Therefore, the permutation transformation for both odd and even matrix dimensions is

$$P_1, P_{21}, P_{213}, P_{3241}, P_{32415}, P_{435261}, P_{4352617}, \dots$$

One can also obtain a general relationship for generating the permutation matrix P for n -even or n -odd.

Similarly, one can obtain the inners from the minors as follows:

$$I = P^T M P. \quad (4)$$

The above is evident from noting (1) and $P^T = P^{-1}$, where P^{-1} = inverse P .

One can easily verify that there exists another type of permutation transformation for (1) which can be expressed for both even and odd " n ," as follows:

$$P_1, P_{12}, P_{231}, P_{2314}, P_{34251}, P_{342516}, P_{531246}, \dots$$

To obtain the corresponding innerwise matrix, for the Hurwitz matrix, we premultiply the latter by the following permutation matrix.

For n -even [the permutation matrix dimension is $(n-1) \times (n-1)$]:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The pattern is²

$$p_{12} = 1, p_{24} = 1, p_{35} = 1$$

$$p_{n/2-1, n-2} = 1, p_{n/2, n-1} = 1$$

$$p_{n/2+1, n-3} = 1, p_{n/2+2, n-5} = 1, \dots$$

For n -odd [the permutation matrix dimension is $(n-1) \times (n-1)$]:

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The pattern is

$$p_{11} = 1, p_{23} = 1$$

$$p_{(n-1)/2, n+1/2} = 1, p_{(n-1)/2+1, n-1} = 1$$

$$p_{(n-1)/2+2, n-3} = 1, \dots$$

The innerwise matrix has the unified form of left of triangles of zeros, as shown in earlier publications [1], [3], [4].

Finally, to show that for the general problem of root clustering the innerwise matrix offers a unifying form which has a pattern that can be utilized advantageously for computational purposes, the following example is discussed.

EXAMPLE

Let

$$F(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0, \quad a_4 > 0.$$

The stability condition in the open left-half plane is [2]

- 1) $a_k > 0, k = 0, 1, 2, 3, 4$;
- 2) in the following Hurwitz matrix,

$$[\Delta_3] = \begin{bmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{bmatrix}, \quad \Delta_1 > 0, \Delta_3 > 0.$$

² In this pattern, the first subscript in p_{jk} refers to row location, and the second one refers to column location.

The inner form of the above matrix can be obtained as discussed above by using the premultiplying matrix as follows:

$$[\Delta_3^1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_3 & a_1 & 0 \\ a_4 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{bmatrix} = \begin{bmatrix} a_3 & a_1 & 0 \\ 0 & a_2 & a_0 \\ a_4 & a_3 & a_1 \end{bmatrix} [\Delta_1^1] \begin{bmatrix} a_0 \\ a_1 \\ 0 \end{bmatrix}$$

The stability condition is that $[\Delta_3^1]$ be positive innerwise (p.i.) plus condition 1) stated above.

For the same polynomial, the stability condition within the unit circle [1] is that

$$F(1) > 0, \quad F(-1) > 0$$

and that $[\Delta_3^+]$ be positive innerwise.

To obtain, for instance, $[\Delta_3^+]$, we have

$$\begin{aligned} [\Delta_3^+] &= X_3 + Y_3 = \begin{bmatrix} a_4 & a_3 & a_2 \\ 0 & a_4 & a_3 \\ 0 & 0 & a_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_0 \\ 0 & a_0 & a_1 \\ a_0 & a_1 & a_2 \end{bmatrix} \\ &= [\Delta_3^-] = \begin{bmatrix} a_4 & a_3 & a_2 + a_0 \\ 0 & a_4 + a_0 & a_3 + a_1 \\ a_0 & a_1 & a_4 + a_2 \end{bmatrix} [\Delta_1^+] \end{aligned}$$

Similarly, $[\Delta_3^-]$ can be obtained.³

The innerwise matrix $[\Delta_3^+]$ can be transformed directly into a minor form, whereby the inner $[\Delta_1^+]$ becomes the principal minor and the determinant values of the matrices are as follows:

$$[\Delta_3^+]^1 = \begin{bmatrix} a_4 + a_0 & 0 & a_3 + a_1 \\ a_3 & a_4 & a_2 + a_0 \\ a_1 & a_0 & a_4 + a_2 \end{bmatrix} [\Delta_1^+]$$

CONCLUSION

In comparing the inner form for the left-half plane and for the unit circle, the first element of the second row in both cases is zero. This is the unifying feature. The minor form has no such unifying pattern. Furthermore, there exists no inner-minor transformation that makes the first entry of the third row of $[\Delta_3^+]$ zero.

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³ In a recent work by Anderson and Jury [5], it is shown that $[\Delta_3^+]$, p.i. can be replaced by the simpler conditions on the linear combinations of the coefficients of $F(z)$.

Comments on "On the Simplification of Linear Systems"

M. R. CHIDAMBARA

The author of the above note¹ is to be complimented on presenting a method of approximating a higher order system using the frequency response approach. It is interesting to note that there is good agree-

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The author is with the School of Automation, Indian Institute of Science, Bangalore 12, India.

¹ T. C. Hsia, *IEEE Trans. Automat. Contr.* (Tech. Notes and Corresp.), vol. AC-17, pp. 372-374, June 1972.

TABLE I

Model	Unit Step Response	$\int_0^\infty \epsilon(t)^2 dt$	Remarks
N_2	$1 - (7/6)e^{-t} + (1/6)e^{-2t}$	3.3×10^{-4}	Predominant poles $-1, -2$ retained in the model
$(C-I)_2$	$1 - (6/5)e^{-t} + (1/5)e^{-2t}$	2.38×10^{-4}	
N_3	$1 - 1.2308e^{-1.043t} + 0.2308e^{-2.4t}$	1.2×10^{-4}	Second-order model with two poles unspecified
$(C-I)_3$	$1 - 1.24e^{-1.043t} + 0.24e^{-2.4t}$	0.6188×10^{-4}	Poles obtained in N_3 assigned to the second-order model
$(C-II)_3$	$1 - 1.21695e^{-1.0299t} + 0.21695e^{-2.4001t}$	0.415×10^{-4}	Both poles unspecified
N_4	$1 - e^{-0.923t}$	1.05×10^{-3}	First-order model
$(C-II)_4$	$1 - e^{-0.914t}$	0.967×10^{-3}	
Original system (fourth order)	$1 - e^{-t} - e^{-2t} + 2e^{-3t} - e^{-4t}$	0.0	Exact system

ment between the transient response of the higher order system and that of the corresponding lower order model, which is derived by matching the magnitudes of the frequency responses. However, a few points discussed in the above note¹ need some clarification.

Strictly speaking, the simplified models developed by Davison [1], [5], viz., D_m and D_n , and by Chidambara [2]–[4], namely, $C1$ and $C2$, should not be used for comparison with the model presented by Hsia, because the basis on which these different models are developed is entirely different from that on which the model is derived by Hsia.

If a comparison of the integral of the squared error is valid at all, reference should be made to the models derived using the two techniques described in [6] in view of which the models $C1$ and $C2$ become obsolete.

Denote the lower order model derived from Technique I [6], which retains the predominant eigenvalues, by $C-I$, and that derived from the other method [6] by $C-II$. It is worth recalling [6] that Technique II allows one to assign any desired eigenvalues to the lower order model. Let N refer to the model derived by method proposed by Hsia.¹

Table I gives a comparison of the quality of the different simplified models for the same fourth-order example¹ [4]. Table II shows a comparison of the transfer functions of the various lower order models.

Table II demonstrates that the simplified models based on Hsia's frequency response approach and Chidambara's transient response method¹ [6] have approximately the same transfer function. Does this imply that there is a one-to-one (at least in an approximate sense) correspondence between the approximations based on frequency response approach and the transient response approach?

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TABLE II

Model	Transfer Function
N_2	$\frac{0.8333s + 2}{(s + 1)(s + 2)}$
$(C-I)_2$	$\frac{0.8s + 2}{(s + 1)(s + 2)}$
N_3	$\frac{0.2917s + 1}{0.399s^2 + 1.375s + 1}$
$(C-I)_3$	$\frac{0.2664s + 1}{0.39955s^2 + 1.3755s + 1}$
$(C-II)_3$	$\frac{0.29583s + 1}{0.40458s^2 + 1.387537s + 1}$
N_4	$\frac{1}{1.0833s + 1}$
$(C-II)_4$	$\frac{1}{1.09409s + 1}$

M. R. Chidambara, "Further comments by M. R. Chidambara," *IEEE Trans. Automat. Contr.* (Corresp.), vol. AC-12, pp. 799–800, Dec. 1967.

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Author's Reply²

T. C. HSIA

I wish to thank Prof. Chidambara for his interest in my note. I am very grateful for his establishing more comparisons of my method to the best of his methods. These results together with those in my note¹ should provide the reader with a good view of the performances of the wide-range of techniques for system simplification. I would like, however, to make the following remarks in response to Prof. Chidambara's comments.

1) The intent of my comparison study was to demonstrate that the new method is valid and useful. The models $C1$, $C2$, D_n , and D_m were chosen in the study simply because they are readily available and are well known to the readers. The use of the integral-square-error criterion seems to me to be the most common, as well as unbiased, in that none of these models was in any way related to this criterion.

2) On the other hand, the models $C-I$ and $C-II$ of Chidambara are indeed derived on the basis of minimization of the step response integral-square-error. So to compare them with my frequency domain method is rather inappropriate. Even then, the N models compare rather well. This result has further demonstrated the good quality of my method.

3) It is noted that the approximation accuracy of the N models can be further improved if more than $(p + q)$ equations in (9) of my note¹ are matched.

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The author is with the Department of Electrical Engineering, University of California, Davis, Calif. 95616.