the line $x_2 = \{(\sqrt{\beta_2} - 1)^2 - 1\}x_1$ (which has been chosen because it gives a tractable result), the relation

$$\frac{(\sqrt{\alpha_1} + 1)^2 + (\sqrt{\beta_2} - 1)^4}{(\sqrt{\alpha_1} + 1)^2 + 1} = \frac{(\sqrt{\beta_2} - 1)^2 + (\sqrt{\beta_2} - 1)^4}{(\sqrt{\beta_2} - 1)^2} \quad (12)$$

must be satisfied. For any value of $\alpha_1 \geq 0$, this is a quadratic in $(\sqrt{\beta_2} - 1)^2$, and the positive root gives the acceptable solution

$$\beta_2 = \left[\left\{ \frac{(\sqrt{\alpha_1} + 1)^2 + 1 + \sqrt{(\sqrt{\alpha_1} + 1)^4 + 2(\sqrt{\alpha_1} + 1)^2 + 5}}{2} \right\}^{\frac{1}{2}} + 1 \right]^2 \quad (13)$$

For $\alpha_1 = 0$, (13) yields $\beta_2 = [1 + \sqrt{2}]^2 + 1 \approx 6.5$, as compared with $\beta = 4$ from (4), but still conservative compared with the optimum result $\beta = 11.6$ obtained by Brockett's variational technique [2].

REFERENCES

- [1] A. C. Tsoi and H. M. Power, "Equivalent predictions of the circle criterion and an optimum quadratic form for a second-order system," *IEEE Trans. Automat. Contr.* (Tech. Notes and Corresp.), vol. AC-17, pp. 565-566, Aug. 1972.
- [2] R. W. Brockett, "Variational methods for stability of periodic equations," in *Differential Equations and Dynamical Systems*, J. Hale and J. P. La Salle, Ed., New York: Academic, 1967, pp. 299-308.
- [3] —, "The status of stability theory for deterministic systems," *IEEE Trans. Automat. Contr.* (Survey Papers), vol. AC-11, pp. 596-606, July 1966.

On the Stability of Nonlinear Time-Varying Systems

Y. V. VENKATESH

Abstract—Concerning the absolute stability of a nonlinear time-varying feedback system, an important correction to the results of the author [1] is presented.

INTRODUCTION

The author's paper [1] gives some new conditions for the absolute stability of a feedback system governed by the set of equations

$$\begin{aligned} \dot{x}(t) &= \exp(At)x(0) - \int_0^t \exp(A(t-\tau))b k(\tau)\phi(\sigma(\tau))d\tau \\ \sigma(t) &= f(t) - \int_0^t g(t-\tau)k(\tau)\phi(\sigma(\tau))d\tau \end{aligned} \quad (1)$$

on the interval $[0, \infty)$. For an explanation of the notation, terminology, and for a statement of the problem, definitions, and assumptions, see [1].

The multiplier employed in [1] to establish absolute stability conditions, when, for instance, $\phi(\cdot) \in P_M$, is of the form

$$\begin{aligned} z(s) &= \alpha + \beta s + \sum_{i=1}^{m_1} \gamma_i (s + \nu_i \mu_i) / (s + \mu_i) \\ &\quad + \sum_{i=1}^{m_2} \eta_i (-s + \zeta_i \theta_i) / (-s + \zeta_i) \end{aligned} \quad (2)$$

where constants $\alpha, \beta, \mu_i, \zeta_i > 0$; $\eta_i, \gamma_i \geq 0$; $0 \leq \nu_i, \theta_i < 1$ for all i ; and $\beta > \sum_{i=1}^{m_2} [\eta_i / \zeta_i (1 - \theta_i)]$, which depends only on the noncausal part of (2). This lower bound on β swamps the effect of the noncausal

part of the multiplier, and is hence undesirable. It turns out, however, that such a restriction on β appearing in Section III of [1] is superfluous.

CORRECTION

Beginning with Section III-B on mathematical preliminaries, a major slip in the integrals I_0, I_1 , etc., of [1] is the employment of $\sigma(t)$, $\phi(\sigma(t))$ for functions vanishing outside the interval $0 \leq t \leq T$. Hence, the first correction to the lemmas and results is: Replace $\sigma(t)$, $\phi(\sigma(t))$ in the integrals I_0, I_1 , etc., with $\sigma^*(t)$, $\phi^*(\sigma(t))$, which are equal to their nonstarred counterparts in $(0, T)$ and to zero outside this interval. Once this is accomplished, note from (29) that $\sigma_2(T)$ is identically zero, which implies that the term (35) is also identically zero. This leads immediately to the following correct form of Result 1.

Result 1: Let α, β be positive constants and $z_1^*(s)$ be as defined in [1]. Then the integral

$$I_{01}^* = \int_0^T k(t)\phi^*(\sigma(t)) \left\{ \alpha \sigma^*(t) + \beta \frac{d\sigma^*}{dt} + Z_1^* \sigma^*(t) \right\} dt$$

is positive [except for a constant term depending only on $\sigma(0)$] if, for some constant $\epsilon_0 > 0$,

$$-\min_i (\zeta_i \theta_i) k(t) \leq \frac{dk}{dt} \delta_{\max} \leq \frac{\alpha}{\beta} k(t) - \epsilon_0, \quad \forall t \in [0, \infty).$$

REMARKS

1) By a similar argument, the lower bound on β appearing in Results 2-6 is seen to be superfluous. Consequently, Theorems 3 and 4, with their corollaries, remain valid without the restriction on β .

2) The upper and lower bounds on coefficients ν_i, ν_i', θ_i , and θ_i' of the multiplier employed in Theorems 3 and 4 imply, in essence, a time domain constraint [of the type appearing in O'Shea, Zames, and Falb (see [1, references])] on the multiplier. The author has obtained absolute stability conditions for (1) in terms of a multiplier containing general causal and noncausal functions with a certain time domain constraint. These will be reported elsewhere.

EXAMPLE (ADAPTED FROM O'SHEA [2])

Consider a system with

$$G(s) = \left(\frac{s + 0.005}{s + 0.0001} \right)^2 \left(\frac{s + 0.1}{s + 2} \right) \left(\frac{s + 1000}{s + 50} \right)$$

The system is asymptotically stable for all constant negative (linear) feedback, regardless of magnitude. With the choice of

$$\begin{aligned} z(s) &= \frac{(s + 0.0001)(-s + 0.1)}{(s + 0.005)(-s + 2)} (s + 40) \\ &= 0.0017 + s + \left(2.0034 - \frac{0.0084}{s + 0.005} \right) + \left(39.89 - \frac{79.6029}{-s + 2} \right), \end{aligned}$$

which satisfies the conditions of the corollary to Theorem 3, it can be verified that $\text{Re } z(j\omega)G(j\omega) \geq 0$ for all real ω . Note that here $\zeta\theta = 0.004439$, $\nu\mu = 0.001617$. Therefore, the system is asymptotically stable for all monotonic nonlinearities if, for some (arbitrarily small) positive ϵ_0 ,

$$-0.004439k(t) \leq \frac{dk}{dt} \delta_{\max} \leq 0.001617k(t) - \epsilon_0, \quad \forall t \in [0, \infty).$$

This constraint on $k(t)$ is apparently very trivial. But it appears

that no improvement is possible using multipliers of the present form. It is quite likely that the system becomes unstable for larger variations of (dk/dt) ; this remains to be demonstrated.

ACKNOWLEDGMENT

The author wishes to thank Dr. M. A. L. Thathachar and M. K. Sundaresan for their critical remarks on [1].

REFERENCES

- [1] Y. V. Venkatesh, "Noncausal multipliers for nonlinear system stability," *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 195-204, Apr. 1970.
- [2] R. P. O'Shea, "An improved frequency time domain stability criterion for autonomous continuous systems," *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 725-731, Dec. 1967.

A Linear Minimal-Time Intercept Problem

M. R. MUKERJEE

Abstract—Sufficient conditions for intercept are obtained for a linear minimal-time intercept problem, with the controllers restricted to the unit hypersphere and unit hypercube. It is shown that an expression called an interceptability matrix can be defined whose positive definiteness is the required sufficient condition.

The minimal-time intercept problem for a pair of linear systems has been studied [1], [2] as a linear differential game. The basic system consists of the pursuer and evader systems, the equations of motion being given by

$$\dot{x}_p(t) = A_p(t)x_p(t) + B_p(t)u(t), \quad x_p(t_0) = x_{p0} \quad (1)$$

$$\dot{x}_e(t) = A_e(t)x_e(t) + B_e(t)v(t), \quad x_e(t_0) = x_{e0} \quad (2)$$

respectively, where x_p, x_e are n_p - and n_e -dimensional state vectors and u, v are p - and q -dimensional control vectors of the pursuer and evader, and A_p, B_p, A_e, B_e are continuous time-varying matrices of appropriate dimensions. An intercept occurs when

$$y(T) = H_p(T)x_p(T) - H_e(T)x_e(T) = 0. \quad (3)$$

The objective of the pursuer is to effect the intercept in the minimum possible time, and that of the evader is to prevent an intercept. The case considered earlier [1], [2] is the special case where y is a scalar. A more general case is investigated here.

Let γ be an arbitrary unit row vector. Then it can be shown, by a modification of a result obtained elsewhere [3] that a necessary and sufficient condition for an intercept to be possible is as follows. Let

$$W(x_{p0}, x_{e0}, \gamma, t) = \gamma(H_p \phi_p(t, t_0)x_{p0} - H_e \phi_e(t, t_0)x_{e0}) + \int_{t_0}^t \max_u \gamma H_p \phi_p(t, \alpha) B_p u d\alpha - \int_{t_0}^t \max_v \gamma H_e \phi_e(t, \alpha) B_e v d\alpha \quad (4)$$

and let

$$\lambda(x_{p0}, x_{e0}, t) = \min_{\gamma} W(x_{p0}, x_{e0}, \gamma, t). \quad (5)$$

Then the smallest positive root of the equation

$$\lambda(x_{p0}, x_{e0}, t) = 0 \quad (6)$$

should be finite. It can also be shown that this time is the minimal time of intercept.

Two restraint sets for the controls are examined here, viz.,

$$\|u\| \leq 1, \quad \|v\| \leq 1 \quad (7)$$

and

$$|u_i| \leq 1, \quad |v_i| \leq 1, \quad (8)$$

i.e., the controls restricted to the unit hypersphere or unit hypercube.

Carrying out the maximization operations in the first case gives

$$u = \frac{(H_p \phi_p B_p)' \gamma'}{\| \gamma H_p \phi_p B_p \|}, \quad v = \frac{(H_e \phi_e B_e)' \gamma'}{\| \gamma H_e \phi_e B_e \|} \quad (9)$$

where the prime indicates transpose. Substituting in (4),

$$W(x_{p0}, x_{e0}, \gamma, t) = \gamma(H_p \phi_p x_{p0} - H_e \phi_e x_{e0}) + \int_{t_0}^t (\| \gamma H_p \phi_p B_p \| - \| \gamma H_e \phi_e B_e \|) d\alpha. \quad (10)$$

A sufficient condition for intercept is that the second term be positive for all t . Thus a sufficient condition is that the matrix

$$M(t, t_0) = \int_{t_0}^t (H_p \phi_p B_p B_p' \phi_p' H_p' - H_e \phi_e B_e B_e' \phi_e' H_e') d\alpha \quad (11)$$

be positive definite for all t . This matrix may be termed an output interceptability matrix. If H_p, H_e are identity matrices (i.e., if the dimensions of pursuer and evader are the same), a state interceptability matrix is obtained, which can be recognized as the difference between the controllability matrices of the pursuer and evader. The same result, with some modification was obtained in [5] and indicates that the pursuer should be "more" controllable than the evader in every Euclidean dimension.

In the second case, using (8),

$$u_i = \text{sgn } \gamma(H_p \phi_p B_p)_i, \quad v_i = \text{sgn } \gamma(H_e \phi_e B_e)_i \quad (12)$$

where the subscript denotes the column. The expression (4) becomes

$$W(x_{p0}, x_{e0}, \gamma, t) = \gamma(H_p \phi_p x_{p0} - H_e \phi_e x_{e0}) + \int_{t_0}^t \left(\sum_i |\gamma(H_p \phi_p B_p)_i| - \sum_j |\gamma(H_e \phi_e B_e)_j| \right) d\alpha. \quad (13)$$

It is seen by expanding that

$$\int_{t_0}^t \gamma(H_p \phi_p B_p B_p' \phi_p' H_p' - H_e \phi_e B_e B_e' \phi_e' H_e') \gamma d\alpha = \int_{t_0}^t \left(\sum_i |\gamma(H_p \phi_p B_p)_i|^2 - \sum_j |\gamma(H_e \phi_e B_e)_j|^2 \right) d\alpha. \quad (14)$$

Thus, if (11) is positive definite, the second term of (13) is also positive for all t . Thus, (11) being positive definite is a sufficient condition for intercept in both cases. The actual optimal strategies can be determined by obtaining the adjoint variables as indicated in [2].

REFERENCES

- [1] Y. C. Ho and S. Baron, "A minimal time intercept problem," *IEEE Trans. Automat. Contr.* (Corresp.), vol. AC-10, p. 200, Apr. 1965.
- [2] E. J. Davison and J. M. Seugnet, "A minimum time intercept problem," *IEEE Trans. Automat. Contr.* (Corresp.), vol. AC-15, pp. 116-118, Feb. 1970.
- [3] M. R. Mukerjee, "On linear differential games," *Int. J. Contr.*, communication.
- [4] B. N. Pshenichnyi, "Linear differential games," *Automat. Remote Contr.* (USSR), pp. 55-68, Jan. 1968.
- [5] Y. C. Ho, A. E. Bryson, Jr., and S. Baron, "Differential games and optimal pursuit-evasion strategies," *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 385-389, Oct. 1965.

Manuscript received July 18, 1972.

The author is with the Department of Electrical Engineering, University of Roorkee, Roorkee, India.