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Hydrodynamics of R-charged D1-branes

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Abstract: We study the hydrodynamic properties of strongly coupled $SU(N)$ Yang-Mills theory of the D1-brane at finite temperature and at a non-zero density of R-charge in the framework of gauge/gravity duality. The gravity dual description involves a charged black hole solution of an Einstein-Maxwell-dilaton system in 3 dimensions which is obtained by a consistent truncation of the spinning D1-brane in 10 dimensions. We evaluate thermal and electrical conductivity as well as the bulk viscosity as a function of the chemical potential conjugate to the R-charges of the D1-brane. We show that the ratio of bulk viscosity to entropy density is independent of the chemical potential and is equal to $1/4\pi$. The thermal conductivity and bulk viscosity obey a relationship similar to the Wiedemann-Franz law. We show that at the boundary of thermodynamic stability, the charge diffusion mode becomes unstable and the transport coefficients exhibit critical behaviour. Our method for evaluating the transport coefficients relies on expressing the second order differential equations in terms of a first order equation which dictates the radial evolution of the transport coefficient. The radial evolution equations can be solved exactly for the transport coefficients of our interest. We observe that transport coefficients of the D1-brane theory are related to that of the M2-brane by an overall proportionality constant which sets the dimensions.

Contents

1. Introduction

There has been recent interest in constructing holographic duals which model phenomena and properties observed in macroscopic low energy physics. Such holographic duals may provide new insights because the properties and phenomena of interest usually lie in a regime which is strongly coupled in the field theory description but semi-classical from the gravity point of view. Transport properties of various systems which admit holographic duals have been evaluated from the gravity description. A universal result which has emerged out of these investigations is that the ratio of shear viscosity η to the entropy density s for field theories which admit gravity duals in the two derivative approximation is given by [1, 2] [1]

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}, \tag{1.1}$$

where $\hbar$ is the Planck's constant and k_B is the Boltzmann's constant. This ratio has been evaluated for well known AdS/CFT pairs like $\mathcal{N} = 4$ super Yang-Mills as well as simple phenomenological gravity models. Other gauge/gravity duals which involve near horizon geometries which are not asymptotically anti-de Sitter backgrounds like that of Dp-branes, $p \geq 2$ [4, 5] have also been studied. This ratio for these backgrounds has also been shown to be $\hbar/4\pi k_B$ [6].

In [7], we began an investigation of macroscopic properties of the $1+1$ dimensional field theory of the D1-branes. In $1+1$ dimensions, there is no shear, therefore it is necessary to study non-conformal field theories to obtain non-trivial hydrodynamic coefficients. D1-branes are interesting as they provide the simplest and the most symmetric non-conformal $1+1$ dimensional field theory which admits a gravity dual. The theory is the maximally supersymmetric Yang-Mills with $SU(N)$ gauge group. It can be obtained as a dimensional reduction of $\mathcal{N} = 4$ SYM from $3+1$ dimensions. In [7], we isolated the sound mode in gravity and evaluated the speed of sound v_s and the bulk viscosity ζ in the following regimes

$$\begin{aligned} &(i) \qquad \sqrt{\lambda}N^{-2/3} \ll T \ll \sqrt{\lambda}, \\ &(ii) \qquad \sqrt{\lambda}N^{-1} \ll T \ll \sqrt{\lambda}N^{-2/3}. \end{aligned} \tag{1.2}$$

Here, $\lambda = g_{\rm YM}^2 N$ is the t' Hooft coupling and T is the temperature. In the above regimes, the field theory of the D1-branes admits a gravity dual [5] which for the purposes of evaluating transport coefficients reduces to an Einstein-dilaton theory in 3 dimensions. In [7], it was shown that

$$v_s = \frac{c}{\sqrt{2}}, \qquad \frac{\zeta}{s} = \frac{\hbar}{4\pi k_B}, \tag{1.3}$$

[1] See [3] for a recent review and list of references on related topics.

for hydrodynamics of the D1-brane theory, here c is the velocity of light. It was also seen that theories arising form D1-branes at cones over Sasaki-Einstein 7-manifolds give rise to the values in (1.3). It was suggested that there might be a class of non-conformal field theories which admit 3d gravitational backgrounds for which $\zeta/s = \hbar/4\pi k_B$. For the rest of the paper, we will work with units in which $\hbar = k_B = c = 1$.

In [8], it was shown that the supergravity fluctuations which determine the hydrodynamic coefficients of the uncharged D1-branes were related by dimensional reduction to that of the M2-branes. The dimensional reduction related the shear viscosity of the conformal hydrodynamics of the M2-brane to that of the bulk viscosity of D1-branes. This explained why $\zeta/s = 1/4\pi$, it can essentially be traced to the relation (1.1) for the M2-branes. It also explained why the value of the speed of sound of the D1-brane theory behaves as though it is a conformal theory in $2+1$ dimensions. One expects a similar connection for the transport coefficients between the D1-brane theory with finite charge density and the corresponding M2-brane theory. This would imply that the ratio ζ/s will be independent of the chemical potential and continues to be $1/4\pi$ since it is related to the ratio η/s of the M2-brane theory. There should also be similar relationships between other transport coefficients like conductivity. This is one of our motivations to explore the hydrodynamics of charged D1-branes. There is a need to develop novel theories for 1+1 dimensional condensed matter systems as many higher dimensional models can't be applied here and there is a profusion of knowledge through experiments about new such systems and their properties [9]. So another reason is to study the macroscopic properties of strongly coupled $1+1$ dimensional field theories which admit gravitational duals. Gravity duals of $1+1$ dimensional systems with a well defined field theory have not been extensively studied [2]. These systems play an important role in many quantum phenomena and it is worthwhile to see what insights the gauge/gravity correspondence gives in this context with a well defined field theory in mind.

In this paper, we study the hydrodynamics of D1-branes at finite charge density in a regime which admits a gravity description. The gravity dual description involves a charged black hole in an Einstein-dilaton-Maxwell scalar system in 3 dimensions which is obtained by a consistent truncation of spinning D1-branes in 10 dimensions. We study two situations:

1. The case in which the charge density corresponding to a single $U(1)$ of the $SO(8)$ R-symmetry of the D1-brane theory is turned on, we call this the single charged D1-brane.

2. The situation in which equal charge densities along the 4 Cartan's are turned on, we call this the equal charged D1-brane.

[2] Holographic duals of 1+1 dimensional systems from a bottom up approach without a known boundary field theory were studied in [10, 11].

In both these cases, we see that both the speed of sound and the ratio of bulk viscosity to entropy density is given by (1.3). The values of these quantities are independent of the chemical potential. We also evaluate the charge conductivity, the charge diffusion constant, the sound diffusion constant and the thermal conductivity for both the situations and compare the results for which the corresponding M2-brane calculation has been done. We see that apart from an overall proportionality constant which sets the dimensionality of the transport coefficients in these theories, the transport coefficients are identical in the two theories. The results are summarized in the following table.

Transport Coefficients	Single-charged D1 brane	Equal-charged D1 brane	Equal-charged M2 brane
σ_{DC}	$\frac{1}{16\pi G_3}\frac{(2k+3)^2}{9\sqrt{1+k}}$	$\frac{1}{16\pi G_3}\frac{(3-k)^2}{9(1+k)^2}$	$\frac{1}{16\pi G_4}\frac{(3-k)^2}{9(1+k)^2}$
ζ	$\frac{r_H^4}{16\pi G_3 L^4}\sqrt{1+k}$	$\frac{r_H^4}{16\pi G_3 L^4}(1+k)^2$	$--$
η	$--$	$--$	$\frac{r_H^4(1+k)^2}{16\pi G_4 L'^4}$
D_c	$\frac{L^3(3-2k)}{6r_H^2\sqrt{1+k}}$	$\frac{L^3(k+3)}{6r_H^2(1+k)^2}$	
D_s	$\frac{L^3}{12r_H^2\sqrt{1+k}}$	$\frac{L^3}{12r_H^2(1+k)^2}$	$\frac{L'^3}{12r_H^2(1+k)^2}$
κ_T	$\frac{r_H^2}{8LG_3}\frac{(2k+3)(1+k)}{k}$	$\frac{r_H^2}{8LG_3}\frac{(3-k)(1+k)}{k}$	$\frac{r_H^2}{8L'G_4}\frac{(3-k)(1+k)}{k}$

Table 1. Transport coefficients of D1-branes and M2-branes.

r_H : radius of the horizon k: (R-charge)^2 in units of r_H.
G_3, G_4: Newton's constant in 3 and 4 dimensions.
L, L': radius of the orthogonal S^7 for D1, M2-branes.
$\sigma_{\rm DC}$: electrical conductivity, ζ: bulk viscosity, η: shear viscosity.
D_c, D_s: charge diffusivity, sound diffusivity, κ_T: thermal conductivity.

Hydrodynamics of uncharged M2-branes were first studied in [12, 13]. We obtained the shear viscosity of the charged M2-branes from the fact that $\eta/s = 1/4\pi$ [14]. The conductivity of charged M2-branes was obtained from [15]. The charge diffusion constant for the M2-branes at non-zero chemical potential has not yet been evaluated in the literature as far as we are aware. However for M2-branes at zero chemical potential, the charge diffusion constant has been evaluated in [16] [3] and it agrees with the $k = 0$ limit of the D1-brane theory answer. The sound diffusion constant for the charged M2-branes has been calculated by using $D_s = \eta/2(\epsilon + p)$ where ϵ, p are the energy density and the pressure of the M2-branes. Notice that the bulk viscosity of the D1-brane theory is proportional to the shear viscosity of the M2-brane theory. Another observation of our study of the transport coefficients of the charged D1-brane is the following relationship between the bulk viscosity and the thermal conductivity

$$\frac{\kappa_T \hat{\mu}^2}{\zeta T} = 4\pi^2 \tag{1.4}$$

where κ_T is the thermal conductivity, T the temperature and $\hat{\mu}$ the chemical potential. This relationship is analogous to the Wiedemann-Franz law and a similar relationship between shear viscosity and the thermal conductivity has been observed in the case of single charged D3-branes [17]. Since the charged D1-brane theory is obtained as a consistent truncation of spinning D1-branes, there is a maximum allowed spin or charge beyond which the solution is thermodynamically unstable [18]. We show that the transport coefficients exhibit critical behaviour at the boundary of the thermodynamical instability. For the single charged case, we observe that the charge diffusion mode becomes unstable at the boundary of instability. This suggests that for this case, the thermodynamical instability can be better understood by studying the charged diffusion mode in more detail.

This paper is organized as follows: In the next section, we introduce the single charged D1-brane background and obtain the consistent truncation of the solution to 3 dimensions. We also review the thermodynamics of this solution and obtain the boundary of thermodynamic instability. In section 3, we study hydrodynamics of a charged fluid in $1 + 1$ dimensions and obtain the dispersion relations of the two hydrodynamic modes, the charge diffusion mode and the sound mode in terms of thermodynamic variables. We then use the thermodynamics of the D1-brane solution to explicitly evaluate the dispersion relations. We also determine the form of the retarded correlation functions of the stress tensor and the charge current using conservation laws. In section 4, we study the supergravity fluctuations of the single charged D1-brane solution and isolate the gauge invariant fluctuations which correspond to the two hydrodynamical modes in the field theory. In section 5, we determine the various transport coefficients from gravity using the relevant Kubo's formula. To do this, we reduce the problem to solving a set of coupled first

[3] See below equation (3.32) in [16].

order non-linear differential equations which are exactly solvable in limit required by the Kubo's formulae. These first order equations dictate the radial evolution of the transport coefficient. In section 6, we discuss the properties of the transport coefficients, their behaviour at the boundary of thermodynamic instability. We then discuss the connection of the D1-brane theory to that of M2-branes. It will be interesting to compare our results with what is known for these systems. Appendix A contains the details of the consistent truncation which is required to obtain the charged D1-brane solution in 3 dimensions. Appendix B contains the details of the evaluation of the transport coefficients for the equal charged D1-brane.

2. The R-charged D1-brane

In this section, we introduce the gravity dual of $SU(N)$ Yang-Mills with 16 supercharges in 1+1 dimensions at finite R-charge density and state its domain of validity. We then discuss its thermodynamic properties. This section will also serve to set up notations and conventions.

In [5], it was argued that $SU(N)$ Yang-Mills with 16 supercharges in $1+1$ dimensions at large N is dual to the near horizon geometry of N D1-branes. Heating up this theory to a finite temperature T, the gravity dual is given in terms of the near horizon geometry of non-extremal D1-brane solution. The gravity dual can be trusted in the domain

$$\sqrt{\lambda}N^{-\frac{2}{3}} << T << \sqrt{\lambda}, \tag{2.1}$$

where $\lambda = \sqrt{g_{\rm YM}^2 N}$ is the t'Hooft coupling of the theory. The only non-trivial viscous transport coefficient of this system was evaluated using this gravitational dual in [7]. We now wish to turn on finite R-charge density in the field theory. By the usual gauge/gravity correspondence, the $SO(8)$ isometry of the S^7 present in the near horizon geometry of the D1-branes corresponds to the $SO(8)$ R-symmetry of the Yang-Mills. Therefore to turn on R-charge density, it is necessary to consider D1-branes with angular momentum. The near horizon supergravity solution of non-extremal D1-branes spinning along one of the Cartan directions of $SO(8)$ is given by [18].

$$\begin{aligned}
ds^2 &= H_1^{-3/4}(-fdt^2 + dz^2) - 2H_1^{-3/4}\frac{L^3 r_0^3}{\Delta r^6} l \sin^2\theta dt d\phi, \\
&\quad + H_1^{1/4}\left(\frac{1}{\tilde{h}}dr^2 + r^2(\Delta d\theta^2 + H\sin^2\theta d\phi^2 + \cos^2\theta d\Omega_5^2)\right), \\
e^\Phi &= H_1^{1/2}, \\
A^{(2)} &= -\left(\frac{dt}{H_1} + \frac{r_0^3}{L^3} l^2 \sin^2\theta d\phi\right)\wedge dz,
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \Delta &= 1 + \frac{l^2 \cos^2\theta}{r^2}, \qquad H = 1 + \frac{l^2}{r^2}, \\ H_1 &= \frac{L^6}{\Delta r^6}, \qquad f = 1 - \frac{r_0^6}{\Delta r^6}, \\ \tilde{h} &= \frac{1}{\Delta}\left(1 + \frac{l^2}{r^2} - \frac{r_0^6}{r^6}\right). \end{aligned} \tag{2.3}$$

The above solution is written in the Einstein frame. $d\Omega_5^2$ is the metric of a unit 5-sphere and

$$L^6 = g_{\rm YM}^2 2^6 \pi^3 N(\alpha')^4, \qquad g_{\rm YM}^2 = \frac{g_s}{2\pi\alpha'} \tag{2.4}$$

with g_s, α' being the string coupling and the string length respectively. $A^{(2)}$ is the gauge potential for the RR 2-form sourced by the D1-branes. Note that the above solution reduces to the non-spinning near horizon solution of the non-extremal D1-brane when one sets the angular velocity $l = 0$. For completeness, we mention that the background in (2.2) is a solution of type IIB supergravity equations of motion in 10 dimensions obtained from the following action

$$S = \frac{1}{16\pi G_{10}} \int d^{10}x\sqrt{g}\left[R - \frac{1}{2}\partial_M\phi\partial^M\phi - \frac{1}{2\cdot 3!}e^{\phi}(F_3)^2\right]. \tag{2.5}$$

To study the hydrodynamics of this solution, one needs to consider perturbations of this solution along the brane directions (t, z) and the radial direction. The fluctuations along the 7-sphere do not play any role. Thus to simplify our analysis, it is convenient to perform a Kaluza-Klein reduction of this solution to 3 dimensions. Using the results of [19], it can be shown that the 10 dimensional solution in (2.2) admits a consistent reduction on the S^7 sphere to the following solution in 3 dimensions

$$\begin{aligned} ds^2 &= \left(-c_T^2 dt^2 + c_X^2 dz^2 + c_R^2 dr^2\right), \\ c_T^2 &= \left(\frac{r}{L}\right)^8 K, \qquad c_X^2 = \left(\frac{r}{L}\right)^8 H, \qquad c_R^2 = \frac{H}{K}\left(\frac{r}{L}\right)^2, \\ A_t &= -\frac{r_0^3 l}{L^2(r^2+l^2)}, \qquad \phi = -3\log\left(\frac{r}{L}\right), \qquad \Psi = 1 + \frac{l^2}{r^2}. \end{aligned} \tag{2.6}$$

Here H and K are defined as

$$H = 1 + \frac{l^2}{r^2}, \qquad K = 1 + \frac{l^2}{r^2} - \frac{r_0^6}{r^6}. \tag{2.7}$$

The details of this Kaluza-Klein reduction are given in Appendix A. The rotation along one of the Cartan directions reduces to the charge denoted by the gauge potential A_t in 3 dimensions. Note that the deformation of round S^7 metric in (2.2) parametrized by Δ results in an additional scalar Ψ in 3 dimensions. It can also be shown using this consistent reduction that the background in (2.6) is a solution of the equations of motion of the following action

$$I = \frac{1}{16\pi G_3}\int d^3x\sqrt{-g}\left(R(g) - \frac{8}{9}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}\Psi^2 e^{-\frac{4}{3}\phi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\Psi^2}\partial_\mu\Psi\partial^\mu\Psi + \frac{2}{3\Psi}\partial_\mu\phi\partial^\mu\Psi + \frac{12}{L^2}e^{\frac{4}{3}\phi}(1+\Psi^{-1})\right), \tag{2.8}$$

where

$$\frac{1}{G_3} = \frac{2\pi^4 L^7}{3!G_{10}}, \qquad G_{10} = 2^3\pi^6 g_s^2(\alpha')^4. \tag{2.9}$$

Thus the 10 dimensional rotating D1-brane solution reduces to a charged black hole of an Einstein-Maxwell-dilaton system along with a scalar. The R-charge is given by the gauge potential A_t corresponds to rotation along the S^7 in 10 dimensions. As a simple consistency check, note that both the action in (2.8) and the solution in (2.6) reduces to the truncation studied in [7] [4] for the uncharged D1-brane. Since the above solution is a consistent truncation to 3 dimensions, any solution to hydrodynamic fluctuations studied in 3 dimensions can by lifted to 10 dimensions. For completeness, we write down the equations of motion of the action given in (2.8).

$$G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{A} + C_{\mu\nu} = 0,$$

$$\mathcal{A} = -\frac{8}{9}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2\Psi^2}\partial_\mu\Psi\partial^\mu\Psi + \frac{2}{3\Psi}\partial_\mu\phi\partial^\mu\Psi - \frac{\Psi^2}{4}e^{-4\phi/3}F_{\mu\nu}F^{\mu\nu} \tag{2.10}$$

$$+\frac{12}{L^2}e^{4\phi/3}(1+\Psi^{-1}), \tag{2.11}$$

$$C_{\mu\nu} = -\frac{8}{9}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2\Psi^2}\partial_\mu\Psi\partial_\nu\Psi + \frac{1}{3\Psi}(\partial_\mu\phi\partial_\nu\Psi + \partial_\nu\phi\partial_\mu\Psi) \tag{2.12}$$

$$-\frac{1}{2}\Psi^2 e^{-4\phi/3}F_{\mu\rho}F_\nu{}^\rho, \tag{2.13}$$

$$\Box\phi + \frac{6}{L^2}e^{4\phi/3}(2+\Psi^{-1}) = 0, \tag{2.14}$$

$$\Box\log\Psi - \frac{\Psi^2}{2}e^{-4\phi/3}F_{\mu\nu}F^{\mu\nu} + \frac{8}{L^2}e^{4\phi/3}(1-\Psi^{-1}) = 0, \tag{2.15}$$

$$\partial_\mu[\sqrt{-g}\Psi^2 e^{-4\phi/3}F^{\mu\nu}] = 0. \tag{2.16}$$

We refer to the solution in (2.6) as the single charged D1-brane. The equal charged D1-brane solution in which equal charge density along all the 4 Cartans of the $SO(8)$ are turned on is given in (A.20) of Appendix A.

2.1 Thermodynamics of the R-charged branes

The thermodynamic properties of spinning D-branes were studied in complete generality in [18] from which we can read out the thermodynamic properties of the

[4] See equations (4.3), (4.5), (4.6), (4.7).

black hole of interest given in (2.6). We now summarize the relevant thermodynamic properties. The Hawking temperature and the entropy density are given by

$$T = \frac{1}{2\pi L^3}\frac{r_H^5}{r_0^3}(3+2k), \qquad s = \frac{1}{4G_3}\frac{r_0^3 r_H}{L^4}, \tag{2.17}$$

where k is given by

$$k = \frac{l^2}{r_H^2}, \tag{2.18}$$

and r_H is the radius of the horizon which is given by the largest root of the equation

$$r_H^6 + r_H^4 l^2 - r_0^6 = 0. \tag{2.19}$$

The energy density and the free energy density is given by

$$\epsilon = \frac{1}{4\pi G_3}\frac{r_0^6}{L^7}, \qquad p = -f = \frac{1}{8\pi G_3}\frac{r_0^6}{L^7} = \frac{\epsilon}{2}. \tag{2.20}$$

Here we have also identified the pressure using its thermodynamic relationship with free energy density. The charge density ρ and its conjugate the chemical potential μ are given by

$$\rho = \frac{1}{8\pi G_3}\frac{r_0^3 l}{L^5}, \qquad \mu = A_t(r)|_{r\to\infty} - A_t(r)|_{r_H} = \frac{l r_H^4}{L^2 r_0^3}. \tag{2.21}$$

Note that we have defined the chemical potential as the voltage difference between the boundary $r \to \infty$ and the horizon. In writing these thermodynamic quantities, we have used the relation (2.19).

For the black hole solution given in (2.6) with very large charge, there exists a thermodynamic instability. This instability is equivalent to the instability occurring in D1-branes which are rotating too fast [18]. Given the energy density of the system, the thermodynamic stability is determined by the condition

$$H_s = \det\begin{pmatrix} \frac{\partial^2\epsilon(s,\rho)}{\partial s^2} & \frac{\partial^2\epsilon(s,\rho_i)}{\partial s\partial\rho} \\ \frac{\partial^2\epsilon(s,\rho_i)}{\partial\rho\partial s} & \frac{\partial^2\epsilon(s,\rho)}{\partial\rho^2} \end{pmatrix} > 0. \tag{2.22}$$

To evaluate it, it is convenient to write the above Hessian as

$$H_s = \left(\frac{\partial T}{\partial r_0}\frac{\partial \mu}{\partial l} - \frac{\partial T}{\partial l}\frac{\partial \mu}{\partial r_0}\right)\left(\frac{\partial s}{\partial r_0}\frac{\partial \rho}{\partial l} - \frac{\partial s}{\partial l}\frac{\partial \rho}{\partial r_0}\right)^{-1}, \tag{2.23}$$

where we have used the chain rule and standard thermodynamic relations. Using the expressions for the thermodynamic variables given in (2.17), (2.20) and (2.21), it can be shown that the Hessian reduces to

$$H_s = 2G_3^2 L^4 \frac{(3-2k)}{r_H^4(1+k)^2}. \tag{2.24}$$

Thus the condition for thermodynamic stability implies the following restriction on the values of the R charge

$$k < \frac{3}{2}. \tag{2.25}$$

Finally, for completeness, we mention that the condition for the validity of the supergravity solution of the non-extremal spinning D1-brane remains the same as that of the non-extremal brane and is given by

$$\sqrt{\lambda}N^{-\frac{2}{3}} << T << \sqrt{\lambda}. \tag{2.26}$$

The bound $k < 3/2$ in terms of field theory chemical potential can be written as

$$\hat{\mu} = \frac{\mu}{L} < \frac{\pi T}{\sqrt{6}}. \tag{2.27}$$

Therefore the transport coefficients evaluated in this paper are valid in the regime given by (2.26) and (2.27) of the field theory.

3. Hydrodynamics of a charged fluid in $1+1$ dimensions

In this section, we show that a charged fluid in 1+1 dimensions has two hydrodynamic modes and derive their dispersion relation. The stress tensor and the charge current of a relativistic fluid in $1+1$ dimensions are given by

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + p)u^\mu u^\nu + P\eta^{\mu\nu} - \zeta(u^\mu u^\nu + \eta^{\mu\nu})\partial_\lambda u^\lambda, \\ j^\mu &= \rho u^\mu - \sigma T(\eta^{\mu\nu} + u^\mu u^\nu)\partial_\nu \left(\frac{\mu}{T}\right), \end{aligned} \tag{3.1}$$

where u^μ is the 2-velocity with $u_\mu u^\mu = -1$ and ζ is the bulk viscosity and σ the conductivity. The remaining variables ϵ, p, ρ, μ refer to the energy density, pressure, charge density and the chemical potential of the system respectively. $\eta^{\mu\nu}$ refers to the flat Minkowski metric with signature $(-1, 1)$. The equations of motion of the fluid are given by the following conservation laws

$$\partial_\mu T^{\mu\nu} = 0, \qquad\qquad \partial_\mu j^\mu = 0. \tag{3.2}$$

We now wish to obtain the linearized hydrodynamics modes, therefore let us consider small fluctuations from the rest frame of the fluid. The 2-velocity is then given by

$$u^0 = 1, \qquad u^z = \delta u^z. \tag{3.3}$$

Note that $u^0 = 1$ up to the linear order due to the constraint $u^\mu u_\mu = -1$. In considering the small fluctuations, one should keep in mind that spatial and temporal variations of the thermodynamic quantities are all of linear order. We can write the stress energy tensor to the linear order as given below

$$T^{00} = \epsilon + \delta T^{00}, \qquad T^{0z} = \delta T^{0z}, \qquad \delta T^{zz} = p - \frac{\zeta}{\epsilon + p}\partial_z \delta T^{0z}. \tag{3.4}$$

In writing the above form of the stress tensor, we have eliminated δu^z using

$$\delta u^z = \frac{\delta T^{0z}}{\epsilon + p}, \qquad \partial_z \delta u^z = \frac{\partial_z \delta T^{0z}}{\epsilon + p}. \tag{3.5}$$

As we are working only to the linear order on taking the spatial derivative of δu_x, the derivative acts only on δT^{0z}. This is because derivatives of thermodynamic quantities are first order and therefore contribute only at second order in the above equation. Similarly the current density can be written as

$$j^0 = \rho + \delta j^0, \qquad j^z = \delta j^z = \rho \frac{\delta T^{0z}}{\epsilon + p} - \sigma T \partial_z \bar{\mu}, \tag{3.6}$$

where $\bar{\mu} = \mu/T$ and we have again used (3.5). It is convenient to work with thermodynamic variables in which the energy density ϵ and the charge density ρ are the independent variables and all other thermodynamic quantities are functions of ϵ and ρ. Then we can write δj^z as

$$\delta j^z = \rho \frac{\delta T^{0z}}{\epsilon + p} - \sigma T \left(\partial_\epsilon \bar{\mu} \partial_z \delta T^{00} + \partial_\rho \bar{\mu} \partial_z \delta j^0 \right). \tag{3.7}$$

Substituting the form of the stress tensor and the current density given in (3.4), (3.6) and (3.7) into the conservation equations (3.2), we obtain

$$\begin{aligned} \partial_0 \delta j^0 + \rho \frac{\partial_z \delta T^{0z}}{\epsilon + p} - \sigma T \left(\partial_\epsilon \bar{\mu} \partial_z^2 \, \delta T^{00} + \partial_\rho \bar{\mu} \partial_z^2 \delta j^0 \right) = 0, \\ \partial_0 \delta T^{00} + \partial_z \delta T^{0z} = 0, \\ \partial_0 \delta T^{0z} + \left(\frac{\partial p}{\partial \epsilon} \partial_z \delta T^{00} + \frac{\partial p}{\partial \rho} \partial_z \delta j^0 \right) - \frac{\zeta}{\epsilon + p} \partial_z^2 \delta T^{0z} = 0. \end{aligned} \tag{3.8}$$

The above three equations determine the linearised hydrodynamic modes. Performing the Fourier transform of the equations given in (3.8) both in position and time, we obtain the following set of algebraic equations

$$\begin{aligned} (-i\omega + \sigma T \partial_\rho \bar{\mu} q^2) \delta j^0 + \frac{i \rho q}{\epsilon + p} \delta T^{0z} + \sigma T \partial_\epsilon \bar{\mu} q^2 \delta T^{00} = 0, \\ -i\omega \delta T^{00} + iq \delta T^{0z} = 0, \\ iq \partial_\epsilon p \delta T^{00} + iq \partial_\rho p \delta j^0 + \left(-i\omega + \frac{\zeta q^2}{\epsilon + p} \right) \delta T^{0z} = 0. \end{aligned} \tag{3.9}$$

The above equations have non-trivial solutions for the fluctuations $\delta j^0, \delta T^{0z}, \delta T^{00}$ only if the following constraint on ω is satisfied.

$$(-i\omega + \sigma T q^2 \partial_\rho \bar{\mu}) \left(\omega^2 - q^2 \partial_\epsilon p + \frac{i \zeta q^2 \omega}{\epsilon + p} \right) + q^2 \partial_\rho p \left(\frac{i \rho \omega}{\epsilon + p} + \sigma T q^2 \partial_\epsilon \bar{\mu} \right) = 0. \tag{3.10}$$

To solve for ω in terms of q, we can assume the following expansions for ω

$$\omega = v_s q - iD_s q^2 + \cdots, \qquad \omega = -iD_c q^2 + \cdots. \tag{3.11}$$

Substituting the first expansion of ω in terms of q given in the above equation in the constraint (3.10) and matching terms of $O(q^3)$ and $O(q^4)$, we obtain the following expressions for the sound speed and its damping coefficient

$$v_s^2 = (\partial_\epsilon p + \frac{\rho}{\epsilon + p}\partial_\rho p), \tag{3.12}$$

$$D_s = \frac{\zeta}{2(\epsilon + p)} + \frac{\sigma T}{2v_s^2}\left(\rho\frac{\partial_\rho \bar{\mu}}{\epsilon + p} + \partial_\epsilon \bar{\mu}\right)\partial_\rho p. \tag{3.13}$$

Similarly substituting the second expansion for ω given in (3.11) in the constraint (3.10) and demanding that the leading coefficient of $O(q^4)$ vanishes, we obtain the following value for the charge diffusion constant D_c

$$D_c = \sigma T \frac{\partial_\epsilon p \partial_\rho \bar{\mu} - \partial_\rho p \partial_\epsilon \bar{\mu}}{\partial_\epsilon p + \rho \frac{\partial_\rho p}{\epsilon + p}}. \tag{3.14}$$

It can be shown that these are the only two modes of the equations of motion of linearized hydrodynamics. To summarize, the two modes are the sound mode and the charge diffusion mode given by the dispersion relations in (3.11).

We can now use the thermodynamic properties of the charged black hole given in (2.17), (2.20) and (2.21) to evaluate the dispersion relations explicitly. From (2.20), note that the pressure just depends on the free energy of the system and is independent of the charge density. Therefore for the R-charged D1-brane, the dispersion relations simplify to

$$\begin{aligned} \omega &= \pm\frac{1}{\sqrt{2}}q - i\frac{\zeta}{2(\epsilon + p)}q^2, \\ \omega &= -i\sigma T \left.\frac{\partial \bar{\mu}}{\partial \rho}\right|_\epsilon q^2. \end{aligned} \tag{3.15}$$

We can further simplify the charge diffusion constant as follows

$$\begin{aligned} D_c &= \sigma\left(\partial_\rho \mu - \frac{\mu}{T}\partial_\rho T\right), \\ &= \sigma\left(\frac{\partial_l \mu}{\partial_l \rho} - \frac{\mu}{T}\frac{\partial_l T}{\partial_l \rho}\right), \\ &= \sigma(16\pi G_3)\frac{3L^3}{2r_H^2}\frac{(3-2k)}{(3+2k)^2}. \end{aligned} \tag{3.16}$$

To obtain the second line, we have used chain rule and also the fact that the energy density ϵ is independent of l. The last line is obtained by evaluating all the

the derivatives of the thermodynamic quantities using (2.17), (2.20) and (2.21). Therefore we see that the charge diffusion mode is given by

$$\omega = -i\sigma(16\pi G_3)\frac{3L^3}{2r_H^2}\frac{(3-2k)}{(3+2k)^2}q^2. \tag{3.17}$$

Note that if the conductivity σ remains finite at the boundary of thermodynamic stability $k = 3/2$, the charge diffusion mode becomes unstable. Later in this paper we will explicitly evaluate the conductivity of the charged D1-brane solution and show that it is indeed finite at $k = 3/2$ and thus at the boundary of thermodynamic stability, the charge diffusion mode becomes unstable.

One way of reading out the transport coefficients is to study the hydrodynamic modes and identify the coefficient of the dissipative parts. From (3.15) and (3.17), we see that we can read out both the bulk viscosity and the conductivity. Another approach is to use Kubo's formula which directly give the transport coefficients in terms of the two point functions. Let us first define the various retarded Green's functions:

$$\begin{aligned} G_{\mu\nu\alpha\beta}(\omega,q) &= -i\int d^2x\theta(t)e^{-i(\omega t+qz)}\langle[T_{\mu\nu}(x),T_{\alpha\beta}(0)]\rangle, \\ G_{\mu\nu\rho}(\omega,q) &= -i\int d^2x\theta(t)e^{-i(\omega t+qz)}\langle[J_{\mu}(x),T_{\nu\rho}(0)]\rangle, \\ G_{\mu\nu}(\omega,q) &= -i\int d^2x\theta(t)e^{-i(\omega t+qz)}\langle[J_{\mu}(x),J_{\nu}(0)]\rangle. \end{aligned} \tag{3.18}$$

Conservation laws and symmetries constrain the form of $G_{\mu\nu\alpha\beta}(\omega,q)$ to be [7]

$$G_{\mu\nu\alpha\beta}(\omega,q) = P_{\mu\nu}P_{\alpha\beta}G_B(\omega,q), \tag{3.19}$$

where $P_{\mu\nu}$ is defined by

$$P_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \tag{3.20}$$

and $k_\mu = (-\omega, q)$. Thus the two point function of the stress tensor is determined just by one function G_B. For future reference, we write down the following component of this correlator

$$G_{zzzz} = \frac{\omega^4}{(\omega^2-q^2)^2}G_B(\omega,q). \tag{3.21}$$

Similarly one can show that conservation laws $k^\mu G_{\mu\nu}(\omega,q) = 0$ determine the form of the retarded two point function of the currents to be [20]

$$G_{\mu\nu}(\omega,q) = P_{\mu\nu}G_J(\omega,q). \tag{3.22}$$

We write down the following component of this two point function

$$G_{zz}(\omega,q) = \frac{\omega^2}{\omega^2-q^2}G_J(\omega,q). \tag{3.23}$$

What is left now is the retarded two point function of the stress tensor and the charge current. Though we will not be requiring the form of this two point function, for completeness, we state that conservation laws and symmetries determine this two point function to be

$$G_{\mu\nu\rho}(\omega,q)=\epsilon_{\mu\sigma}k^{\sigma}P_{\nu\rho}G_S(\omega,q), \tag{3.24}$$

where $\epsilon_{\mu\nu}$ is the antisymmetric tensor with $\epsilon_{tz}=-\epsilon_{zt}=1$.

The transport coefficients, bulk viscosity ζ and the conductivity σ are given by the following Kubo's formulae

$$\begin{aligned}\zeta&=\lim_{\omega\to 0}\frac{i}{\omega}G_{zzzz}(\omega,q=0)=\lim_{\omega\to 0}\frac{i}{\omega}G_B(\omega,0), \\ \sigma(\omega)&=\frac{i}{\omega}G_{zz}(\omega,q=0)=\frac{i}{\omega}G_J(\omega,0).\end{aligned} \tag{3.25}$$

The DC conductivity can be obtained by further taking the limit

$$\sigma_{\rm DC}=\lim_{\omega\to 0}\frac{i}{\omega}G_{zz}(\omega,q=0)=\lim_{\omega\to 0}\frac{i}{\omega}G_J(\omega,0). \tag{3.26}$$

Note that all these formulae involve the $q=0$ limit. This is a useful feature which we will exploit in solving for the hydrodynamic modes from gravity. We will also be interested in the thermal conductivity of the charged D1-brane fluid. The thermal conductivity can be evaluated using its relation to the electrical conductivity [17], which is given by

$$\kappa_T=\left(\frac{\epsilon+P}{\rho}\right)^2\frac{\sigma}{T}. \tag{3.27}$$

4. Hydrodynamic modes in gravity

In this section, we study linearised fluctuations of the gravity solution in (2.6) and isolate the gauge invariant combinations of fluctuations which correspond to the sound mode and the diffusion mode. These we have obtained in the previous section using general hydrodynamic considerations. We consider linearised wave like perturbations of the single charged D1-brane solution of the form $g_{\mu\nu}\to g_{\mu\nu}+\delta g_{\mu\nu}$, $A_\mu\to A_\mu+\delta A_\mu$, $\phi\to\phi+\delta\phi$ and $\Psi\to\Psi+\delta\Psi$. Due to translational invariance along the D1-brane directions, we can assume that all the perturbations can be expanded using its Fourier mode as

$$\begin{aligned}\delta g_{\mu\nu}(t,z,r)&=e^{-i(\omega t-qz)}h_{\mu\nu}(r), \qquad \delta A_\mu(t,z,r)=e^{-i(\omega t-qz)}a_\mu(r), \\ \delta\phi(t,z,r)&=e^{-i(\omega t-qz)}\varphi(r), \qquad \delta\Phi(t,z,r)=e^{-i(\omega t-qz)}\xi(r).\end{aligned} \tag{4.1}$$

We further parameterize the radial dependence of the metric and the gauge perturbations as

$$h_{tt}=-c_T^2H_{tt}, \qquad h_{tz}=c_X^2H_{tz}, \qquad h_{zz}=c_X^2H_{zz}, \qquad a_\mu=\frac{l^2r_0^3}{L^2}B_\mu, \tag{4.2}$$

where c_X and c_T are defined in (2.6). We fix the gauge by imposing $\delta g_{r\mu} = 0,\ \delta A_r = 0$. The linearized equations of motion for the perturbations are given by

$$\begin{aligned}
&3r^2H^2KH''_{zz} + 3rH\left[(2H+1)H + (3H+1)K\right]H'_{zz} + 6Kr\xi' \\
&-4HK(3H+1)r\varphi' + 6r^3H^2(H-1)(H-K)B'_t + 12(4H-1)\xi \\
&-6(H-1)(H-K)H_{tt} - 8\left\{6H^2(H+1) + (H-1)(H-K)\right\}\varphi = 0, \qquad (4.3)\\
&rHH''_{tz} + (5H+2)H'_{tz} + 2r^2(H-1)(H-K)B'_z = 0, \qquad (4.4)\\
&3r^2H^2KH''_{tt} + 3rH\left[3H(2H+1) - (H+1)K\right]H'_{tt} + 6Kr\xi' \\
&-4HK(3H+1)r\varphi' - 6r^3H^2(H-1)(H-K)B'_t + 12\left(2H+1+\frac{2l^2}{r^2}\frac{K}{H}\right)\xi \\
&+6(H-1)(H-K)H_{tt} - 8\left\{6H^2(H+1) - (H-1)(H-K)\right\}\varphi = 0, \qquad (4.5)\\
&H^3B''_t + \frac{H^2}{r}(4-H)B'_t + \frac{H}{r^3}\left(\frac{4}{H}\xi' - \frac{8}{3}\varphi' + H'_{zz} - H'_{tt}\right) \\
&\qquad + \frac{8}{H}\frac{(H-1)}{r^4}\xi - \frac{L^6}{r^6}\frac{H^3}{K}q(\omega B_z + qB_t) = 0, \qquad (4.6)\\
&HKB''_z + \frac{1}{r}\left\{2H(2H+1) - (5H-2)K\right\}B'_z + \frac{2}{r^3}H'_{tz} \\
&\qquad\qquad + \frac{L^6}{r^6}\frac{H^2}{K}\omega(qB_t + \omega B_z) = 0, \qquad (4.7)\\
&r^2\varphi'' + \left[1 + \frac{2}{K}(2H+1)\right]r\varphi' - \frac{3}{2}r(H_{tt} + H_{zz})' \\
&-\frac{6}{KH}\xi + \frac{1}{K^2}\left[8K(2H+1) + \frac{L^6}{r^4}(\omega^2H - q^2K)\right]\varphi = 0, \qquad (4.8)\\
&r^2HK\xi'' + \left[2H(2H+1) + K(5H-4)\right]r\xi' - \frac{l^2}{r}HK(H_{zz} + H_{tt})' \\
&+4r^3H^2(H-1)(H-K)B'_t + \frac{16}{3}\frac{l^2}{r^2}\left(2H^2 + K - H\right)\varphi - 4(H-1)(H-K)H_{tt} \\
&+\left[(\omega^2H - q^2K)\frac{H}{K}\frac{L^6}{r^4} + 4\left\{H(2H+3) - 3 - (4-H)(H-1)\frac{K}{H}\right\}\right]\xi = 0. \qquad (4.9)
\end{aligned}$$

Here H and K are defined in (2.7). Equations of motion obtained from the variations $\delta g_{\mu r}$ and δA_r lead to the following 4 constraints.

$$\begin{aligned}
&rH(qKH'_{tt} - \omega HH'_{tz}) + q(2H+1)(H-K)H_{tt} - \frac{4}{3}qK(3H+1)\varphi + 2q\frac{K}{H}\xi \\
&\qquad\qquad -2r^2H(H-1)(H-K)(qB_t + \omega B_z) = 0, \qquad (4.10)\\
&rH^2(qH_{tz} + \omega H_{zz})' + 2\omega\xi - \frac{4}{3}\omega H(3H+1)\varphi \\
&\qquad\qquad -\frac{H}{K}(H-K)(2H+1)(\omega H_{zz} + 2qH_{tz}) = 0, \qquad (4.11)\\
&3rH^2K(3H+1)H'_{tt} + 3rH^3(K+2H+1)H'_{zz} + 4rH^2K(3H+1)\varphi' - 6rHK\xi' \\
&+6r^3H^3(H-1)(H-K)B'_t + 12\left\{(2H+1)H + 2(H-1)(H-K)\right\}\xi
\end{aligned}$$

$$
\begin{aligned}
&-8H\left\{6H^2(H+1)+(H-1)(H-K)\right\}\varphi-6H(H-1)(H-K)H_{tt}\\
&+3\frac{H^4}{K}\frac{L^6}{r^4}\left(-q^2\frac{K}{H}H_{tt}+2\omega qH_{tz}+\omega^2H_{zz}\right)=0, \qquad (4.12)\\
&r^3H^2\left(\omega B_t'+q\frac{K}{H}B_z'\right)+2qH_{tz}-\omega\left(H_{tt}-H_{zz}+\frac{8}{3}\varphi-\frac{4}{H}\xi\right)=0. \qquad (4.13)
\end{aligned}
$$

It can be shown that the constraints (4.10), (4.11), (4.12) and (4.13) are consistent with the dynamical equations of motion (4.3), (4.4), (4.5), (4.6), (4.7), (4.8) and (4.9). That is, on evolving the constraints using the equations of motion, one does not generate new constraints. We have verified that on differentiating the constraints with respect to r, one just obtains a linear combination of the dynamical equations of motion as well as the constraints.

Though we have fixed the gauge $\delta g_{\mu r}=0$, $\delta A_r=0$, there are still residual gauge degrees of freedom arising from diffeomorphisms $x^\mu\to x^\mu+\epsilon^\mu$ with $\epsilon^\mu=\epsilon^\mu(r,\omega,q)e^{-i\omega t+iqz}$ and $U(1)$ gauge transformations $A_\mu\to A_\mu+\partial_\mu\chi$ with $\chi=\tilde{\chi}(\omega,q)$ $e^{-i\omega t+iqz}$. Under diffeomorphism, the metric, the gauge field and the scalars transform as

$$
\begin{aligned}
g_{\mu\nu}&\to g_{\mu\nu}-\nabla_\mu\epsilon_\nu-\nabla_\nu\epsilon_\mu, \qquad (4.14)\\
A_\mu&\to A_\mu-\partial_\mu\epsilon^\rho A_\rho-\epsilon^\sigma\partial_\sigma A_\mu,\\
\phi&\to\phi-\partial_\mu\phi\epsilon^\mu, \qquad \Psi\to\Psi-\partial_\mu\Psi\epsilon^\mu,
\end{aligned}
$$

where $\epsilon^\mu(r,\omega,q)$ is determined by the gauge condition $\delta g_{\mu r}=0$. The residual $U(1)$ gauge transformations on a given Fourier mode of the gauge field act as follows

$$
A_t\to A_t-i\omega\tilde{\chi}, \qquad A_z\to A_z+iq\tilde{\chi}. \qquad (4.15)
$$

Instead of fixing the gauge completely, it is more convenient to work in variables which are invariant under these residual gauge transformations. To do this, we first work out the change of the fluctuations under diffeomorphisms explicitly. This is given by

$$
\begin{aligned}
&H_{tt}\to H_{tt}-\frac{2}{c_T^2}(i\omega\epsilon_t+\Gamma^r_{tt}\epsilon_r),\\
&H_{tz}\to H_{tz}+\frac{1}{c_X^2}(i\omega\epsilon_z-iq\epsilon_t), \qquad H_{zz}\to H_{zz}-\frac{2}{c_X^2}(iq\epsilon_z-\Gamma^r_{zz}\epsilon_r).\\
&B_t\to B_t-\frac{2}{r^3H^2}\epsilon^r+i\omega\frac{L^2A_t}{lr_0^3}\epsilon^t, \qquad B_z\to B_z-iq\frac{L^2A_t}{lr_0^3}\epsilon^t.\\
&\varphi\to\varphi-\frac{\phi'}{c_R^2}\epsilon_r, \qquad \xi\to\xi-\frac{H'}{c_R^2}\epsilon_r, \qquad (4.16)
\end{aligned}
$$

where Γ's refer to the Christoffel symbols of the single charged D1-brane solution. Similarly under the $U(1)$ transformations, the gauge field fluctuations change as

$$
B_t\to B_t-i\omega\tilde{\chi}, \qquad B_z\to B_z+iq\tilde{\chi}. \qquad (4.17)
$$

From the gauge transformations in (4.16) and (4.17), we can show that the following are gauge invariant variables both under diffeomorphisms as well as $U(1)$ gauge transformations.

$$\begin{aligned}
Z_P &= -q^2\frac{K}{H}H_{tt} + 2\omega q H_{tz} + \omega^2 H_{zz} - \frac{2V}{3H}\varphi, \\
G_P &= qB_t + \omega B_z + \frac{2q}{3r^2H^2}\varphi, \\
S_P &= 2(1-H)\varphi + 3\xi.
\end{aligned} \tag{4.18}$$

where

$$V = q^2(K + 2H + 1) - \omega^2(3H + 1) \tag{4.19}$$

and k is defined in (2.18). Note that the gauge invariant variables given in (4.18) are not unique, in fact any linear combinations of the above variables are also gauge invariant.

After tedious but straightforward manipulations, it can be shown that the dynamical equations and the constraint equations can be used to write down 3 second order coupled linear differential equations for the gauge invariant variables Z_P, G_P and S_P. Before we present these equations, we redefine quantities so that we are dealing only with dimensionless variables as follows:

$$\begin{aligned}
&\frac{r_H^2}{r^2} = u, \qquad \hat{q} = r_H q, \qquad \hat{\omega} = r_H\omega, \\
&\hat{Z}_P = r_H^2 Z_p, \qquad \hat{G}_P = r_H^3 G_P, \qquad \hat{S}_P = S_P \\
&r_0^6 = r_H^6(1+k) \quad \text{with} \quad k = \frac{l^2}{r_H^2}, \qquad \hat{L} = \frac{L}{r_H}.
\end{aligned} \tag{4.20}$$

We also define the expression

$$\alpha_t = q^2\frac{K}{H} - \omega^2. \tag{4.21}$$

In the equations below, for brevity of notation, we continue to refer to the hatted dimensionless quantities in terms of their original symbols. The equations for the gauge invariant quantities are given below where the prime denotes derivative with respect to the dimensionless quantity u.

$$\begin{aligned}
Z_P'' &+ \left[\frac{(K-2H-1)}{uK} + \frac{2}{uHV}\{q^2(H-K)(2H+1) + (H-1)(q^2-\omega^2)\}\right] Z_P' \\
&= \left[\frac{L^6 H}{4K^2}\alpha_t + \frac{(H-K)}{u^2HVK}\{q^2(K(4H+5) - (2H+1)^2) - \omega^2(H-1)\}\right] Z_P \\
&\quad + \frac{2q(H-K)(H-1)}{u^3HVK}[q^2\{K(4H+5) - (2H+1)^2\} \\
&\quad - \omega^2(H-1) + 6\omega^2(H+1)(H-K)]G_P
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{3u^2VH^3K}\left[-q^4(H-K)^2(3H+10)+6q^2\omega^2(H-K)^2(H+2)\right.\\
&+(9H^2-2H+1)H(q^2-\omega^2)^2-6H(H-1)(H-K)(q^2-\omega^2)^2\\
&\left.-2(H-K)q^2(q^2-\omega^2)(2H^2-H+1)\right]S_P,
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
G_P''&-\frac{G_P'}{uH^2K\alpha_t}[2q^2K^2(1-H)+\omega^2H\{K(4H-1)-H(2H+1)\}]\\
&+\frac{q}{H^2V\alpha_t}[q^2(2H-K+1)-\omega^2(H+1)]Z_P'-\frac{2q}{3H^3}S_P'\\
=&\ \frac{G_P}{VH^3K^2\alpha_t}\left[\frac{L^6}{4}H^4V\alpha_t^2+\frac{K}{u^2}(H-1)(H-K)\{q^2(2H+1)(2q^2K\right.\\
&\left.-\omega^2(H+K))-\omega^2\alpha_tH(3H+1)\}\right]+\frac{q}{uH^2VK}(2H+1)(H-K)Z_P\\
&+\frac{q}{3uH^5KV\alpha_t}[(1+3H-6H^2)H^2(q^2-\omega^2)^2\\
&+q^4(H-K)\{2H+(3H+2)HK+2(H-1)K^2\}+3q^2HV(H-1)(H-K)\\
&+\omega^2H(H-K)\{q^2(H^2-6KH-2)+2H\alpha_t+2V(H+1)\}]S_P,
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
S_P''&-\frac{6(H-1)(q^2-\omega^2)}{uV\alpha_t}Z_P'-\frac{6q}{u^2\alpha_t}(H-1)(H-K)G_P'\\
&-\frac{1}{uHK}[H(2H+1)+K(H-2)]S_P'=\\
&-\frac{3}{u^2VK}(H-1)(H-K)Z_P-\frac{6q}{u^3VH\alpha_t}(H-1)^2(H-K)(q^2-\omega^2)G_P\\
&+\frac{1}{u^2H^3KV\alpha_t}\left[\frac{L^6u^2}{4K}VH^4\alpha_t^2+(4-H)(H-1)H^2K\alpha_t^2\right.\\
&-(q^2-\omega^2)^2H^2(8H^3+5H^2-7H+2)\\
&+q^2H(H-K)\{\alpha_t(H+1)(4H^2-9H+6)+q^2(8H^3+5H^2-7H+2)\}\\
&+\omega^2(H-K)\{q^2H(2+11H-13H^2-8H^3)-H(H-1)\alpha_t(2H^2-3H-6)\\
&\left.+4\omega^2H(H-1)(2H+1)\}\right]S_P.
\end{aligned}
\tag{4.24}
$$

At present, it seems that there are 3 gauge invariant modes in contrast with the 2 modes in 1 + 1 hydrodynamics as shown in the previous section. We will show subsequently that one of these modes can be decoupled from the rest and consistently set to zero and plays no role in determining the transport coefficients.

4.1 Properties of the fundamental equations

Though the equations given in (4.22), (4.23) and (4.24) seem a set of complicated coupled differential equations, we will show that for the transport properties of interest, namely the conductivity and the bulk viscosity can be obtained from them using analytical methods. For this purpose, we need to discuss various properties relevant to these equations.

(i) $l = 0$ limit

In this limit, the charged D1-brane reduces to the uncharged D1-brane. An important check for the system of equations in (4.22), (4.23) and (4.24) is that they decouple and one of the mode reduces to the sound mode studied in [7]. Setting $l = 0$, we see the parameters which enter these equations reduce to

$$r_0 \to r_H, \qquad H \to 1, \qquad K \to f = 1 - \frac{r_0^6}{r^6}, \tag{4.25}$$

$$\alpha_t \to q^2(f - \lambda),$$

$$V \to q^2(3 + f - 4\lambda), \qquad \text{with} \quad \lambda = \frac{\omega^2}{q^2}. \tag{4.26}$$

Note that with these parameters, the mode Z_P reduces to the sound mode studied in [7], also the definition of H_{tt} is negative of H_{tt} in [7]. Substituting these values of the parameters into the fundamental equations for the gauge invariant fluctuations, we see that the variable S_P can be consistently set to zero and the equation for Z_P decouples from G_P. The equation for Z_P reduces to

$$Z_P'' + \left[\frac{6+f}{rf} - \frac{12(1-f)}{r(3+f-4\lambda)}\right] Z_P' - \left[\frac{q^2L^6}{f^2r^6}(f-\lambda) - \frac{36(1-f)^2}{r^2f(f+3-4\lambda)}\right] Z_P = 0. \tag{4.27}$$

It can be seen that this is the equation for the sound mode obtained in [7]. To write the equation for the gauge fluctuation in the $l = 0$ limit, it is convenient to redefine it as

$$\begin{aligned} H_P = \frac{lr_0^3}{L^2}G_P &= q\left(\frac{lr_0^3}{L^2}B_t\right) + \omega\left(\frac{lr_0^3}{L^2}B_z\right) + \frac{2qlr_0^3}{3r^2H^2L^2}\varphi, \\ &= q\delta A_t + \omega\delta A_z + \frac{2qlr_0^3}{3r^2H^2L^2}\varphi. \end{aligned} \tag{4.28}$$

Thus in the $l \to 0$ limit, the dilaton fluctuation decouples from the gauge invariant combination H_P. Substituting G_P in terms of H_P, it can be seen that the sound mode Z_P decouples from the gauge mode and reduces to

$$H_P'' - \frac{3u^2\lambda}{(\lambda - f)f}H_P' + \frac{L^3}{4f^2}(\omega^2 - q^2f)H_P = 0. \tag{4.29}$$

It can be easily verified that this is the equation which is obtained by examining the gauge field equation

$$\partial_\mu(\sqrt{-g}e^{-4\phi/3}F^{\mu\nu}) = 0, \tag{4.30}$$

where the metric and the dilaton background values are that of the uncharged D1-brane. The background gauge field in this case vanishes and the field strength $F^{\mu\nu}$ is just that of the fluctuations δA_z and δA_t. Thus we have seen that in the $l = 0$ limit, we obtain two modes, the mode Z_P corresponds to the sound mode and the mode H_P corresponds to the charge diffusion mode. This is what is expected for the uncharged D1-brane. The dispersion relation for the quasi-normal mode of Z_P was obtained in [7] and it is given by

$$\omega = \pm\frac{1}{\sqrt{2}}q - i\frac{L^3}{12}q^2 + ... \tag{4.31}$$

Note that we are measuring all quantities in units of $r_H = r_0$ here. Then identifying the sound speed and the bulk viscosity from the above dispersion relation, it was seen that

$$v_s^2 = \frac{1}{2}, \qquad \frac{\zeta}{s} = \frac{1}{4\pi}, \tag{4.32}$$

where s represents entropy density for the uncharged D1-brane. In this paper, we will show that the ratio ζ/s continues to be $\frac{1}{4\pi}$ for the case of the charged D1-brane also. The quasi-normal mode for the gauge field equation (4.29) is given by

$$\begin{aligned} H_P = A(1-u^3)^{-i\frac{L^3}{6}\omega}\Bigg(1 + i\omega\frac{L^3}{2}\left[\frac{1}{2}\ln\frac{1+u+u^2}{3}\right. \\ \left. +\frac{1}{\sqrt{3}}\left\{\tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right) - \frac{\pi}{3}\right\}\right] + i\frac{q^2L^3}{2\omega}(1-u) + O(\omega^2, q^4, \omega q^2)\Bigg) \end{aligned} \tag{4.33}$$

where A is an arbitrary constant. Note that the above solution satisfies the ingoing boundary condition at the horizon $u = 1$. Imposing the Dirichlet condition at the boundary $u = 0$, we obtain the charge dispersion relation

$$\omega = -i\frac{L^3}{2}q^2 + \cdots. \tag{4.34}$$

Here again, we are measuring all quantities in units of $r_H = r_0$. Using the expression for the charge diffusion constant in terms of conductivity given in (3.16), we find the conductivity for the D1-brane system in absence of charge density is given by

$$\sigma = \frac{1}{16\pi G_3}. \tag{4.35}$$

ii. $q = 0$ limit

Note that the formula for conductivity as well as the Kubo's formula for bulk viscosity involves the $q \to 0$ limit. It is therefore useful to examine the fundamental equations in this limit. The following simplifications occurs in this limit

$$\alpha_t \to -\omega^2, \qquad V \to -(3H+1)\omega^2. \tag{4.36}$$

Examining the equation for the gauge field (4.23), we see that it decouples from Z_P and S_P and it reduces to

$$G_P'' + [K(4H-1) - H(2H+1)]\frac{G_P'}{uHK} - \frac{(H-1)(H-K)}{u^2H^2K}G_P + \frac{L^6}{4}\frac{H}{K^2}w^2G_P = 0. \tag{4.37}$$

The equation for Z_P and S_P are coupled, they reduce to

$$
\begin{aligned}
&Z_P'' + \left\{ \frac{K-2H-1}{uK} + \frac{2(H-1)}{uH(3H+1)} \right\} Z_P' + \frac{L^6\omega^2 H}{4K^2} Z_P \\
&\quad = \frac{(H-1)(H-K)}{u^2HK(3H+1)} Z_P - \frac{(9H^2-2H+1)-6(H-1)(H-K)}{3u^2H^2K(3H+1)} \tilde{S}_P, \\
&\tilde{S}_P'' - \frac{\{K(H-2)+H(2H+1)\}}{uHK} \tilde{S}_P' + \frac{6(H-1)}{u(3H+1)} Z_P' \\
&\quad = \frac{3(H-1)(H-K)}{u^2K(3H+1)} Z_P - \left\{ \frac{L^6\omega^2 H}{4K^2} + \frac{2H}{u^2K} + \frac{(H^2-1)(3H-2)}{u^2H^2(3H+1)} \right\} \tilde{S}_P,
\end{aligned}
\tag{4.38}
$$

where

$$\tilde{S}_P = \omega^2 S_P. \tag{4.39}$$

It is now possible to decouple the equations for Z_P and $\tilde{S}_P$ by redefining $\tilde{S}_P$ as

$$\tilde{S}_P = \hat{S}_P + \frac{3H(1-H)}{3H+1} Z_P. \tag{4.40}$$

In terms of $\hat{S}_P$, the equations in (4.38) reduce to

$$
\begin{aligned}
&\hat{S}_P'' - \frac{\{K(H-2)+H(2H+1)\}}{uHK} \hat{S}_P' \\
&\quad = -\left\{ \frac{L^6\omega^2 H}{4K^2} + \frac{2H}{u^2K} + \frac{(H^2-1)(3H-2)}{u^2H^2(3H+1)} \right\} \hat{S}_P, \\
&Z_P'' + \left\{ \frac{K-2H-1}{uK} + \frac{2(H-1)}{uH(3H+1)} \right\} Z_P' - \frac{(H-1)(H-K)}{u^2HK(3H+1)} Z_P + \frac{L^6\omega^2 H}{4K^2} Z_P \\
&\quad = -\frac{(9H^2-2H+1)-6(H-1)(H-K)}{3u^2H^2K(3H+1)} \left(\hat{S}_P + \frac{3H(1-H)}{3H+1} Z_P \right).
\end{aligned}
\tag{4.41}
$$

Note that Z_P decouples from the equation for $\hat{S}_P$. We can now set $\hat{S}_P$ consistently to zero and study only the decoupled equation for Z_P. Simplifying the equation for Z_P, we obtain

$$
\begin{aligned}
&Z_P'' + \left\{ \frac{K-2H-1}{uK} + \frac{2(H-1)}{uH(3H+1)} \right\} Z_P' + \frac{L^6\omega^2 H}{4K^2} Z_P \\
&\quad = \frac{(H-1)(K(3H-7)+(3H+1)(2H+1))}{u^2HK(3H+1)^2} Z_P.
\end{aligned}
\tag{4.42}
$$

Thus we have shown that setting $q = 0$, we can obtain two decoupled equations (4.37) and (4.42) which correspond to the charge diffusion mode and the sound mode. Thus to obtain conductivity and the bulk viscosity of the charged D1-brane fluid, it is sufficient to study the equations (4.37) and (4.42).

iii. Behaviour at the horizon

To obtain the behaviour of the functions G_P and Z_P at the horizon, we define $x = \ln(1-u)$. Then both the equations (4.37) and (4.42) reduce to the oscillator equation

$$(\partial_x^2 + \frac{L^6}{4}\frac{1+k}{(2k+3)^2}\omega^2)Y = 0. \tag{4.43}$$

The ratio, $\frac{1+k}{(2k+3)^2}$ is obtained due to the behaviour of the coefficient proportional to ω^2 in both the equations. Thus the behaviour near the horizon is given by

$$G_P, Z_P \to (1-u)^{\pm i\frac{L^3\sqrt{1+k}}{2(2k+3)}}, \quad \text{for} \quad u \to 1. \tag{4.44}$$

Since classically horizons do not radiate, we need to choose the ingoing boundary condition

$$(1-u)^{-i\frac{L^3\sqrt{1+k}}{2(2k+3)}}, \tag{4.45}$$

to solve these equations.

iv. Behaviour at the boundary

Examining the coefficients of the equation for G_P given in (4.37) for $u \to 0$, the boundary, the equation reduces to

$$G_P'' + 2kG_P' + \frac{L^6}{4}\omega^2 G_P = 0. \tag{4.46}$$

Thus the solution for G_P at the boundary, $u \to 0$, admits a Taylor series expansion of the form

$$G_P \sim A(1 + O(u^2)) + Bu(1 + O(u^2)) + \qquad u \to 0, \tag{4.47}$$

where A and B are integration constants. Similarly examining the coefficients of the equation for Z_P given in (4.42), we see that, at the boundary, the equation reduces to

$$Z_P'' - \frac{2}{u}Z_p' + \frac{8k}{9u}Z_P = 0. \tag{4.48}$$

The above equation admits an expansion of the form

$$Z_P \sim A(1 + \cdots) + Bu^3(1 + \cdots). \tag{4.49}$$

The behaviour at the boundary is necessary to obtain the transport coefficients. In fact, the transport coefficients are proportional to the ratio B/A, that is the ratio of the normalizable mode by the non-normalizable mode.

5. Transport coefficients from gravity

We first summarize the method put forward by [21, 22] to evaluate transport coefficients from gravity.

1. Let $Z_k(r)$ be the gauge invariant variables constructed from the fluctuating gravity fields. In general, they satisfy coupled second order linear differential equations. We choose linear combination $Z(r)$ such that they satisfy decoupled second order linear differential equations. These decoupled gauge invariant variables correspond to the hydrodynamic modes of the field theory.

2. A local solution of the second order differential equations near the horizon $r = r_0$ will in general be a superposition of incoming and outgoing waves. Classically the horizon does not radiate, therefore, we choose the incoming wave boundary condition at the horizon.

3. The solution which obeys incoming wave boundary condition at the horizon can be written as a linear combination of two local solutions $f_1(r)$ and $f_2(r)$ at the boundary $r \to \infty$ as

$$Z(r) = Af_1(r) + Bf_2(r), \tag{5.1}$$

where A and B are the connection coefficients of the corresponding differential equations. Coefficients A and B depend on the parameters ω, q which enter the differential equation. Near the boundary, the solution (5.1) admits an expansion

$$Z(r) = A(1 + \cdots) + Br^{-\Delta}(1 + \cdots), \tag{5.2}$$

where the ellipses denote higher powers of r which are suppressed as $r \to \infty$ and $\Delta > 0$.

4. The action of the quadratic fluctuations can also be organized in terms of the gauge invariant observables. Evaluating the action on shell, it reduces to a boundary term which is of the form

$$S^{(2)} = \lim_{r\to\infty} \int d\omega dq F(r, \omega, q) Z'(r) Z(r) + \text{contact terms}, \tag{5.3}$$

where the contact terms do not involve derivatives of $Z(r)$ and

$$F(r, \omega, q) \to r^{\Delta+1} f(\omega, q), \qquad \text{as,} \qquad r \to \infty. \tag{5.4}$$

5. We can now use the fact that $Z(r)$ is a linear combination of the fluctuation gravity fields and apply the prescription in [21, 22] to compute the retarded correlator for the corresponding operator O in the field theory. We obtain

$$\langle OO \rangle_R \sim \frac{B}{A} \sim \frac{r^{\Delta+1}}{Z} \frac{dZ(r)}{dr}\bigg|_{r\to\infty,\text{finite term}}. \tag{5.5}$$

We have not written an equality but used $\sim$ as we have not yet kept track of the proportionality constant which depends on $F(r, \omega, q)$ in the limit $r \to \infty$.

We have used the expansion in (5.2) and the property (5.4) to write the last relation in (5.5). Note that from the last expression in (5.5), we need to extract the finite piece to obtain the ratio B/A.

6. To apply Kubo's formula, we only need the retarded correlator with $q = 0$. Thus it suffices to evaluate the following ratio to obtain the transport coefficients of interest,

$$\lim_{r\to\infty} \frac{1}{Z}\frac{dZ(r)}{dr}\bigg|_{q=0} . \tag{5.6}$$

In fact, for the DC conductivity and the bulk viscosity, we need to take a $\omega \to 0$ limit which is given by

$$\mathrm{Re}\left(\lim_{r\to\infty,\omega\to 0} \frac{1}{i\omega Z}\frac{dZ(r)}{dr}\bigg|_{q=0}\right) . \tag{5.7}$$

Since it is only the ratio

$$\mathcal{R}(r) = \frac{1}{Z}\frac{dZ(r)}{dr}. \tag{5.8}$$

at $r \to \infty$ that determines the retarded correlators, one can determine the differential equation satisfied by $\mathcal{R}(r)$ from the second order ordinary linear differential equation satisfied by $Z(r)$. We will see that this is a first order, ordinary but non-linear differential equation. The boundary conditions for this differential equation are determined from the ingoing boundary conditions satisfied by $Z(r)$ at the horizon. This equation, in fact, governs the radial evolution of the transport coefficients. We will show that for the DC conductivity and for the bulk viscosity, this equation is exactly solvable enabling us to determine the analytic expressions for these transport coefficients. The fact that the evaluation of transport coefficients can be reduced to solving a first order but non-linear differential equation has been observed recently for the case of $\mathcal{N} = 4$ super-Yang Mills by [23] and has been argued to be true in general in [24].

The rest of this section is organized as follows: We first show that the radial evolution of the transport coefficients are determined by first order non-linear ordinary differential equations. These equations are exactly solvable for the DC conductivity and the bulk viscosity. We then evaluate the effective action to determine the proportionality constant relating the ratio $\mathcal{R}$ in (5.8) to the transport coefficients.

5.1 Radial evolution of the transport coefficients

Radial evolution of conductivity

Let us obtain the equation that governs the radial evolution of conductivity. Note that the equation for G_P can be written as

$$\frac{1}{HK}(HKG_P')' - \frac{(H-1)(H-K)}{u^2H^2K}G_P + \frac{L^6}{4}\frac{H}{K^2}w^2G_P = 0. \tag{5.9}$$

One can now think of this mode as a minimally coupled scalar with a mass term proportional to $(H-1)$. Thus except for the term proportional to $(H-1)$, it falls in the class of equations of motion studied in [25] for which the radial evolution of the transport coefficients was easy to obtain in the $\omega \to 0$ limit [5]. To remove this term from (5.9), we perform the following redefinition

$$G_P = \frac{2H+1}{H}G. \tag{5.10}$$

Then the equation for G reduces to

$$G'' + \left(\frac{8H^2+1}{uH(2H+1)} - \frac{2H+1}{uK}\right)G' + \frac{L^6}{4}\frac{H}{K^2}w^2G = 0. \tag{5.11}$$

Thus, the redefinition in (5.10) removes the mass term and reduces the equation to that of a minimally coupled massless scalar. To obtain the R-charge retarded correlator, we need to impose ingoing boundary conditions at the horizon, $u=1$ on G_P. From the redefinition in (5.10), we see that this translates to ingoing boundary condition on G. From the discussion in around (4.45), we see that we have to impose the condition

$$G \sim (1-u)^{-i\frac{L^3\sqrt{1+k}}{2(2k+3)}}, \qquad u \to 1. \tag{5.12}$$

As we have discussed earlier conductivity is proportional to the ratio

$$\mathcal{R}_{G_P} = \frac{1}{i\omega G_P}\frac{dG_P}{du} = \frac{i}{\omega}\frac{k}{H(2H+1)} + \frac{1}{i\omega G}\frac{dG}{du}, \tag{5.13}$$

where we have used the redefinition given in (5.10) and also changed the variable from r to u. Thus we need to evaluate the ratio $\frac{1}{i\omega G}\frac{dG}{du}$ at the boundary subject to the condition (5.12) at the horizon. Let us define this ratio as

$$f_G = \frac{1}{i\omega G}\frac{dG}{du}. \tag{5.14}$$

The boundary condition for this ratio at the horizon, $u=1$ is then given by

$$f_G|_{r_H} \to \frac{L^3\sqrt{1+k}}{2(3+2k)(1-u)} + \ldots, \tag{5.15}$$

[5] See equation (38) in [25].

where the ellipses refer to sub-leading terms at $u = 1$. The differential equation satisfied by the ratio f_G can be obtained from the differential equation in (5.11). This is given by

$$f_G' + \left(\frac{8H^2+1}{uH(2H+1)} - \frac{2H+1}{uK}\right) f_G - i\frac{L^6 H}{4K^2}\omega + i\omega f_G^2 = 0. \tag{5.16}$$

This is a first order non-linear differential equation which governs the radial evolution of conductivity. From this equation, it is easy to obtain the DC conductivity and the pole at $\omega \to 0$ present in the imaginary part of the conductivity. We first decompose the above equation into its real and imaginary parts.

$$\begin{aligned}\mathrm{Re} f_G + \left(\frac{8H^2+1}{uH(2H+1)} - \frac{2H+1}{uK}\right) \mathrm{Re} f_G - 2\omega \mathrm{Im} f_G \mathrm{Re} f_G &= 0, \\ \mathrm{Im} f_G' + \left(\frac{8H^2+1}{uH(2H+1)} - \frac{2H+1}{uK}\right) \mathrm{Imf}_G - \omega \left(\mathrm{Im} f_G^2 - \mathrm{Re} f_G^2 + \frac{L^6 H}{4K^2}\right) &= 0.\end{aligned}$$

These equations simplify and decouple in the limit $\omega \to 0$. This decoupling would not have been possible in the original equation for G_P given in (5.9) due to the presence of the mass term proportional to $(H-1)$. But removing this term through the re-definition in (5.10) enables us to calculate DC conductivity exactly as follows. The solution for f_G satisfying the boundary condition (5.15) in the $\omega \to 0$ limit is given by

$$\mathrm{Re} f_G = \frac{(2k+3)^2}{\sqrt{1+k}} \frac{H}{K(2H+1)^2}, \qquad \mathrm{Im} f_G = 0. \tag{5.17}$$

We can now use (5.13) to evaluate the ratio which is proportional to the real part of the DC conductivity. This is given by

$$\begin{aligned}\mathrm{Re}\left(\mathcal{R}_{G_P}\right)_{u\to 0, \omega\to 0} &= \frac{L^3}{2}\left[\mathrm{Re} f_G + \mathrm{Re}\left(\frac{i}{\omega}\frac{k}{H(2H+1)}\right)\right]_{u\to 0, \omega \to 0}, \\ &= \frac{L^3}{2}\frac{(2k+3)^2}{9\sqrt{1+k}}.\end{aligned} \tag{5.18}$$

Now from the solution for $\mathrm{Im} f_G$ given in (5.17) in the $\omega \to 0$ limit, we see that $\mathrm{Im} f_G = \mathcal{O}(\omega)$. Thus imaginary part of the conductivity in the $\omega \to 0$ limit is given by

$$\mathrm{Im}\left(\mathcal{R}_{G_P}\right)_{u\to 0, \omega \to 0} = \mathrm{Im}\left(\frac{i}{\omega}\frac{k}{H(2H+1)}\right)_{u\to 0} = \frac{k}{3\omega}. \tag{5.19}$$

Therefore we see that the imaginary part of the conductivity has a pole at $\omega \to 0$ limit which is expected because of the translational invariance of the system. Translational invariance implies that there are no-impurities, which in turn implies infinite conductivity at $\omega = 0$ by Drude's formula. In fact, using the Kramers-Kronig relation

$$\mathrm{Im}\,\sigma(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\mathrm{Re}\,\sigma(\omega')}{\omega' - \omega} d\omega', \tag{5.20}$$

we see that the real part of the conductivity contains a delta function if and only if the imaginary part has a pole. Since we have found a pole in the imaginary part of the conductivity, it follows that the real part has a delta function singularity at $\omega = 0$. Therefore the value for the DC conductivity [6] is valid at $\omega \to 0^+$.

As a further check on our analytical manipulations, we have solved the differential equation for conductivity given in (5.9) numerically subject to the ingoing boundary conditions at the horizon and evaluated the ratio $\mathcal{R}_{G_P}$. For very small values of ω, we find very good agreement with the formula given in (5.18) and (5.19). This is shown in figure. 1 of section 5.2.

Radial evolution of bulk viscosity

The bulk viscosity is determined from the equation for Z_P given in (4.42) which can be written as

$$\begin{aligned} &\frac{1}{K}\frac{d}{dy}\left(K\frac{dZ_P}{dy}\right) + \frac{2(H-1)}{3yH(3H+1)}\frac{dZ_P}{dy} + \frac{\omega^2 L^6 H}{36u^4K^2}Z_P \\ &\quad = \frac{(H-1)(K(3H-7)+(3H+1)(2H+1))}{9u^6HK(3H+1)^2}Z_P. \end{aligned} \tag{5.21}$$

where $y = u^3$. Again, we see that the equation resembles a minimally coupled scalar equation except for the terms proportional to $(H-1)$. We can remove these terms by the following redefinition for Z_P.

$$Z_P = \frac{3H+1}{H}Z. \tag{5.22}$$

Then the equation for Z reduces to the simple form

$$Z'' + \frac{K-2H-1}{uK}Z' + \frac{\omega^2 L^3 H}{4K^2}Z = 0. \tag{5.23}$$

To obtain the retarded two point function of the stress tensor we need to impose ingoing boundary condition on Z_P at $u = 1$. Using the redefinition in (5.22), we see that this translates to the ingoing boundary condition on G at the horizon. Therefore, we need to impose

$$Z \to (1-u)^{-i\frac{L^3\sqrt{1+k}}{2(2k+3)}}, \qquad u \to 1 \tag{5.24}$$

From the earlier discussion, we see that the bulk viscosity is proportional to the real part of the following ratio evaluated at the boundary.

$$\mathrm{Re}(\mathcal{R}_{Z_P}) = \mathrm{Re}\left(\frac{1}{i\omega 3u^2 Z_P}\frac{dZ_P}{du}\right), \tag{5.25}$$

[6] Recently [26] has made a proposal for the value of the DC conductivity for conformal systems with chemical potential in arbitrary dimensions. We thank Sean Hartnoll for bringing this reference to our attention.

$$= \mathrm{Re}\left(\frac{1}{i\omega}\frac{k}{3u^2(1+ku)}\right) + \mathrm{Re}\left(\frac{1}{i\omega 3u^2 Z}\frac{dZ}{du}\right),$$
$$= \mathrm{Re}\left(\frac{1}{i\omega 3u^2}\frac{dZ}{du}\right),$$

Here, we have used the redefinition of Z_P given in (5.22). We are dividing by $3u^2$ so that we can extract out the ratio B/A in the expansion of Z_P near the boundary given in (4.49). Note that since the bulk viscosity is proportional to the real part of the ratio $\frac{1}{i\omega 3u^2}\frac{dZ_P}{du}$, it is determined by the behaviour of Z. Therefore let us define the ratio

$$f_Z = \frac{1}{i\omega Z}\frac{dZ}{du}. \tag{5.26}$$

Using the ingoing boundary condition for Z at the horizon, boundary condition for f_Z at the horizon is given by

$$f_Z|_{u\to 1} \to \frac{L^3\sqrt{1+k}}{2(3+2k)(1-u)} + \ldots, \tag{5.27}$$

where ellipses refer to sub-leading terms at $u = 1$. The differential equation satisfied by f_Z can be obtained from the differential equation for Z in (5.23) and is given by

$$f_Z' + \frac{K-2H-1}{uK}f_Z - i\frac{L^6 H}{4K^2}\omega + i\omega f_Z^2 = 0. \tag{5.28}$$

Again separating into the real and imaginary parts we obtain

$$\mathrm{Re}f_Z' + \frac{K-2H-1}{uK}\mathrm{Re}f_Z - 2\omega\mathrm{Im}f_Z\mathrm{Re}f_Z = 0,$$
$$\mathrm{Im}f_Z' + \frac{K-2H-1}{uK}\mathrm{Im}f_Z + \omega(\mathrm{Re}f_Z^2 - \mathrm{Im}f_Z^2) - \frac{\omega L^6 H}{4K^2} = 0. \tag{5.29}$$

The solution of these equations in the $\omega \to 0$ limit which obeys the boundary conditions in (5.27) is given by

$$\mathrm{Re}f_Z = \frac{L^3\sqrt{1+k}u^2}{2K}, \qquad \mathrm{Im}f_Z = 0. \tag{5.30}$$

It is now easy to obtain the ratio which is proportional to the bulk viscosity. It is given by

$$\mathrm{Re}(\mathcal{R}_{Z_P})|_{u\to 0} = \mathrm{Re}\left(\frac{1}{i\omega 3u^2}\frac{dZ}{du}\right)\bigg|_{u\to 0}, \tag{5.31}$$
$$= \mathrm{Re}\frac{f_Z}{3u^2},$$
$$= \frac{L^3}{2}\frac{\sqrt{1+k}}{3}. \tag{5.32}$$

Again as a further check on our manipulations, we evaluate the ratio $\mathcal{R}_{Z_P}$ directly by solving the differential equation (4.42) numerically subject to ingoing boundary conditions at the horizon. We find the result for $\omega \to 0$ in very good agreement with the expression given in (5.31). This is shown in figure 2 of section 5.2.

Note that the problem of obtaining the DC conductivity and the bulk viscosity has been reduced to solving a first order but non-linear differential equation. This equation governs the radial evolution of the ratio which is proportional to the respective transport coefficient. In the $\omega \to 0$ limit, the solution of these transport coefficients were easy to obtain exactly.

5.2 Comparison with numerical analysis

In this section, we solve the equations of motion for the charge diffusion and sound mode numerically and find the transport coefficients. We will actually find the ratio $\mathcal{R}_{G_P}$ and $\mathrm{Re}\,\mathcal{R}_{Z_P}$ which is proportional to the conductivity and the viscosity. Further more, we work in a normalization in which $L^3 = 2$ for convenience. Since we have analytic expressions for DC value of conductivity as well as viscosity at very small ω, we can check our numerics with these results. We also know the exact expression for conductivity and viscosity in the limit $l = 0$ and this gives us another check on our numerical results.

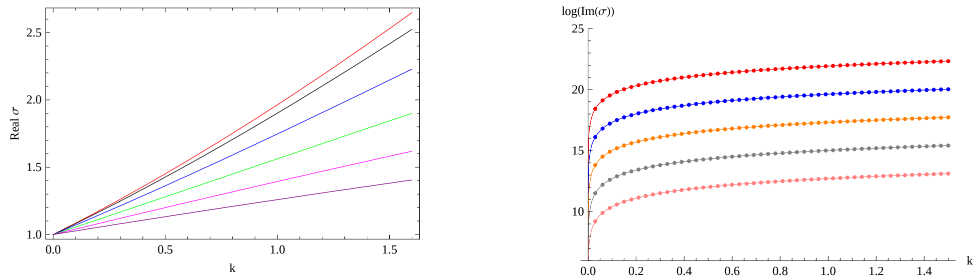

Figure 1: Plots of real (on left) and log of imaginary part of conductivity vs k for the single charged case. On left, the different colors red, black, blue, green, magenta and purple correspond to $\omega = 10^{-10}$, 0.2, 0.4, 0.6, 0.8 and 1.0 respectively. On right, the different colors red, blue, orange, gray and pink correspond to $\omega = 10^{-10}$, 10^{-9}, 10^{-8}, 10^{-7} and 10^{-6} respectively. The dots are the numerical values and the solid lines are curves Im $\sigma = \frac{k}{3\omega}$. σ is in units of $(16\pi G_3)^{-1}$ and ω is in units of $2r_H^2/L^3$.

In figure (1), we plot real and imaginary parts of conductivity vs k for the single charged case. For $k = 0$, the real part of conductivity approaches 1. This is in accord with our analytic calculation for $l = 0$ case. We compare the k dependence obtained numerically for the real part of conductivity for very small ω ($\omega = 10^{-10}$) with the DC conductivity. We find good agreement between them as the absolute value of the difference between numerical and analytically obtained values is at most 10^{-5}. We expect that the errors in our numerics remain in the same order for all other numerical curves, which tell dependence of AC conductivity on k for different values of ω. We don't have analytic expressions for non-trivial ω to compare them with. We find that conductivity increases monotonically with k, thought the slope decreases as we increase ω. Similarly we see that for small ω, our numerical results for imaginary part of conductivity fit well with the analytic expression. The absolute difference in this case is at most 10^{-6}. For small ω, imaginary part of conductivity grows linearly with k.

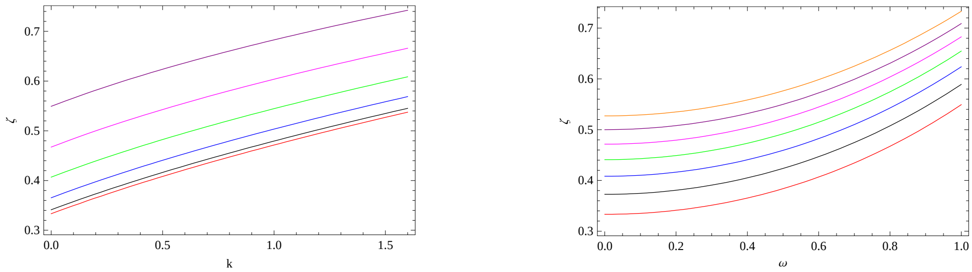

Figure 2: Plots of viscosity vs k (on left) and ω for single charged case. On left, the different colors red, black, blue, green, magenta and purple correspond to $\omega = 10^{-10}$, 0.2, 0.4, 0.6, 0.8 and 1.0 respectively. On right, the different colors red, black, blue, green, magenta, purple and orange correspond to $k = 0$, 0.5, 1.0, 1.5, 2.0, 2.5 and 2.9 respectively. ζ is in units of $r_H^4/(16\pi G_3 L^4)$ and ω is in units of $2r_H^2/L^3$.

In figure (2), we plot viscosity against k and ω for the single charged case. We note from the plot that the smallest value of ζ is at $\omega = k = 0$. We also compare the analytic expression for the viscosity as a function of k for $\omega \to 0$ with the numeric plot of ζ vs k for $\omega = 10^{-10}$ (red curve in left plot in figure (2)). We find the absolute difference between analytical and numerical values to be less than 10^{-4}. From the curves, we see that the curve for ζ vs k for a given ω shifts as a whole as one changes ω. From the right plot, we see that the amount of shift increases non-linearly with ω.

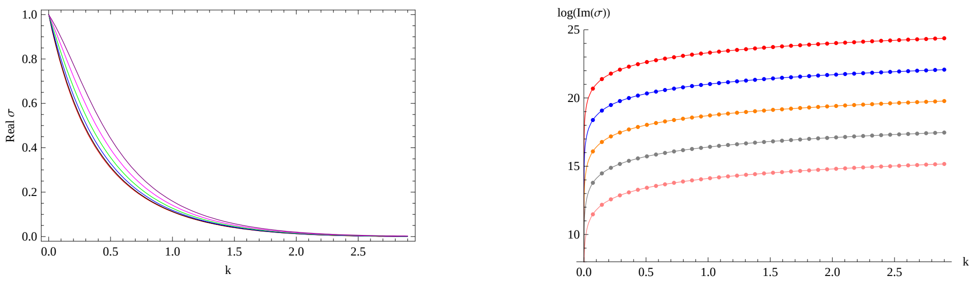

Figure 3: Plots of real (on left) and log of imaginary part of conductivity vs k for the equal charged case. On left, the different colors red, black, blue, green, magenta and purple correspond to $\omega = 10^{-10}$, 0.2, 0.4, 0.6, 0.8 and 1.0 respectively. On right, the different colors red, blue, orange, gray and pink correspond to $\omega = 10^{-10}$, 10^{-9}, 10^{-8}, 10^{-7} and 10^{-6} respectively. The dots are the numerical values and the solid lines are curves Im $\sigma = \frac{4k}{3\omega}$. σ is in units of $(16\pi G_3)^{-1}$ and ω is in units of $2r_H^2/L^3$.

In figure (3), we plot the real and imaginary part of conductivity against k for the equal charged case. Here too, we have analytic expressions for the DC conductivity which we compare with the σ vs k plot for $\omega = 10^{-10}$, red curve in the left plot. We find a good agreement with the absolute difference between the numeric and analytic values being less than 10^{-6}. This bound on error is also same for the plots of imaginary conductivity vs k on the right. We observe here that there is little change in the curves of σ vs k as one changes ω. The σ vs k behaviour here is very different from the same in single charged case. Latter, the curves were monotonically increasing, but here, conductivity decreases with increasing k.

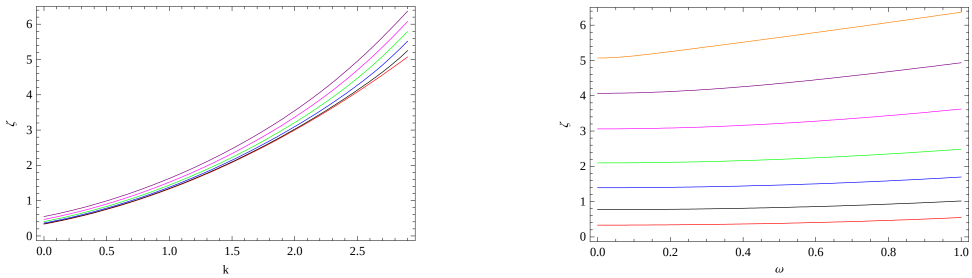

Figure 4: Plots of viscosity vs k (on left) and ω for equal charged case. On left, the different colors red, black, blue, green, magenta and purple correspond to $\omega = 10^{-10}$, 0.2, 0.4, 0.6, 0.8 and 1.0 respectively. On right, the same correspond to $k = 0$, 0.5, 1.0, 1.5, 2.0 and 2.5 respectively. Orange curve on right correspond to $k = 2.9$. ζ is in units of $r_H^4/(16\pi G_3 L^4)$ and ω is in units of $2r_H^2/L^3$.

Now we plot the behaviour of viscosity vs k and ω in figure (4). Again as in the single charged case, the minimum value of viscosity is at $k = \omega = 0$, which is same as before and saturates the conjectured lower bound on bulk viscosity. The red curve on the left, which stands for viscosity vs k at $\omega = 10^{-10}$ is compared to analytic value of viscosity obtained in the $\omega \to 0$ limit. We get the absolute difference between analytic and numerical values to be less than 10^{-9} here, suggesting excellent agreement. We find little dependence of viscosity on ω, particularly at smaller values of k.

5.3 Evaluation of the transport coefficients

In this section, we use the standard prescription of the gauge/gravity correspondence to evaluate the retarded two point functions which determines the conductivity and bulk viscosity. This will determine the proportionality constant between the ratios $\mathcal{R}_{G_P}$, $\mathcal{R}_{Z_P}$ and the transport coefficients. For this, we first need to expand the bulk action given in (2.8) along with the Gibbons-Hawking boundary term to second order in the fluctuations $H_{\mu\nu}, B_\mu, \varphi$ and ξ. In this section, we will not be using the dimensionless variables given in (4.20). All quantities in this section will have their respective dimensions, whenever needed, we will restore the dimensions of the ratios $\mathcal{R}_{G_P}$ and $\mathcal{R}_{Z_P}$. Using equations of motion and the constraints (Equations (4.3) to

(4.13)), we can write the bulk action (2.8) expanded to second order in fluctuations as a total derivative in r.

$$
\begin{aligned}
S^{(2)}_{\text{bulk}} &= \frac{1}{16\pi G_3}\int d\omega dq dr \frac{d\mathcal{L}_B}{dr},\\
\frac{L^7}{r^7}\mathcal{L}_B &= \frac{K}{4}\left(2H_{tt}H_{tt}{}' - H_{zz}H_{tt}{}' - H_{tt}H_{zz}{}' + 2H_{zz}H_{zz}{}'\right) - \frac{3}{2}HH_{tz}H'_{tz}\\
&\quad -\frac{r}{2\omega}(H-1)(H-K)\left(qB_t + wB_z\right)\left(r^3HKB'_z + 2H_{tz}\right) - \frac{8}{9}K\varphi\varphi'\\
&\quad -\frac{K}{2H^2}\xi\xi' + \frac{K}{3H}(\xi\varphi' + \varphi\xi') + \frac{K}{rH^3}(2-H)\xi^2 + \frac{2K}{3rH^2}(H-1)\varphi\xi\\
&\quad -\frac{1}{2rH}H_{tt}H_{zz}\{H(2H+1) + (4H+1)K\}\\
&\quad +\frac{1}{4rH}(H_{tt}+H_{zz})\left\{K(3H+1)H_{zz} + H(2H+K+1)H_{tt}\right\}\\
&\quad +\frac{1}{rK}H_{tz}{}^2\{H(2H+1) - (5H+2)K\} - \frac{4K}{3r}(H_{tt}-H_{zz})\varphi\\
&\quad +\frac{K}{3rH}\{(7H+1)H_{tt} - (H-1)H_{zz}\}\varphi - \frac{K}{2rH^2}(H_{tt}+H_{zz})\xi. \qquad (5.33)
\end{aligned}
$$

Note that here the prime denotes derivative with respect to r. The Gibbons-Hawking term expanded to second order in fluctuations is given by

$$
\begin{aligned}
S^{(2)}_{\text{GH}} &= \frac{1}{8\pi G_3}\int d^2x\sqrt{-h}K_{ext},\\
\frac{8L^7}{r^6}\sqrt{-h}K_{ext} &= \frac{4}{K}\{K(8H+3) - H(2H+1)\}H^2_{tz} + 8rHH_{tz}H'_{tz}\\
&\quad -\frac{1}{H}(H_{tt}-H_{zz})^2\{K(1+4H) + H(2H+1)\}\\
&\quad -2rK(H_{tt}-H_{zz})(H'_{tt}-H'_{zz}). \qquad (5.34)
\end{aligned}
$$

We now combine the $S^{(2)}_{\text{bulk}}$ and $S^{(2)}_{\text{GH}}$. Using the constraints, we can rewrite it in terms of the gauge invariant quantities Z_P, G_P and S_P as follows

$$
\begin{aligned}
S &= S^{(2)}_{\text{bulk}} + S^{(2)}_{GH} = \frac{1}{16\pi G_3}\int d\omega dq\mathcal{L},\\
\frac{L^7}{r^7}\mathcal{L} &= \frac{l^2r_0^6K}{2r^8\alpha_t H}\left(\frac{q}{V}Z_P + r^2HG_P\right)\left(\frac{q}{V}Z_P + r^2HG_P\right)' - \frac{3H^2K}{2V^2}Z_PZ'_P\\
&\quad -\frac{K}{2}\left(\frac{Z_P}{V} + \frac{S_P}{3H}\right)\left(\frac{Z_P}{V} + \frac{S_P}{3H}\right)' + \text{ contact terms.} \qquad (5.35)
\end{aligned}
$$

where 'contact terms' represent those terms in the action which do not contain any derivatives in r and the counter terms which render the complete boundary action finite. Next we define a new variable

$$
\mathcal{S} = S_P + \frac{3H(1-H)}{V}Z_P. \qquad (5.36)
$$

Note that this is also a gauge invariant variable. It has the following useful property

$$\mathcal{S} \to \frac{\hat{S}}{\omega^2}, \qquad \text{as} \quad q \to 0. \tag{5.37}$$

where $\hat{S}$ is defined in (4.40). Thus on taking $q \to 0$ limit, we can consistently set $\mathcal{S}$ to zero. We can now rewrite the boundary Lagrangian using $\mathcal{S}$ as

$$\begin{aligned}\frac{L^7}{r^7}\mathcal{L} = & \frac{l^2 r_0^6 HK}{2r^4\alpha_t}\left(\frac{q}{r^2VH}Z_P + G_P\right)\left(\frac{q}{r^2VH}Z_P + G_P\right)' - \frac{2H^2K}{V^2}Z_P Z_P' \\ & -\frac{K}{18H^2}\mathcal{S}\mathcal{S}' - \frac{K}{6V}(Z_P\mathcal{S}' + \mathcal{S}Z_P') + \text{ contact terms.}\end{aligned} \tag{5.38}$$

To evaluate the transport coefficients using the Kubo's formula in (3.25) and (3.26), it is sufficient to look at the boundary Lagrangian at $q \to 0$ limit. In this limit, we can set consistently $\mathcal{S} = 0$. So the boundary Lagrangian can now be simplified as

$$\mathcal{L} = -\frac{r^7}{L^7}\left\{\frac{l^2 r_0^6}{2r^4\omega^2}G_P G_P' + \frac{1}{8\omega^4}Z_P Z_P' + \text{contact terms}\right\}. \tag{5.39}$$

At $q = 0$, the expression for G_P reduces to

$$G_P = \omega B_z = \omega\frac{L^2}{l r_0^3}A_z. \tag{5.40}$$

Substituting this in (5.39), the boundary action involving the gauge field can be written as

$$S^{(2)}_{\text{gauge}} = \frac{r_H^2}{16\pi G_3 L^3}\int d\omega dq (A_z^{(0)})^2 i\omega\mathcal{R}_{G_P}|_{u=0}. \tag{5.41}$$

Here we have converted the derivative in r to derivative in u and used the definition of $\mathcal{R}_{G_P}$. $A_z^{(0)}$ refers to the boundary value of the gauge field. This field couples to the R-current of the D1-brane theory by the coupling

$$S_{\text{coupling}} = i\int d^2x (J^t A_t^{(0)} + J^z A_z^{(0)}). \tag{5.42}$$

Then using the gauge/gravity prescription, we can obtain the retarded Green's function of the R-current by

$$G_{zz} = -\frac{\delta^2 S^{(2)}}{\delta A_z^0 \delta A_z^0}. \tag{5.43}$$

Using this prescription and the boundary action for the gauge field given in (5.41), we obtain the following expression for the R-current correlator from gravity

$$G_{zz} = -\frac{2r_H^2}{16\pi G_3 L^3}i\omega\mathcal{R}_{G_P}|_{u=0}. \tag{5.44}$$

Finally we can compute the DC conductivity using the Kubo's formula

$$\begin{aligned}
\sigma_{DC} &= \mathrm{Re}\left(\lim_{\omega\to 0}\frac{i}{\omega}G_{zz}(\omega,q=0)\right),\\
&= \frac{2r_H^2}{16\pi G_3 L^3}\lim_{\omega\to 0}\mathrm{Re}\mathcal{R}_{G_P}|_{u=0},\\
&= \frac{2r_H^2}{16\pi G_3 L^3}\left(\frac{L^3}{2r_H^2}\frac{(2k+3)^2}{9\sqrt{1+k}}\right),\\
&= \frac{1}{16\pi G_3}\frac{(2k+3)^2}{9\sqrt{1+k}}.
\end{aligned} \tag{5.45}$$

Here, in the third step, we have used the result (5.18) and reinstated the proper dimensions for the ratio $\mathcal{R}_{G_P}$ which has the dimensions of length. As a check of the final answer note that at $k=0$, it reduces to the value evaluated using the quasi-normal mode analysis in (4.35).

Similarly we can determine viscosity using Kubo's formula. At $q=0$, the fluctuation Z_P reduces to

$$Z_P = \omega^2 H_{zz} + 2\frac{(3H+1)}{3H}\varphi. \tag{5.46}$$

Substituting this in (5.39), the boundary action involving quadratic terms in the fluctuation H_{zz} is given by

$$S^{(2)}_{H_{zz}} = \frac{1}{16\pi G_3}\frac{3r_H^6}{4L^7}\int d\omega dq (H^{(0)}_{zz})^2 i\omega\mathcal{R}_{Z_P}|_{u=0}. \tag{5.47}$$

Again we have converted the derivative in r to a derivative in u and used the definition of $\mathcal{R}_{Z_P}$. $H^{(0)}_{zz}$ refers to the boundary value of the fluctuation. The boundary fluctuations of the metric couples with the stress tensor of the field theory by the following action [27]

$$S_{\text{coupling}} = \frac{i}{2}\int d^2x (H^{(0)}_{tt}T^{tt} + H^{(0)}_{zz}T^{zz} + 2H^{(0)}_{tz}T^{tz}). \tag{5.48}$$

Then using the standard gauge/gravity prescription, the two point function of the stress tensor is given by

$$G_{zzzz} = -4\frac{\delta^2 S^{(2)}}{\delta H_{zz}(\omega)\delta H_{zz}(-\omega)}. \tag{5.49}$$

Using this prescription and the quadratic action for the metric fluctuation given in (5.47), we see the above two point function is given by

$$G_{zzzz} = -\frac{1}{16\pi G_3}\frac{3r_H^6}{L^7}i\omega\mathcal{R}_{Z_P}|_{u=0}. \tag{5.50}$$

We now can compute the bulk viscosity using the Kubo's formula

$$\begin{aligned}
\zeta &= \mathrm{Re}\left(\lim_{\omega\to 0}\frac{i}{\omega}G_{zz,zz}(\omega, q=0)\right),\\
&= \frac{1}{16\pi G_3}\frac{6r_H^6}{L^7}\lim_{\omega\to 0}\mathrm{Re}\,\mathcal{R}_{Z_P}|_{u=0},\\
&= \frac{r_H^4}{16\pi G_3 L^4}\sqrt{1+k},\\
&= \frac{1}{4\pi}s.
\end{aligned}\tag{5.51}$$

Here again, in the third line, we have used the expression for $\mathcal{R}_{Z_P}$ given in (5.31). In the last line, we have written the expression for ζ using the definition of entropy density for the single charged D1-brane given in (2.17). Thus we see that the ratio of bulk viscosity to entropy density remains $1/4\pi$ when the charge density is turned on.

6. Properties of the transport coefficients

We first summarize the results of the transport coefficients of the single charged D1-brane.

$$\begin{aligned}
\sigma &= \frac{1}{16\pi G_3}\frac{(2k+3)^2}{9\sqrt{1+k}},\\
\zeta &= \frac{r_H^4}{16\pi G_3 L^4}\sqrt{1+k}.
\end{aligned}\tag{6.1}$$

In this section, we restrict ourselves to only the DC conductivity except in subsection (6.2). Using these two results, we can find three more transport coefficients. The charge diffusion constant is related to conductivity by (3.16) and is given by

$$D_c = \frac{L^3}{r_H^2}\frac{3-2k}{6\sqrt{1+k}}. \tag{6.2}$$

The thermal conductivity is also related to the conductivity by (3.27) and is given by

$$\kappa_T = \left(\frac{\varepsilon+p}{\rho}\right)^2\frac{\sigma}{T} = \frac{r_H^2}{8LG_3}\frac{(2k+3)(1+k)}{k}. \tag{6.3}$$

Finally the sound diffusion constant can be obtained by (3.12) and is given by

$$D_s = \frac{\zeta}{2(\epsilon+p)} = \frac{L^3}{12r_H^2\sqrt{1+k}}. \tag{6.4}$$

As we have noted earlier, the ratio of bulk viscosity to entropy density is independent of the chemical potential and is given by

$$\frac{\zeta}{s} = \frac{1}{4\pi}. \tag{6.5}$$

This property also holds for the equal charged D1-brane solution as shown in appendix B. Using the formula for the bulk viscosity (6.1), the thermal conductivity in (6.3), the Hawking temperature in (2.17) and the chemical potential in (2.21), we can show the following relationship between these quantities is true

$$\frac{\kappa_T \mu^2}{\zeta T} = (2\pi L)^2 \,. \tag{6.6}$$

This relationship is more striking when we write the chemical potential μ in terms of its dimensions. Note that the normalization of the gauge field we have used in (2.8) is such that it is dimensionless. This is convenient for the gravity analysis, but it is conventional for the gauge field to have dimensions of inverse length. Since the chemical potential is basically the value of the gauge field at the horizon (2.21), it must have the dimensions of inverse length. Let us therefore restore its dimensions by defining

$$\hat{\mu} = \frac{\mu}{L}. \tag{6.7}$$

Then the relationship in (6.6) can be written as

$$\frac{\kappa_T \hat{\mu}^2}{\zeta T} = 4\pi^2. \tag{6.8}$$

This relationship is similar to the Wiedemann-Franz law seen between thermal conductivity and electrical conductivity. A similar relationship between thermal conductivity and the shear viscosity for the single charged D3 brane was observed by [17].

6.1 Transport coefficients at criticality

In this section, we discuss the reason for this property as well as behaviour of the transport coefficients near the boundary of thermodynamic stability $k = 3/2$. We first note that the charge diffusion constant D_c for the single charged D1-brane given in (6.2) vanishes at the boundary of thermodynamic instability. This indicates that this mode becomes unstable at $k = 3/2$ and for this case the thermodynamic instability can be studied by examining this mode more carefully. As we will see in appendix B, this feature does not hold for the equal charged D1-brane. It was also not seen in the analysis of [17] for the single charged D3-brane. Thus this feature seems to be specific for the single charged D1-brane and it is worth exploring this further.

To determine the critical behaviour of the transport coefficients at the boundary of thermodynamic instability, we follow the analysis done by [17]. We first define the dimensionless chemical potential m as

$$\mathsf{m} = \frac{\hat{\mu}}{2\pi T_H} = \frac{\mu}{2\pi L T_H} = \frac{\sqrt{k}}{(3+2k)}. \tag{6.9}$$

Note that μ/T is the natural variable that occurs in charge current (3.1). We can invert the relation in (6.9) to write k as

$$k = \frac{1 - 12\mathsf{m}^2 - \sqrt{(1-24\mathsf{m}^2)}}{8\mathsf{m}^2}. \tag{6.10}$$

Thus, we can re-express the transport coefficients as

$$\begin{aligned}
\sigma &= \frac{1}{16\pi G_3}\left(\frac{1-12\mathsf{m}^2-\sqrt{1-24\mathsf{m}^2}}{72\sqrt{2}\mathsf{m}^4}\right)\left(\frac{1-4\mathsf{m}^2+\sqrt{1-24\mathsf{m}^2}}{1+\mathsf{m}^2}\right)^{1/2}, \\
\zeta &= \frac{\pi L^2T^2}{4G_3}\left(\frac{1+6\mathsf{m}^2+\sqrt{1-24\mathsf{m}^2}}{18}\right)\left(\frac{1-4\mathsf{m}^2-\sqrt{1-24\mathsf{m}^2}}{8\mathsf{m}^2}\right)^{1/2}, \\
\kappa_T &= \frac{\pi L^2T}{4G_3}\left(\frac{1+6\mathsf{m}^2+\sqrt{1-24\mathsf{m}^2}}{18\mathsf{m}^2}\right)\left(\frac{1-4\mathsf{m}^2-\sqrt{1-24\mathsf{m}^2}}{8\mathsf{m}^2}\right)^{1/2}, \\
D_c &= \frac{1}{24\pi T}\sqrt{1-24\mathsf{m}^2}\left(\frac{1+6\mathsf{m}^2-\sqrt{1-24\mathsf{m}^2}}{\mathsf{m}^2(1+\mathsf{m}^2)}\right), \\
D_s &= \frac{1}{48\pi T}\left(\frac{5+\sqrt{1-24\mathsf{m}^2}}{1+\mathsf{m}^2}\right).
\end{aligned} \tag{6.11}$$

The boundary of thermodynamic stability lies at $k = \frac{3}{2}$ or $\mathsf{m}_c = \frac{1}{\sqrt{24}}$. Expanding the transport coefficients near this point, we see that the D_c exhibits a square root branch cut at the critical point. The other transport coefficients are finite at the critical point m_c, however their first derivatives including that of D_c diverges as $(\mathsf{m}_c - \mathsf{m})^{-1/2}$. Thus the critical index is 1/2 which indicates that the system exhibits mean field behaviour. A similar behaviour was observed for the shear viscosity and conductivity for the single charged D3 branes in [17].

From the above expressions for the transport coefficients in (6.11), note that that ζ and κ_T are written as $T^2 f(\mathsf{m})$ and $Tg(\mathsf{m})$ respectively. This demonstrates that the system has a hidden $2+1$ conformal invariance since the entropy density is proportional to T^2 in $2+1$ dimensions. Also note that the charge and sound diffusivity can be written in the scaling form $\frac{1}{T}f(\mathsf{m})$. The conductivity just depends on the dimensionless ratio m and assumes the scaling form $f(\mathsf{m})$. From examining the scaling form, it is easy to see that as $T \to \infty$, keeping the chemical potential μ constant, all the expressions for the transport coefficients reduce to the uncharged case as expected. Another point worth mentioning is that on expressing G_3, L in terms of the Yang-Mills coupling and the rank N, the transport coefficients ζ and κ_T are proportional to $N^2/\sqrt{\lambda}$. If at all this system holographically describes a $1+1$ dimensional system seen in nature, the scaling behaviour of the transport coefficients seen in (6.11) is a possible test.

6.2 Behaviour of conductivity

In figure (5), we plot the conductivity vs quantity $1/\mathsf{m}$, which is proportional to temperature, if chemical potential is held constant. For the single charged case, we

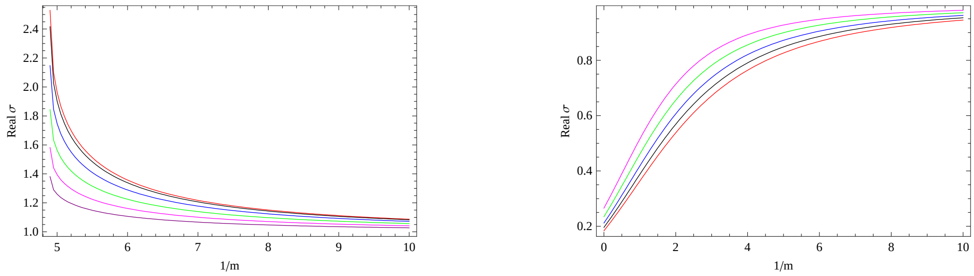

Figure 5: Plots of real part of conductivity vs $1/\mathsf{m}$ for single charged (on left) and equal charged case. The different colors in the left plot, red, black, blue, green, magenta and purple correspond to $\omega = 10^{-10}$, 0.2, 0.4 , 0.6, 0.8 and 1 respectively. The colours in the right plot, red, black, blue, green and magenta correspond to $\omega = 10^{-10}$, 0.4 , 0.6, 0.8 and 1 respectively. σ is in units of $(16\pi G_3)^{-1}$.

can't go to lower values of $1/\mathsf{m} < 1/\mathsf{m}_c$. We note that for both the single charged case and equal charged case, the conductivity saturates to 1, as temperature increases. This is expected from our uncharged brane analysis. As $\mathsf{m} \to 0$, the behaviour of DC conductivity is

$$\begin{aligned} 16\pi G_3 \sigma_{DC} &\to 1 + \frac{15}{2}\mathsf{m}^2 + \cdots \quad \text{for single charged case,} \\ & 1 - 24\mathsf{m}^2 + \cdots \quad \text{for equal charged case.} \end{aligned} \tag{6.12}$$

The low temperature behaviour in equal charged case is $\sigma_{DC} \sim \mathsf{m}^{-2} \sim T^2$ for $\mathsf{m}^{-1} \to 0$.

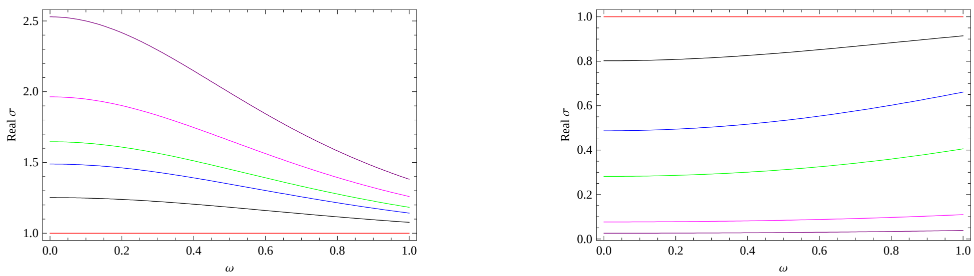

Figure 6: Plots of real part of conductivity vs ω for the single charged (on left) and equal charged case. The different colours in the left plot, red, black, blue, green, magenta and purple correspond to $\mathsf{m} = 10^{-3}$, 0.15, 0.18, 0.19, 0.2, $2\sqrt{6}$ respectively. The different colours in the right plot, red, black, blue, green, magenta, purple and orange correspond to m =10^{-3}, 0.1, 0.2, 0.3, 0.6, 1 and 100 respectively. σ is in units of $(16\pi G_3)^{-1}$ and ω is in units of $2r_H^2/L^3$.

In figure (6), we show the dependence of conductivity on frequency for various fixed values of m for the single and equal charged case. Here, the behaviour of the curves are in contrast with each other in two cases. While for the single charged case, we find the curves fit well with the expression $\sim a(\mathsf{m}) + \frac{b(\mathsf{m})}{c(\mathsf{m})+\omega^2}$ for some ω

independent functions $a(\mathsf{m})$, $b(\mathsf{m})$ and $c(\mathsf{m})$ of m. On the right, we see that the conductivity increases for intermediate values of m as ω is increased.

6.3 The relation to the M2-brane theory

It has been observed that the thermodynamic properties of the near horizon geometry of M2-branes is similar to that of the D1-branes [28, 29]. We now recall the thermodynamic properties of uncharged M2-branes and compare them to uncharged D1-branes. These properties were obtained from [18]. The near horizon geometry of M2-branes is AdS_4 times S^7, let the radius of S^7 be L' and the Newton's constant in 4 dimensions be G_4. The thermodynamic properties of non-extremal uncharged D1-branes and non-extremal uncharged M2-branes with non-extremal parameter r_0 is given by:

	D1-branes	M2-branes
s	$\frac{1}{4G_3}\frac{r_0^4}{L^4}$	$\frac{1}{4G_4}\frac{r_0^4}{L'^4}$
T	$\frac{3}{2\pi}\frac{r_0^2}{L^3}$	$\frac{3}{2\pi}\frac{r_0^2}{L'^3}$
ϵ	$\frac{1}{4\pi G_3}\frac{r_0^6}{L^7}$	$\frac{1}{4\pi G_4}\frac{r_0^6}{L'^7}$
$p=-f$	$\frac{1}{2}\epsilon$	$\frac{1}{2}\epsilon$

Table 2. Thermodynamics of uncharged D1-branes and M2-branes.

From the equation of state $p=\frac{\epsilon}{2}$, it seems that the non-conformal D1-brane theory behaves as though it is a conformal theory in $2+1$ dimensions.

This similarity of thermodynamic properties of uncharged D1-branes and M2-branes was also seen to extend to the transport properties. In [7], it was noted that the bulk viscosity to entropy density of non-extremal D1-branes is given by $1/4\pi$. This fact was explained by the observation in [30]. Consider conformal hydrodynamics of a charged fluid in $2+1$ dimensions [7]. The stress tensor and the current are

[7] [30] considered the case of the uncharged fluid and obtained a general relation between conformal hydrodynamics between 2σ dimensions and non-conformal hydrodynamics in d dimensions. The case when $\sigma = 3/2, d = 2$ corresponds to the relation between D1-branes and M2-branes. In general, the relation found in [30] relates conformal hydrodynamics in fractional dimensions to non-conformal hydrodynamics in integer dimensions.

given by

$$\begin{aligned}\tilde{T}^{ab} &= \tilde{\epsilon} u^a u^b + \tilde{p}(\eta^{ab} + u^a u^b) - 2\eta\sigma^{ab}, \\ \tilde{j}^a &= \tilde{\rho} u^a - \tilde{\sigma} T(\eta^{ab} + u^a u^b)\partial_b \left(\frac{\mu}{T}\right),\end{aligned} \tag{6.13}$$

where $a, b \in \{0, 1, 2\}$, η_{ab} is the Minkowski metric in $2+1$ dimensions and

$$\sigma_{ab} = P_a^c P_b^d \partial_{(c} u_{d)} - \frac{1}{2} P_{ab} \partial \cdot u, \qquad P_{ab} = \eta_{ab} + u_a u_b. \tag{6.14}$$

Let us now dimensionally reduce these equations with the ansatz $u^a = (u^\mu, 0)$ where $\mu \in \{0, 1\}$ along with the assumption that there is no dependence along the direction 2 for any thermodynamic variable. Then the non-trivial components of the stress tensor and the current can be written as

$$\begin{aligned}\tilde{T}^{\mu\nu} &= (\tilde{\epsilon} + \tilde{p}) u^\mu u^\nu + \tilde{p} g^{\mu\nu} - 2\eta\tilde{\sigma}^{\mu\nu} - \eta P^{\mu\nu} \partial \cdot u, \\ \tilde{T}^{2\mu} &= \tilde{T}^{22} = 0, \\ \tilde{j}^\mu &= \tilde{\rho} u^\mu - \tilde{\sigma} T(g^{\mu\nu} + u^\mu u^\nu)\partial_\nu \left(\frac{\mu}{T}\right), \\ j^2 &= 0,\end{aligned} \tag{6.15}$$

where

$$\tilde{\sigma}_{\mu\nu} = P_\mu^\rho P_\nu^\sigma \partial_{(\rho} u_{\sigma)} - P_{\mu\nu} \partial \cdot u = 0. \tag{6.16}$$

To show the above expression vanishes, one can explicitly evaluate the components or else use the fact that it is a traceless symmetric tensor in $1+1$ dimensions and is orthogonal to the velocity vector u^μ. Thus the stress tensor and the charge current in $1+1$ dimensions is given by

$$\begin{aligned}T^{\mu\nu} &= R\tilde{T}^{\mu\nu} = (R\tilde{\epsilon} + R\tilde{p}) u^\mu u^\nu + R\tilde{p}\eta^{\mu\nu} - R\eta P^{\mu\nu} \partial \cdot u, \\ j^\mu &= R\tilde{j}^\mu = R\tilde{\rho} u^\mu - R\tilde{\sigma} T(\eta^{\mu\nu} + u^\mu u^\nu)\partial_\nu \left(\frac{\mu}{T}\right),\end{aligned} \tag{6.17}$$

where R is the radius of compactification. On comparing this form of the stress tensor to that given in (3.1) we see that we can identify

$$\epsilon = R\tilde{\epsilon}, \qquad p = R\tilde{\epsilon}, \qquad \sigma = R\tilde{\sigma}, \qquad \zeta = R\eta. \tag{6.18}$$

The entropy density $\tilde{s}$ in $2+1$ dimensions is related to the entropy density in $1+1$ dimensions by

$$s = R\tilde{s}. \tag{6.19}$$

From this, we can conclude that for a fluid dynamics in $1+1$ dimensions, which is related by compactification on a circle of radius R to conformal hydrodynamics in $2+1$ dimensions, the relation

$$p = \frac{\epsilon}{2} \tag{6.20}$$

will continue to hold true due to (6.18). Furthermore, we have

$$\frac{\zeta}{s} = \frac{\eta}{\tilde{s}}. \tag{6.21}$$

Thus the ratio of bulk viscosity to entropy density in $1+1$ dimensions is identical to the ratio of shear viscosity to entropy density of the conformal $2+1$ hydrodynamics.

In [30], it was shown that the equations of gravity fluctuations for the uncharged D1-brane which determine the hydrodynamical transport coefficients is a dimensional reduction of the gravity fluctuations of the uncharged M2-brane background. This fact and (6.21) explains the reason why the ratio of bulk viscosity to entropy density for the D1-brane is given by $1/4\pi$. It also explains the fact that speed of sound for the D1-brane theory is same as that of the M2-brane theory. One expects this argument to go through for the charged D1-branes and this is the reason we observe that the speed of sound is $1/\sqrt{2}$ and the bulk viscosity to entropy density is $1/4\pi$. As an evidence for this argument, we will now show that the 3 dimensional truncated action given in (A.19) which supports the equal charged D1-brane solution can be obtained by dimensional reduction of the following 4 dimensional action.

$$S = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g_4}\left(R_4 + \frac{6}{L'^2} - L'^2 F^{\mu\nu}F_{\mu\nu}\right), \tag{6.22}$$

where g_4 and R_4 are the 4 dimensional metric and the Ricci curvature respectively. G_4 in the four dimensional Newton's constant and L' is the radius of AdS_4. This is the action which admits the solution of the equal charged M2-brane. Note that the near horizon geometry of the equal charged M2-brane is just a Reissner-Nordström black hole in AdS_4. We address the equal charged case since the single charged M2-brane has not been studied in the literature. We now compactify the action in (6.22) using the following ansatz

$$\begin{aligned} ds^2 &= ds^2_{(2+1)} + e^{-\frac{4}{3}\phi}dy^2, \\ A_y &= 0. \end{aligned} \tag{6.23}$$

As usual, all fields do not have any dependence on the compact direction y. Substituting this ansatz in the action (6.22), we obtain

$$S = \frac{2\pi R_y}{16\pi G_4}\int d^3x\sqrt{-\tilde{g}_3}e^{-\frac{2}{3}\phi}\left(\tilde{R}_3 + \frac{6}{L'^2} - L'^2F^{\mu\nu}F_{\mu\nu}\right). \tag{6.24}$$

To bring the action in the Einstein form, we perform the following re-definition

$$\tilde{g}_{\mu\nu} = e^{\frac{4}{3}\phi}g_{\mu\nu}. \tag{6.25}$$

We then obtain

$$S = \frac{2\pi R_y}{16\pi G_4}\int d^3x\sqrt{-g_3}\left(R_3 - \frac{8}{9}(\partial\phi)^2 + \frac{6}{L'^2}e^{\frac{4}{3}\phi} - L'^2e^{\frac{-4}{3}\phi}F^{\mu\nu}F_{\mu\nu}\right). \tag{6.26}$$

Now comparing (A.19) and the above action, we see that they are the same on identifying

$$\Phi = \exp\left(\frac{2}{3}\phi\right), \qquad L' = \frac{L}{2}, \qquad A_\mu \to \frac{A_\mu}{L'}. \tag{6.27}$$

This observation indicates that the supergravity fluctuations which determine the transport coefficients of the equal charged D1-brane theory can be obtained by dimensional reduction of the fluctuations which determine the transport coefficient of the equal charged M2-brane theory. As a result, the transport coefficients of the M2-brane theory is related to that of the D1-brane theory.

Finally, we mention that from (6.18) the conductivity of the M2-brane theory is related to that of the M2-brane theory. The conductivity of the equal charged M2-brane theory has been evaluated in [15] [8] and is given by

$$\sigma_{\rm M2} = \frac{1}{16\pi G_4}\frac{(3-k)^2}{9(1+k)}. \tag{6.28}$$

Note that apart from the dimensions set by G_4, the dependence of the conductivity is identical to that of the equal charged D1-brane theory given in (B.16). In table 1, we have compared the transport properties of the equal charged M2-brane and the equal charged D1-brane.

7. Conclusions

In this paper, we have studied the transport properties of the $1+1$ dimensional $SU(N)$ gauge theory with 16 supercharges of the D1-branes at finite chemical potential in the framework of the gauge/gravity duality. We evaluated the bulk viscosity, electrical conductivity , thermal conductivity, the charge and sound diffusivity for two cases. One in which the chemical potential conjugate to one of the $U(1)$ R-charges is turned on and another in which equal charges conjugate to all the 4 Cartans of the $SO(8)$ R-symmetry are turned on. In both the situations, we find that the ratio of bulk viscosity to the entropy density is independent of the chemical potential and is equal to $1/4\pi$. We showed that for the single charged D1-brane theory, the charge dissipative mode becomes unstable and the transport properties exhibit critical behaviour at the boundary of thermodynamic instability. We also demonstrated that the shear viscosity and thermal conductivity satisfy a relationship similar to the Wiedemann-Franz law. We have observed that the transport coefficients of the D1-branes theory is same as that of the M2-brane theory apart from an overall normalization which determines the dimensions and suggests a plausible reason for this behaviour. The summary of the transport coefficients obtained in this paper and their comparison with the transport coefficients of the M2-brane theory is given in

[8] see equation (83) of [15] and identify q^2 as k

table 1. A technical result of our analysis is the following: we reduced the problem of solving the second order differential equation which determines the transport coefficient to a first order non-linear differential equation. This equation governs the radial evolution of the transport coefficient. We were able to solve these equations analytically for the transport coefficients of interest in this paper.

A possible extension of this work is to compute the transport coefficients when all the 4 chemical potentials corresponding to the 4 Cartans of the $SO(8)$ R-symmetry are turned on. This would provide a complete knowledge of the transport coefficients of the D1-brane gauge theory. It will also be interesting to understand the thermal stability of the full system with all the R-charges turned on. Another direction is to understand the connection of the D1-brane theory with that of the M2-brane theory better. This would involve an analysis similar to [8]. We need to show that the hydrodynamic fluctuations in gravity which determine the transport for the charged M2-brane and D1-brane are related by compactification. From the point of the view of the theories of the M2-branes and D1-branes, it is interesting to note that unlike the presently unknown theory of the M2-branes, the theory of the D1-brane is a regular gauge theory in $1+1$ dimensions. We have seen that the D1-brane gauge theory provides physical information regarding the M2-brane theory. It is worthwhile to explore and utilize this fact to understand the M2-brane theory further.

1+1 relativistic hydrodynamics occurs in the short time description of the plasma formed after highly relativistic collisions [31]. The equation of state of this plasma does not obey $p = \epsilon/2$, however it will be interesting to see if the transport properties of this plasma show the behaviour seen here. Another area where relativistic 1+1 hydrodynamics could be important is in carbon nano-tubes and graphene nano-ribbons. These materials can be described as a graphene layer rolled up and a graphene layer whose linear dimensions is much larger than that of its width respectively [32, 33]. These systems are relativistic since they are obtained by a dimensional reduction of $2+1$ dimensional graphene which is described by a massless Dirac equation. It will be interesting to compare the transport properties of these materials with that of the field theory studied here. The system we study here has a gap set by the Yang-Mills coupling. Hydrodynamics of other 1+1 dimensional systems with a gap have been studied in [34, 35] [9]. Even though we have analysed only a bosonic system, we can think of it describing a 1+1 dimensional condensed matter system or a quasi 1+1 dimensional system made up of strongly interacting bosonic quasiparticles which are themselves made up of elementary electrons, just like Cooper pairs. It would be interesting to evaluate an effective Lagrangian for such quasiparticles from the action of the gauge theory dual to our gravity system and then compare it to effective action for one dimensional effective condensed matter systems like Luttinger liquids. A curious observation is that our plots for conductivity vs temperature and

[9]We thank Subir Sachdev for bringing these references to our notice.

frequency for equal charged case qualitatively looks similar to a system of carbon nanotubes-polyepoxy composites [36].

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A. Consistent truncation to 3 dimensions

We first show that the solution (2.6) in 3 dimensions is a consistent truncation of the spinning D1-brane solution in 10 dimensions given in (2.2). For this, we use the results of [19] who gave the most general ansatz for the consistent Kaluza-Klein reduction of a 10 dimensional solution on the seven sphere [10]. The ansatz is as follows:

$$\begin{aligned} ds_{10}^2 &= Y^{\frac{1}{8}}\left[\Delta_C^{\frac{3}{4}} ds_3^2 + g^{-2}\Delta_C^{-\frac{1}{4}} T_{ij}^{-1}\mathcal{D}\mu^i\mathcal{D}\mu^j\right], \\ e^{-2\phi} &= \Delta_C^{-1} Y^{1/2}, \\ \hat{F}_{(3)} &= F^1 + F^2 + F^3. \end{aligned} \tag{A.1}$$

where [11]

$$\begin{aligned} F^1 &= gU\epsilon_3, \\ F^2 &= g^{-1}T_{ij}^{-1} * \mathcal{D}T_{jk}\wedge(\mu^k\mathcal{D}\mu^i), \\ F^3 &= -\frac{1}{2g^2}T_{ik}^{-1}T_{jl}^{-1} * F_{(2)}^{ij}\wedge\mathcal{D}\mu^k\wedge\mathcal{D}\mu^l, \\ \mathcal{D}\mu^i &= d\mu^i + gA^{ij}\mu^j, \\ \mathcal{D}T_{ij} &= dT_{ij} + gA^{ik}T_{kj} + gA^{jk}T_{ki}, \\ F_{(2)}^{ij} &= dA^{ij} + gA^{ik}\wedge A^{kj}, \end{aligned} \tag{A.2}$$

and

$$\mu^i\mu^i = 1 \qquad \Delta_C = T_{ij}\mu^i\mu^j \qquad U = 2T_{ik}T_{jk}\mu^i\mu^j - \Delta_C T_{ii} \qquad Y = \det(T_{ij}). \tag{A.3}$$

[10] See section 5. of [19].

[11] Note that the sign of F^1 here is negative of that in [19], this is a result of a different convention for the volume form ϵ_3.

$*$ is the Hodge dual in the three dimensions. The μ's are defined as follows,

$$\begin{aligned} \nu_1 &= \sin\theta, & \nu_2 &= \cos\theta\sin\psi_1, \\ \nu_3 &= \cos\theta\cos\psi_1\sin\psi_2, & \nu_4 &= \cos\theta\cos\psi_1\cos\psi_2, \\ \mu_{2a-1} &= \nu_a\sin\phi_a, & \mu_{2a} &= \nu_a\cos\phi_a. \end{aligned} \tag{A.4}$$

Here, $a = 1,\cdots 4$ and $i,j = 1,\cdots 8$. Then [19] shows that on substituting the above ansatz in to the ten dimensional equations of motion, there is a consistent reduction to the equations of motion for the three dimensional fields. The equations of motion for the three dimensional fields can be derived from the following three-dimensional Lagrangian:

$$\begin{aligned} \mathcal{L} &= R*1 - \frac{1}{32}Y^{-2}*dY\wedge dY - \frac{1}{4}\tilde{T}^{-1}_{ij}*\mathcal{D}\tilde{T}_{jk}\wedge\tilde{T}^{-1}_{kl}\wedge\mathcal{D}\tilde{T}_{li} \\ &-\frac{1}{4}Y^{-1/4}\tilde{T}^{-1}_{ik}\tilde{T}^{-1}_{jl}*F^{ij}_{(2)}\wedge F^{kl}_{(2)} - \frac{g^2}{2}Y^{1/4}\{2\tilde{T}_{ij}\tilde{T}_{ij} - (\tilde{T}_{ii})^2\}*1, \end{aligned} \tag{A.5}$$

where $\tilde{T}_{ij} = Y^{-1/8}T_{ij}$.

Single charged D1-brane

We now show that the spinning D1-brane solution in 10 dimensions given in (2.2) can be written in the form given in (A.1). For this, we choose

$$\begin{aligned} g &= \frac{1}{L}, \qquad A^{12} = -\frac{r_0^3 l}{L^2r^2H}dt, \\ H &= 1+\frac{l^2}{r^2}, \qquad T_{ij} = X_{(i)}\delta_{ij}, \\ X_{(i)} &= \frac{L^2}{r^2H} \quad (i=1,2), \qquad X_{(i)} = \frac{L^2}{r^2} \quad (i\neq 1,2), \\ \epsilon_{tzr} &= 1. \end{aligned} \tag{A.6}$$

For convenience, we also write down the following

$$\begin{aligned} \Delta_C &= \frac{L^8}{r^8}\frac{1}{HH_1}, \qquad H_1 = \frac{L^6}{\Delta r^6}, \\ \Delta &= 1+\frac{l^2}{r^2}\cos^2\theta, \qquad Y = \frac{L^{16}}{r^{16}}\frac{1}{H^2} \\ U &= -2\frac{L^4}{r^4H}\left(3+\frac{2l^2}{r^2}\cos^2\theta\right). \end{aligned} \tag{A.7}$$

This results in the metric

$$\begin{aligned} ds^2 &= H_1^{-3/4}(-fdt^2+dz^2) + H_1^{1/4}\left(\frac{1}{\tilde{h}}dr^2 + r^2(\Delta d\theta^2 + \tilde{\Delta}\sin^2\theta d\phi^2 + \cos^2\theta d\Omega_5^2)\right) \\ &-2H_1^{-3/4}\frac{L^3r_0^3}{\Delta r^6}l\sin^2\theta dt d\phi, \end{aligned} \tag{A.8}$$

which is same as that given by equation (2.2). Note that the exponent of dilaton is negative of that given in main text. This is just due to difference in conventions between [18] and [19]. The dilaton and the three form are given by

$$
\begin{aligned}
e^{\phi} &= H_1^{-1/2}, \\
\hat{F}_{(3)} &= -2\frac{r^5}{L^6}\left(3+\frac{2l^2}{r^2}\cos^2\theta\right)dt\wedge dz\wedge dr + 2\sin\theta\cos\theta\frac{l^2r^4}{L^6}dt\wedge dz\wedge d\theta \\
&\quad -2\sin\theta\cos\theta\frac{r_0^3 l}{L^3}dz\wedge d\theta\wedge d\phi.
\end{aligned}
\tag{A.9}
$$

This also agrees with the expression for the dilaton modulo the sign and the two form gauge potential given in (2.2). One can also check that by reading out the three dimensional metric by comparing (A.1) to (A.8) , we obtain the three dimensional truncated solution in (2.6). We can now proceed to obtain the three dimensional Lagrangian for the Kaluza-Klein ansatz in (A.6). We first define the scalars $Z_1 = Y^{-1/8}X_1 = Y^{-1/8}X_2$ and $Z_2 = Y^{-1/8}X_j$ for $j \neq 1,2$. Then the Lagrangian in (A.5) reduces to

$$
\begin{aligned}
\mathcal{L} = \sqrt{-g}\Bigg[&R - \frac{1}{32Y^2}\partial_\mu Y\partial^\mu Y - \frac{1}{2}\left(\frac{1}{Z_1^2}\partial_\mu Z_1\partial^\mu Z_1 + \frac{3}{Z_2^2}\partial_\mu Z_2\partial^\mu Z_2\right) \\
&-\frac{1}{4Y^{1/4}}\frac{1}{Z_1^2}F_{\mu\nu}F^{\mu\nu} + \frac{12}{L^2}Y^{1/4}Z_2(Z_1+Z_2)\Bigg].
\end{aligned}
\tag{A.10}
$$

On identifying

$$
Z_1 = \Psi^{-3/4}, \qquad Z_2 = \Psi^{1/4}, \qquad Y^{1/4} = \frac{e^{\frac{4}{3}\phi}}{\Psi^{1/2}},
\tag{A.11}
$$

the above action reduces to the one given by equation (2.8).

Equal charged D1-brane

We now wish to obtain the truncated 3 dimensional solution as well as the action when one turns on equal charges along the 4 Cartans of the $SO(8)$ R-symmetry. We start with the 10 dimensional D1-brane solution with equal spins along the 4 Cartan's. This is given by [18].

$$
\begin{aligned}
ds^2 &= H_2^{-3/4}(-fdt^2+dz^2) + H_2^{1/4}\left(\frac{dr^2}{hf} + \Lambda_{\alpha\beta}d\eta^\alpha d\eta^\beta\right) \\
&\quad -2\frac{H_2^{-3/4}}{h^3}\frac{L^3r_0^3}{r^6}l\sum_{i=1}^{4}\nu_i^2 dt d\phi_i, \\
A_2 &= -\left(H_2^{-1}dt + \frac{r_0^3}{L^3}l\sum_{i=1}^{4}\nu_i^2 d\phi_i\right)\wedge dz,
\end{aligned}
$$

$$
\begin{aligned}
e^{\phi} &= H_2^{1/2}, \qquad H_2 = \frac{L^6}{r^6 h^3}, \\
h &= 1 + \frac{l^2}{r^2}, \qquad f = 1 - \frac{r_0^6}{h^3 r^6}, \qquad \bar{f} = 1 - \frac{r_0^6}{h^4 r^6} \\
\Lambda_{\alpha\beta} d\eta^\alpha d\eta^\beta &= r^2 h [d\theta^2 + \cos^2\theta d\psi_1^2 + \cos^2\theta \cos^2\psi_1 d\psi_2^2 + \sin^2\theta d\phi_1^2 \\
&\quad + \cos^2\theta \sin^2\psi_1 d\phi_2^2 + \cos^2\theta\cos^2\psi_1 \sin^2\psi_2 d\phi_3^2 \\
&\quad + \cos^2\theta \cos^2\psi_1 \cos^2\psi_2 d\phi_4^2].
\end{aligned} \tag{A.12}
$$

We will now compare the 10 dimensional solution with the form of the Kaluza-Klein ansatz given in (A.1). For this, we first assume that the three dimensional metric is of the form

$$
ds_3^2 = Z\left[-\hat{f} dt^2 + dz^2 + \frac{H_2}{h\bar{f}} dr^2\right], \tag{A.13}
$$

and

$$
T_{ij} = \Phi \delta_{ij}, \qquad A^{ij} = a(r)\sigma^{ij} dt, \tag{A.14}
$$

where $\sigma^{2a-1,2a} = -\sigma^{2a,2a-1} = 1$ and zero otherwise. μ_i are given in (A.4). With this ansatz, the gauge field is given as

$$
\begin{aligned}
F_3 &= gUZ\left(\frac{L^3}{r^3 h^2}\sqrt{\frac{Z\hat{f}}{\bar{f}}}\right) dt \wedge dz \wedge dr \\
&\quad + \frac{a'}{g^2\Phi^2}\left(\frac{r^3 h^2}{L^3}\sqrt{\frac{\bar{f}}{Z\hat{f}}}\right) dz \wedge d\phi_a \wedge \nu_a d\nu_a.
\end{aligned} \tag{A.15}
$$

Comparing with field strength of solution in (A.12), we get

$$
\frac{L^3}{r^3 h^2}\sqrt{\frac{Z\hat{f}}{\bar{f}}} = \frac{1}{g\Phi^2 r h Z H_2}, \qquad \frac{a'}{g} Z = 2\frac{r_0^3 l}{L^9} r^5 h^2. \tag{A.16}
$$

By comparing the metric in (A.1) and the spinning D1-branes solution (A.12), we get

$$
\begin{aligned}
\Phi^{7/4} Z\hat{f} - \frac{a^2}{\Phi^{1/4}} &= H_2^{-3/4} f, \qquad \Phi^{7/4} Z = H_2^{-3/4}, \\
g^2 \Phi^{1/4} r^2 h &= H_2^{-1/4}, \qquad \frac{a}{g\Phi^{1/4}} = -\frac{H_2^{-3/4}}{h^3}\frac{lL^3 r_0^3}{r^6}.
\end{aligned} \tag{A.17}
$$

A solution to the equations in (A.16) and (A.17) is given by

$$
\begin{aligned}
g &= L^{-1}, \qquad \Phi^{-1} = g^2 r^2 h, \\
a &= -\frac{g^2 r_0^3 l}{r^2 h}, \qquad Z = (g^2 r^2 h)^4, \\
\hat{f} &= \bar{f}.
\end{aligned} \tag{A.18}
$$

Now using these equations, the effective 3-dimensional action as given by equation (A.5) reduces to

$$\mathcal{L}_3 = \sqrt{-g}\left[R - \frac{2}{\Phi^2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{\Phi^2}F_{\mu\nu}F^{\mu\nu} + 24\frac{\Phi^2}{L^2}\right], \tag{A.19}$$

with the 3 dimensional solution

$$\begin{aligned}
ds_3^2 &= \frac{h^4}{L^8u^4}\left[-fdt^2 + dz^2 + \frac{L^6}{4h^4f}du^2\right],\\
h &= 1 + ku,\\
A &= -\frac{r_0^3 l}{L^2 h}udt,\\
f &= 1 - \frac{r_0^6u^3}{h^4},\\
\Phi &= \frac{L^2u}{h}.
\end{aligned} \tag{A.20}$$

We have changed the radial variable to $u = \frac{r_H^2}{r^2}$, where r_H is the radius of the horizon. Here, $k = \frac{l^2}{r_H^2}$. We have divided by r_H to turn some quantities like L, r_0 above to be dimensionless. The variable f above is same as $\bar{f}$ in earlier part of the analysis. The radius of the seven sphere is L. Parameter r_0 is related to k and the radius of the horizon as

$$r_0^6 = (1+k)^4 r_H^6. \tag{A.21}$$

Note that there is no extra scalar in this case. The general compactification given in [19] contains 36 scalars, one singlet under SO(8) and the rest which transforms as **35**. In this equal charged case, we turn on only the singlet which is the dilaton. In the single charged case, one more scalar Ψ is turned on and this explicitly breaks the $SO(8)$ symmetry.

B. Transport coefficients for the equal charged D1-brane

In this part of the appendix, we evaluate the conductivity and bulk viscosity for the equal charged D1-brane. We will be brief here since we have provided the details for the single charged D1-brane in the main text.

We first provide a table listing the thermodynamic properties of the equal charged D1-brane

Hawking Temperature(T)	$\frac{r_0^3}{4\pi L^3 r_H}(\frac{6-2k}{1+k})$
Entropy Density(s)	$\frac{1}{4G_3}\frac{r_0^3 r_H}{L^4}$
Energy Density(ϵ)	$\frac{1}{4\pi G_3}\frac{r_0^6}{L^7}$
Pressure ≡(-free energy density(f))	$\frac{1}{8\pi G_3}\frac{r_0^6}{L^7} = \frac{\epsilon}{2}$
Charge Density(ρ)	$\frac{r_0^3 l}{8\pi G_3 L^5}$
Chemical Potential(μ)	$\frac{l r_H(1+k)}{L^2}$

Table 3. Thermodynamic properties of the equal charged D1-brane

In evaluating the above thermodynamic quantities, we have used the relation in (A.21). Note that the Hawking temperature of this black hole is given by

$$T = \frac{r_0^3}{4\pi L^3 r_H}\left(\frac{6-2k}{1+k}\right). \tag{B.1}$$

From this expression, we see that the black hole is stable only for

$$k < 3. \tag{B.2}$$

We can also examine the Hessian to see if the equal charged solution admits the thermodynamic instability seen in the case of the single charged solution. Using the expression of the Hessian given in (2.23), we obtain

$$H_s = \frac{8G_3^2 L^4(k+3)}{r_H^4(1+k)^2}. \tag{B.3}$$

Since $k \geq 0$, the Hessian for this case is always positive and therefore this solution does not exhibit the usual thermodynamic instability. Thus the range of the allowed values of k is $0 < k < 3$. This is the same range found for the case of equal charged M2-branes [37]. Using the expressions for the thermodynamic variables given in table 3, we can evaluate the relationship between the charge diffusion constant and the conductivity from the formula in (3.14). It is given by

$$D_c = \sigma(16\pi G_3)\frac{3(k+3)}{2r_H^2(3-k)^2}. \tag{B.4}$$

Hydrodynamic modes from gravity

To obtain the two hydrodynamic modes of the charged fluid from gravity, we analyze linearized wave like perturbations in the background of the equal charged D1 brane solution given in (A.20). It is a solution of the action given in (A.19). The perturbations are defined as follows:

$$
\begin{aligned}
g_{tt} &\to g_{tt}^0(1+H_{tt}), & g_{tz} &\to g_{zz}^0 H_{tz}, \\
g_{zz} &\to g_{zz}^0(1+H_{zz}), & A_t &\to A_t^0 + \frac{lr_0^3}{L^2}B_t, \\
A_z &\to \frac{lr_0^3}{L^2}B_z, & \Phi &\to \Phi^0 + L^2\varphi.
\end{aligned}
\tag{B.5}
$$

where the superscript '0' refers to the background values. Due to translational invariance along the t and the z directions, we can assume that the dependence of the perturbations along these directions is of the form as $\sim \exp[\frac{2i}{L^3}(-\omega t + qz)]$. Note that here we will be using the dimensionless variables defined in (4.20).

To write the gauge invariant modes, we first introduce the following functions

$$
\begin{aligned}
V &= q^2(4h^3 - r_0^6 u^3) - 4\omega^2 h^3, \\
\alpha &= q^2 f - \omega^2.
\end{aligned}
\tag{B.6}
$$

The two gauge invariant variables which are invariant both under diffeomorphism as well as $U(1)$ gauge transformations are given by

$$
\begin{aligned}
Z_P &= -q^2 f H_{tt} + 2\omega q H_{tz} + \omega^2 H_{zz} + \frac{V}{h^2 u}\varphi, \\
G_P &= qB_t + \omega B_z + q\varphi.
\end{aligned}
\tag{B.7}
$$

These gauge invariant variables satisfy the following equations of motion

$$
\begin{aligned}
V\alpha f u Z_P'' &- 2h^2(q^2-\omega^2)\{q^2(h^2+5h-12) - 2h\omega^2\}(1-f)Z_P' \\
&+ q^2h^2(1-f)^2\{q^2(2h^2+2h-8) - w^2(h^2+6h-8) + q^2 fh(3h-8)\}Z_P' \\
&+ 8(h-2)h^2(q^2-\omega^2)^2 Z_P' - 2qkr_0^6\frac{u^3V^2f}{h^7}G_P' \\
&= -8kqr_0^{12}\frac{u^5(4-h)}{h^6}[3q^4f + 4\omega^4 - q^2\omega^2(4+3f-h+fh)]G_P \\
&+ \frac{u\alpha}{fh}[4(q^2-\omega^2)^2 - q^2(1-f)(h+4)(q^2-\omega^2) \\
&+ q^2h(1-f)^2(3u^{-2}h^2f(h-4)+q^2)]Z_P,
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
G_P'' &+ \frac{qh}{V\alpha}\{q^2(2+h-hf) - 2\omega^2\}Z_P' + \left[\frac{2k}{h\alpha}(q^2-\omega^2) + \frac{\omega^2(4-h)(1-f)}{\alpha h f u}\right]G_P' \\
&= \frac{3qr_0^6u^2}{h^3 fV}Z_P + \frac{4kh(1-f)}{fuV\alpha}\{6q^4f - q^2\omega^2(4+6f-h+fh) + 4\omega^4\}G_P \\
&+ \frac{\alpha}{h^4f^2}G_P.
\end{aligned}
\tag{B.9}
$$

Note that these equations decouple in the $q \to 0$ limit.

Conductivity for the equal charged case

In the limit $q = 0$, the equation for the gauge invariant current mode is

$$G_P''(u) + \frac{(hf + h + 2f - 4)}{fhu} G_P'(u) + \frac{\omega^2}{h^4 f^2} G_P(u) - \frac{4k(1-f)}{h^2 fu} G_P(u) = 0. \tag{B.10}$$

We can remove the coefficient of G_P proportional to $(1-f)$ by the following redefinition

$$G_P = \frac{4-h}{h} G. \tag{B.11}$$

As we have seen for the case of the single charged solution conductivity is essentially determined by the ratio

$$\mathcal{R}_{G_P} = \frac{1}{i\omega G_P} \frac{dG_P}{du}, \tag{B.12}$$

which in turn is determined by the ratio $g = \frac{G_P'(u)}{i\omega G_P(u)}$. Ingoing boundary conditions at the horizon for G_P corresponds to the following boundary condition on g

$$\lim_{u \to 1} g(u) = \frac{1}{(3-k)(1+k)(1-u)}. \tag{B.13}$$

The equation of motion satisfied by g is given by

$$g' + i\omega g^2 - \left\{ \frac{(4-h)(1-f)}{fhu} + \frac{2k}{(4-h)} \right\} g - i \frac{\omega L^6}{4h^4 f^2} = 0. \tag{B.14}$$

The solution for the real part of g in $\omega \to 0$ limit is

$$\mathrm{Re}\, g = \frac{(3-k)^2}{(1+k)^2} \frac{1}{(4-h)^2 f}. \tag{B.15}$$

Just as in the previous case for the single charged, the real part of DC conductivity here is proportional to the value of $\mathrm{Re}\mathcal{R}_{G_P}$ at the boundary $(u = 0)$, which is given by

$$\mathrm{Re}\, \mathcal{R}_{G_P} = \frac{L^3}{2} \frac{(3-k)^2}{9(1+k)^2}. \tag{B.16}$$

Note that here we have reinstated the factors of $L^3/2$ which we have absorbed in defining ω. The imaginary part of $\mathrm{Re}\mathcal{R}_{G_P}$ is given by

$$\mathrm{Im}\mathcal{R}_{G_P} = \frac{L^3}{2} \mathrm{Im} \left[\frac{d}{du} \ln \frac{4-h}{i\omega h} \right]_{u \to 0} = \frac{L^3}{2} \frac{4k}{3\omega}. \tag{B.17}$$

In Figure 3. we have compared these expressions with that determined by numerically solving the equation for G_P. We find that they agree to less than one part in 10^{-6}.

The DC conductivity is related to $\mathrm{Re}G_P$ by a proportionality constant which can be determined from the boundary effective action as was done in the single charged case in section 5.3. This results in

$$\sigma = \frac{1}{16\pi G_3}\frac{(3-k)^2}{9(1+k)^2}. \tag{B.18}$$

Bulk viscosity for the equal charged case

In this section, we calculate the bulk viscosity for the equal charged case and show that the ratio $\frac{\zeta}{s}$ is constant. In the $q=0$ limit, the equation for the sound mode decouples from the current mode. It turns out to be

$$Z_P''(u) + \frac{2(h-2)}{fhu}Z_P'(u) - \frac{(1-f)}{fu}Z_P'(u) + \frac{\omega^2}{h^4 f^2}Z_P(u) = 0. \tag{B.19}$$

Let us define the ratio

$$g = \frac{Z_P'(u)}{i\omega Z_P(u)}. \tag{B.20}$$

Since the sound mode satisfies ingoing boundary condition at the horizon, the function g should satisfy

$$\lim_{u\to 1} g = \frac{1}{(3-k)(1+k)(1-u)}. \tag{B.21}$$

The appropriate solution for g in $\omega \to 0$ limit is

$$\mathrm{Re}\, g = (1+k)^2\frac{u^2}{h^4 f}. \tag{B.22}$$

The bulk viscosity is proportional to the real part of the following ratio evaluated at the horizon:

$$\mathcal{R}_{Z_P} = \frac{1}{3i\omega u^2 Z_P}Z_P'(u). \tag{B.23}$$

We can evaluate this from the expression for $\mathrm{Re}\, g$ in the $\omega \to 0$ limit which results in

$$\mathrm{Re}\,\mathcal{R}_{Z_P}|_{u\to 0,\omega\to 0} = \frac{L^3}{2}\frac{(1+k)^2}{3}. \tag{B.24}$$

Here we have reinstated the factor of $L^3/2$ which we have absorbed in the definition of ω. We have verified that the above expression using the numerical solution for the equation for Z_P to one part in 10^{-9}. This is shown in figure 4. Evaluating the proportionality constant relating the bulk viscosity to the ratio $\mathrm{Re}\,\mathcal{R}_{Z_P}$, we obtain

$$\zeta = \frac{r_H^4}{16\pi G_3 L^4}(1+k)^2. \tag{B.25}$$

The entropy density of the equal charged solution (A.20) is given by

$$s = \frac{r_0^3 r_H}{4G_3 L^4}. \tag{B.26}$$

Using (A.21) and the expression of the bulk viscosity (B.25), we get the ratio

$$\frac{\zeta}{s} = \frac{1}{4\pi}. \tag{B.27}$$

We can now evaluate the thermal conductivity of this solution using (3.27), this results in

$$\kappa_T = \frac{r_H^2}{8LG_3}\frac{(1+k)(3-k)}{k}. \tag{B.28}$$

It can also be verified that this system also satisfies the Wiedemann-Franz like behaviour.

$$\frac{\kappa_T \hat{\mu}^2}{\zeta T} = 4\pi^2. \tag{B.29}$$

The remaining transport coefficients D_c and D_s which are related to the conductivity and the bulk viscosity can be evaluated and are listed in table 1. In the end, we mention that we have verified that the transport coefficients of the equal charged solution does not exhibit the critical behaviour seen in the case of the single charged solution in the domain of $0 < k < 3$. This is consistent with the fact that the Hessian does not show any sign of thermodynamic instability. When written in terms of

$$\mathsf{m} = \frac{\mu}{2\pi LT} = \frac{\sqrt{k}}{3-k},$$

the various transport coefficients are

$$\begin{aligned}
\sigma &= \frac{1}{16\pi G_3}\left[\frac{1+2\sqrt{1+12\mathsf{m}^2}}{3(1+16\mathsf{m}^2)}\right]^2, \\
\zeta &= \frac{\pi L^2 T^2}{4G_3}\left[\frac{1+\sqrt{1+12\mathsf{m}^2}}{6}\right]^2, \\
\kappa_T &= \frac{\pi L^2 T}{4G_3}\left(\frac{1+6\mathsf{m}^2+\sqrt{1+12\mathsf{m}^2}}{18\mathsf{m}^2}\right), \\
D_c &= \frac{\sqrt{1+12\mathsf{m}^2}}{24\pi T}\frac{(1+24\mathsf{m}^2-\sqrt{1+12\mathsf{m}^2})}{\mathsf{m}^2(1+16\mathsf{m}^2)}, \\
D_s &= \frac{1}{24\pi T}\left(\frac{1+2\sqrt{1+12\mathsf{m}^2}}{1+16\mathsf{m}^2}\right).
\end{aligned} \tag{B.30}$$

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