stand for the system function and the asterisk denotes complex conjugate.

## III. Criterion of Comparison

We shall choose a positive real function given by

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sigma^2(t)\,dt$$

as the criterion of comparison for open- and closed-loop sensitivities. For closed-loop sensitivity reduction the inequality to be satisfied is

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sigma_c^2(t)\,dt \le \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\sigma_o^2(t)\,dt. \quad (14)$$

Equation (14) is equivalent in frequency domain to

$$\int_{-\infty}^{\infty} S_c(\omega)\,d\omega \le \int_{-\infty}^{\infty} S_o(\omega)\,d\omega. \quad (15)$$

Substituting for $S_o$ from (13) in (15), we get

$$|1+HH_f|^2 \ge 1. \quad (16)$$

Equation (16) is analogous to Kalman's [3] equation for deterministic systems.

## IV. Conclusion

A condition for closed-loop sensitivity reduction of linear systems with random inputs is derived. It is analogous to Kalman's result for deterministic systems.

V. V. S. Sarma
Sahjendra N. Singh[1]
Dept. of Elec. Engrg.
Indian Institute of Science
Bangalore 12, India

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[1] Now with the Dept. of Elec. Engrg., The Johns Hopkins University, Baltimore, Md. 21218.

# Controllability versus Sensitivity in Linear Discrete Systems

*Abstract*—This correspondence deals with the controllability of the sensitivity system associated to a given linear discrete system. It is proved that, apart from exceptional cases, the sensitivity system is always uncontrollable provided that the number of the parameters is sufficiently large. The results presented here have a structural nature since they involve only the dimensions of the state, control, and parameter vectors.

## I. Introduction

The aim of this correspondence is to extend to the discrete-time case the results given in [1], [2] on the structural uncontrollability of the sensitivity system. Consider the time-invariant linear discrete system

$$x(k+1) = F(p)x(k) + G(p)u(k) \quad (1)$$

where $u$, $x$, and $p$ are the control, state, and parameter vectors of dimension $m$, $n$, and $q$, respectively, and $F(\cdot)$ and $G(\cdot)$ are differentiable at the point $\bar{p}$ (nominal value of the parameter). The sensitivity system is defined as follows:

$$y(k+1) = A(\bar{p})y(k) + B(\bar{p})u(k) \quad (2)$$

where

$$y(k) = \begin{bmatrix} x(k) \\ x^1(k) \\ \vdots \\ x^q(k) \end{bmatrix}, \quad x^i(k) = [\partial x(k)/\partial p_i]|_{p=\bar{p}}, \quad i = 1,\cdots,q$$

$$A(\bar{p}) = \begin{bmatrix} F(\bar{p}) & 0 & \cdots & 0 \\ F^1(\bar{p}) & F(\bar{p}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ F^q(\bar{p}) & 0 & \cdots & F(\bar{p}) \end{bmatrix}, \quad F^i(\bar{p}) = [\partial F(p)/\partial p_i]|_{p=\bar{p}}, \quad i = 1,\cdots,q$$

$$B(\bar{p}) = \begin{bmatrix} G(\bar{p}) \\ G^1(\bar{p}) \\ \vdots \\ G^q(\bar{p}) \end{bmatrix}, \quad G^i(\bar{p}) = [\partial G(p)/\partial p_i]|_{p=\bar{p}}, \quad i = 1,\cdots,q.$$

Manuscript received May 12, 1969; revised September 9, 1969.