Optimal Control with Bounded Control Effort

The solution of the problem of optimum control with saturation constraint on the control effort proposed by Letov[1] was found invalid in general.[2] A necessary condition for which the linear optimal control law must satisfy was given by (35) in Rekasius and Hsia.[2] The purpose of this correspondence is to present a different method by which (35) can be obtained.

Let the boundaries of the linear control region defined by (12) in Rekasius and Hsia[2] be denoted by S. Rewriting (12) in the form

$$u^*(x) = \pm 1, \tag{1}$$

then S can be represented by, in view of (32) and (33)[2]

$$S:\ k_n\left(\sum_{i=1}^{n-1}\gamma_i x_i + x_n\right) = \pm 1. \tag{2}$$

Furthermore, S is expressed in the following parametric form[3]

$$S:\ \begin{aligned} x_1 &= \tau_1 \\ x_3 &= \tau_2 \\ &\vdots \\ x_{n-1} &= \tau_{n-1} \\ x_n &= \pm\frac{1}{k_n} - \sum_{i=1}^{n-1}\gamma_i\tau_i. \end{aligned} \tag{3}$$

It has been stated[2] that all optimal trajectories inside the linear region should ever be tangent to the boundary S. This implies that the scalar product of the direction D of the trajectories and the normal N to the boundaries S should be nonzero everywhere on S, i.e.,[3]

$$D\cdot N \neq 0 \text{ on } S. \tag{4}$$

By use of (3), the optimal system [eqs. (1) and (30)[2]] on S can be written in the following form:

$$\begin{aligned} \dot{x}_1 &= \tau_2 \\ \dot{x}_2 &= \tau_3 \\ &\vdots \\ \dot{x}_{n-1} &= \pm\frac{1}{k_n} - \sum_{i=1}^{n-1}\gamma_i\tau_i \\ \dot{x}_n &= \sum_{i=1}^{n-1}(a_n\gamma_i - a_i)\tau_i \mp \frac{a_n}{k_n} \pm 1. \end{aligned} \tag{5}$$

It can be shown[3] that $D\cdot N$ is

$$D\cdot N = \det\begin{vmatrix} \dfrac{dx_1}{dt} & \dfrac{\partial x_1}{\partial\tau_1} & \cdots & \dfrac{\partial x_1}{\partial\tau_{n-1}} \\ \dfrac{dx_2}{dt} & \dfrac{\partial x_2}{\partial\tau_1} & \cdots & \dfrac{\partial x_2}{\partial\tau_{n-1}} \\ \vdots & & & \\ \dfrac{dx_n}{dt} & \dfrac{\partial x_n}{\partial\tau_1} & \cdots & \dfrac{\partial x_n}{\partial\tau_{n-1}} \end{vmatrix}. \tag{6}$$

Upon substituting (3) and (5) into (6), one obtains $D\cdot N$ (in expanded form) evaluated on S

$$\begin{aligned} D\cdot N = {} & (-1)^n(\gamma_1\gamma_{n-1} - a_n\gamma_1 + a_1)\tau_1 \\ & + (-1)^n(\gamma_2\gamma_{n-1} - a_n\gamma_2 - \gamma_1 + a_2)\tau_2 \\ & \vdots \\ & + (-1)^n(\gamma_{n-1}^2 - a_n\gamma_{n-1} - \gamma_{n-2} + a_{n-1})\tau_{n-1} \\ & + (-1)^n\left(\frac{1}{k_n} + \frac{a_n}{k_n} - 1\right)(\pm 1). \end{aligned} \tag{7}$$

For $D\cdot N = 0$ everywhere on S, it is necessary that the coefficients of all τ_1's in (7) are zero. Thus, one obtains the condition

$$\begin{aligned} \gamma_1\gamma_{n-1} - a_n\gamma_1 + a_1 &= 0 \\ \gamma_2\gamma_{n-1} - a_n\gamma_2 - \gamma_1 + a_2 &= 0 \\ &\vdots \\ \gamma_{n-1}^2 - a_n\gamma_{n-1} - \gamma_{n-2} + a_{n-1} &= 0 \end{aligned} \tag{8}$$

which is identical to eq. (35).[2] Consequently, $D\cdot N$ becomes

$$D\cdot N = (-1)^n(\pm 1)\left(\frac{1}{k_n} + \frac{a_n}{k_n} - 1\right). \tag{9}$$

Again it is required that $D/N \neq 0$. This yields the condition that

$$\frac{1}{k_n} + \frac{a_n}{k_n} - 1 \neq 0. \tag{10}$$

However, this condition is automatically satisfied by the optimal control law because the optimal system is asymptotically stable. The case $D\cdot N = 0$ would mean that the optimal trajectories always lie on the hyperplane S other than go to the origin.

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[2] Z. V. Rekasius and T. C. Hsia, "On an inverse problem in optimal control," *IEEE Trans. on Automatic Control*, vol. AC-9, pp. 370–375, October 1964.
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Stability with Nonlinearity in a Sector

Recent investigations of the stability of a class of nonlinear systems have extended the results of Popov.[1] The block diagram representation of the problem is shown in Fig. 1 where the linear part having a transfer function $G(s)$ (satisfying certain conditions) is followed by a nonlinear gain $K(\sigma)$ ranging from 0 to ∞. Then the condition for absolute stability of the system is given by the Popov criterion,[2]

$$\mathrm{Re}\,[(1 + j\omega\beta)G(j\omega)] \geq 0 \quad \text{for all real } \omega \quad \text{for some } \beta \geq 0. \tag{P}$$

This result has been extended to the case of the nonlinearity in a finite range, $0 < K(\sigma) < K$ by several researchers.[3-5] Extensions to systems with multiple nonlinearities have been carried out by Narendra and Goldwyn[4] and by Ibrahim and Rekasius.[6]

For the system,

$$\begin{aligned} \dot{x} &= Fx - G\dot{\phi}(\sigma) \\ \sigma &= H'x \\ \phi(\sigma) &= K(\sigma)\sigma \\ 0 &< \frac{\phi_i(\sigma_i)}{\sigma_i} < \infty,\ K \text{ diagonal.} \end{aligned} \tag{1}$$

Narendra and Goldwyn have derived the condition

$$\mathrm{Re}\,[(I + i\omega B)\mathcal{G}(i\omega)] \geq 0 \quad \text{for all real } \omega \tag{$\mathrm{P_a}$}$$

for some diagonal $B \geq 0$ and

$$\mathcal{G}(s) = H'(sI - F)^{-1}G,$$

as the condition for absolute stability.

This note indicates how by a simple transformation the extension to the finite range case can be obtained directly from the Popov criterion (P) and the extended criterion ($\mathrm{P_a}$). In both cases the conditions turn out to be identical with previous results.

Problem With Nonlinearity in a Finite Range

In terms of transfer functions the finite range problem (Fig. 2) can be stated as follows.

Given the linear system $G_1(s)$ with a nonlinear gain $\phi_1(\sigma_1)/\sigma_1$ in the feedback path,

$$K_1 < \frac{\phi_1(\sigma_1)}{\sigma_1} < K_1 + K_2, \tag{2}$$

find conditions for absolute stability. This problem is converted to the infinite range case as follows.

Let

$$\phi(\sigma) = \phi_1(\sigma_1) - K_1\sigma_1$$

and

$$\sigma = \sigma_1 - \frac{1}{K_2}\phi(\sigma).$$

Then

$$\frac{\phi_1(\sigma_1)}{\sigma_1} = \frac{\phi(\sigma)/\sigma}{1 + \dfrac{1}{K_2}\dfrac{\phi(\sigma)}{\sigma}} + K_1 \tag{3}$$

so that

$$\begin{aligned} K_1 < \frac{\phi_1(\sigma_1)}{\sigma_1} < K_1 + K_2 &\text{ implies} \\ 0 < \frac{\phi(\sigma)}{\sigma} < \infty. \end{aligned} \tag{4}$$

Manuscript received September 27, 1965; revised November 4, 1965.

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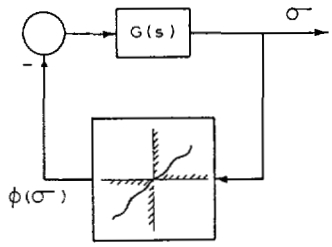

Fig. 1. The infinite range problem

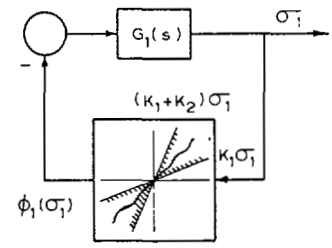

Fig. 2. The finite range problem

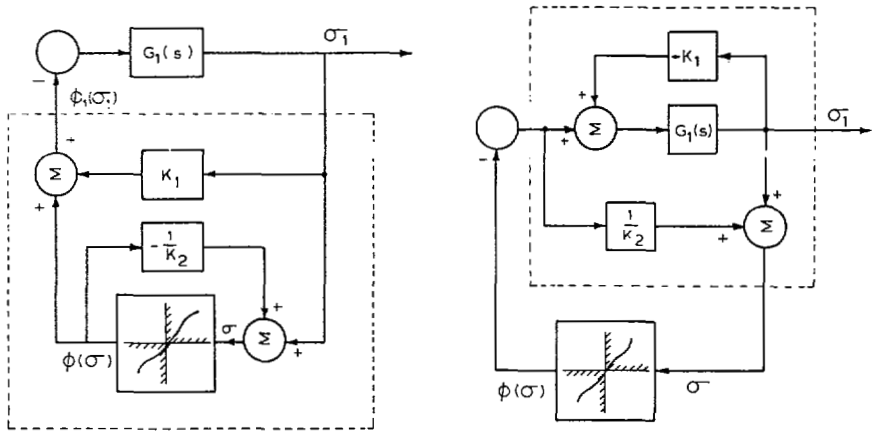

Fig. 3. (a) Transformation of the finite range problem to (b) the infinite range problem.

Thus the nonlinear gain $\phi_1(\sigma_1)/\sigma_1$ in a finite range can be replaced by the nonlinear gain $\phi(\sigma)/\sigma$ in the infinite range with a feedback $-(1/K_2)$ around it, in parallel with a gain K_1 [Fig. 3(a)].

If now we set

$$G(s) = \frac{G_1(s)}{1 + K_1G_1(s)} + \frac{1}{K_2} \tag{5}$$

the system is equivalent to a linear transfer function $G(S)$ with nonlinear feedback gain $\phi(\sigma)/\sigma$ [Fig. 3(b)]

$$0 < \frac{\phi(\sigma)}{\sigma} < \infty.$$

Since the Popov condition is now directly applicable, sufficient condition for asymptotic stability is given by

$$\mathrm{Re}\,[(1 + i\omega\beta)\ \mathrm{G}(i\omega)] \geq 0.$$

Substituting for $G(i\omega)$ from (5), we get the condition for stability for the finite range case as

$$\mathrm{Re}\left\{(1 + i\omega\beta) \cdot \left[\frac{G_1(i\omega)}{1 + K_1G_1(i\omega)} + \frac{1}{K_2}\right]\right\} \geq 0. \quad (\mathrm{P_b})$$

If

$$0 < \frac{\phi(\sigma_1)}{\sigma_1} < K,$$

by setting $K_1 = 0$, $K_2 = K$, we get the condition as

$$\mathrm{Re}\left[(1 + i\omega\mathrm{B})(G_1(i\omega) + \frac{1}{K})\right] \geq 0. \quad (\mathrm{P_b}')$$

Use of a similar transformation in the case of multiple nonlinearities leads to the condition

$$\mathrm{Re}\,[(I + i\omega\beta)\{\mathrm{G}(i\omega) + K^{-1}\}] \geq 0 \quad \text{for all real } \omega \quad (\mathrm{P_c})$$

for K diagonal and

$$0 < \frac{\phi_i(\sigma_i)}{\sigma_i} < K_{ii}.$$

As the transformation (3) is valid for monotone increasing nonlinearities, extensions to the finite range from the infinite range can be obtained in a similar manner.[7] It should be noted that transformation (3) does not postulate the existence of an inverse for the nonlinear function

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Application of Liapunov's Direct Method for Determining Stable Switching Boundaries in the Phase Plane

In order to determine stable switching boundaries in the phase plane for a second-order nonlinear differential equation, Liapunov's direct method is used. The state model for a thermal process is derived by considering the first and second laws of thermodynamics and the sign of the control force is selected to satisfy Liapunov's theorem after an appropriate Liapunov function is selected. The Liapunov function can be easily generated with analog or digital computers and these in turn may be used to control a relay which switches on the correct boundary in the phase plane.

State Model

In the conduction of heat in the unsteady state, the first and second laws of thermodynamics can be applied to derive the mathematical model for the case of heat being supplied by an internal source within a body.[1]

$$\theta - ac = \frac{dc}{dt} \tag{1}$$

where

θ = heat generated by the internal source during dt
ac = heat transferred from the body to surroundings during dt
dc/dt = rate of heat being stored within the body.

The control "force" to be applied is $f(t) = d\theta/dt$.

If further regulation of the temperature of the body to some constant temperature R is desired, and if $R - c = x_1$ is the temperature error, we may derive the state model as follows.

The change in internal energy must equal the net heat flow from the body to its surroundings or for no heat source within the body

$$-\frac{dc}{dt} = a(c - c_\infty).$$

With a heat source θ within the body and the surrounding temperature, c_∞ arbitrarily taken as zero, we have

$$\theta - ac = \frac{dc}{dt}. \tag{1}$$

Differentiating the above equation with respect to time, we have

$$\frac{d^2c}{dt^2} = -\frac{adc}{dt} + \frac{d\theta}{dt}.$$

Letting the control force be $f(t) = d\theta/d^t$ and the temperature error x_1 from some reference temperature R be $x_1 = R - c$,

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[1] F. Kreith, *Principles of Heat Transfer*. Scranton, Pa.: International Textbook Co., 1958, ch. 4.