Time-Domain Criteria for the $L_2$-Stability of Nonstationary Feedback Systems

M. K. SUNDARESHAN and M. A. L. THATHACHAR

Abstract—Criteria for the $L_2$-stability of linear and nonlinear time-varying feedback systems are given. These are conditions in the time domain involving the solution of certain associated matrix Riccati equations and permitting the use of a very general class of $L_2$-operators as multipliers.

I. Introduction

The problem of deriving criteria for the $L_2$-stability of systems containing a linear time-varying operator and a memoryless non-linearity in cascade, in a negative feedback loop, is not amenable to simple treatment, owing to the difficulty in obtaining positivity conditions for time-varying operators. Recently, Williams [1], [2], as well as Estrada and Deesur [3], have obtained conditions for the positivity of systems with a state-space description, in terms of an associated matrix Riccati equation. Here, we follow this approach and give conditions for the positivity and stability of time-varying nonlinear time-varying feedback systems. The positivity condition obtained here is the same as that in [1], [2], and give conditions for the positivity and stability of time-varying single-output systems, the extension to the general case is straightforward.

II. Problem Formulation

Notation and Definitions: Here, detailed definitions will be omitted as these can be found in an earlier paper by the authors [4]. Some new notation will be introduced.

Let $\mathcal{H}$ denote the class of linear causal operators $H$ in $L_2$ with an external (input-output) description, i.e., $H \in \mathcal{H}$ if there exists a map $K^*$ such that $y(t) = H u(t) = \int_0^t K(t, \tau)u(\tau) \, d\tau$, $\forall u(\cdot) \in L_2$, where $y(\cdot) : R^+ \to R$ is the input to $H$ and $y_H(\cdot) : R^+ \to R$ is the output of $H$ and $k(t, \tau) = 0$, $\forall t > \tau$.

A. Note that $H \in \mathcal{H}$ if $H = I_2 \to L_2$ [7].

Let $\mathcal{H}_1$ denote the class of linear causal operators $H$ in $L_2$ with an internal (state-space) description, i.e., $H \in \mathcal{H}_1$ if there exists a map $K^*$ such that $y(t) = H u(t) = \int_0^t K(t, \tau)u(\tau) \, d\tau$, $\forall u(\cdot) \in L_2$, where $y(\cdot) : R^+ \to R^+$ is the input to $H$ and $y_H(\cdot) : R^+ \to R$ is the output of $H$ and $k(t, \tau) = 0$, $\forall t > \tau$.

Let $\mathcal{H}_0$ denote the class of linear causal operators $H$ in $L_2$ with an internal (state-space) description, i.e., $H \in \mathcal{H}_0$ if there exists a map $K^*$ such that $y(t) = H u(t) = \int_0^t K(t, \tau)u(\tau) \, d\tau$, $\forall u(\cdot) \in L_2$, where $y(\cdot) : R^+ \to R^+$ is the input to $H$ and $y_H(\cdot) : R^+ \to R$ is the output of $H$ and $k(t, \tau) = 0$, $\forall t > \tau$.

A. Note that $H \in \mathcal{H}_0$ if $H = I_2 \to L_2$ [7].

Let $\mathcal{H}_1$ denote the class of linear causal operators $H$ in $L_2$ with an internal (state-space) description, i.e., $H \in \mathcal{H}_1$ if there exists a map $K^*$ such that $y(t) = H u(t) = \int_0^t K(t, \tau)u(\tau) \, d\tau$, $\forall u(\cdot) \in L_2$, where $y(\cdot) : R^+ \to R^+$ is the input to $H$ and $y_H(\cdot) : R^+ \to R$ is the output of $H$ and $k(t, \tau) = 0$, $\forall t > \tau$.

Let $\mathcal{H}_0$ denote the class of linear causal operators $H$ in $L_2$ with an internal (state-space) description, i.e., $H \in \mathcal{H}_0$ if there exists a map $K^*$ such that $y(t) = H u(t) = \int_0^t K(t, \tau)u(\tau) \, d\tau$, $\forall u(\cdot) \in L_2$, where $y(\cdot) : R^+ \to R^+$ is the input to $H$ and $y_H(\cdot) : R^+ \to R$ is the output of $H$ and $k(t, \tau) = 0$, $\forall t > \tau$.

A. Note that $H \in \mathcal{H}_0$ if $H = I_2 \to L_2$ [7].

Let $\mathcal{H}_1$ denote the class of linear causal operators $H$ in $L_2$ with an internal (state-space) description, i.e., $H \in \mathcal{H}_1$ if there exists a map $K^*$ such that $y(t) = H u(t) = \int_0^t K(t, \tau)u(\tau) \, d\tau$, $\forall u(\cdot) \in L_2$, where $y(\cdot) : R^+ \to R^+$ is the input to $H$ and $y_H(\cdot) : R^+ \to R$ is the output of $H$ and $k(t, \tau) = 0$, $\forall t > \tau$.

Let $\mathcal{H}_0$ denote the class of linear causal operators $H$ in $L_2$ with an internal (state-space) description, i.e., $H \in \mathcal{H}_0$ if there exists a map $K^*$ such that $y(t) = H u(t) = \int_0^t K(t, \tau)u(\tau) \, d\tau$, $\forall u(\cdot) \in L_2$, where $y(\cdot) : R^+ \to R^+$ is the input to $H$ and $y_H(\cdot) : R^+ \to R$ is the output of $H$ and $k(t, \tau) = 0$, $\forall t > \tau$.

A. Note that $H \in \mathcal{H}_0$ if $H = I_2 \to L_2$ [7].

III. Main Results

Lemma: An operator $H \in \mathcal{H}_0$ is positive (e) if a)

$$d_H(t) > 0, \quad \forall t \in R^+,$$  

and b) there exists a real symmetric bounded nonnegative-definite $n \times n$ matrix $R(t)$ satisfying the Riccati equation

$$[R(t) - R(t)H(t)S(t)H(t)R(t)] \geq [R(t) - R(t)H(t)S(t)H(t)R(t)] - [R(t)A(t) + A(t)R(t)S(t)].$$  

where $y(t) = H u(t) = \int_0^t K(t, \tau)u(\tau) \, d\tau$, $\forall u(\cdot) \in L_2$, and $H \in \mathcal{H}_0$ is strongly positive (e) if there exists an $e > 0$ such that (3.1) and (3.2) are satisfied with $d_H(t)$ replaced by $[d_H(t) - e]$

Proof:

a) Positivity (e) of $H$: It is required to prove that

$$(u_H(t), y_H(t)) \geq 0, \quad \forall u_H(\cdot) \in L_2(t), \quad \forall T \in R^+.$$  

Since $H \in \mathcal{H}_0$, following Williams [1], [2], (3.3) is equivalent to the existence of a quadratic function $S(x_H(t)) = \frac{1}{2}x_H(t)R_H(t)x_H(t)$, satisfying

$$S(x_H(t)) + (u_H(t), y_H(t)) \geq S(x_H(T)).$$  

Consider

$$S(x_H(T)) - S(x_H(0)) = (u_H(t), y_H(t))$$  

$$= \int_0^T [\frac{1}{2}x_H(t)R_H(t)x_H(t) - u_H(t)y_H(t)] \, dt$$  

$$= \int_0^T [\frac{1}{2}x_H(t)R_H(t) + R_H(t)A_H(t) + A_H(t)R_H(t)]x_H(t)$$  

$$+ x_H(t)R_H(t)u_H(t) - u_H(t)x_H(t)x_H(t) - d_H(t)u_H(t)^2 dt,$$

since $H \in \mathcal{H}_0$. If the RHS is $\leq 0$, then (3.4) is satisfied. We will show this by proving that, if $R_H(t)$ is a solution of (3.2), the integrand on the RHS is $\leq 0$. Now, the integrand on the RHS of (3.5) is
Theorem 1 (Linear System): If there exists an operator $M \in \mathcal{D}_{dA}$ such that $M^{-1} \in \mathcal{D}_{dA}$, the composition $L = MG \in \mathcal{D}_{LM}$ and if the following conditions:

a) $d_{LM}(t) > 0 \quad \forall t \in R^+$, $d_{LM}(t) > 0 \quad \forall t \in R^+$; (3.9)

b) there exist real symmetric bounded nonnegative-definite matrices $R(t)$ and $R(t)$, solutions of the Riccati equations,

$$
\dot{R}(t) = \frac{1}{2}[d(t) - \varepsilon(t)][R(t)R(t) - \varepsilon(t)] - [R(t)A(t) + A(t)R(t)]
$$

and

$$
\dot{R}(t) = \frac{1}{2}[d(t) - \varepsilon(t)][R(t)R(t) - \varepsilon(t)] - [R(t)A(t) + A(t)R(t)]
$$

(3.10) satisfy the system described by Fig. 1 is $L$-stable for all $G \in \mathcal{D}_{EA} \cap \mathcal{D}_{LM}$ and $N = E$.

Theorem 2 (Nonlinear System): If there exists an operator $M \in \mathcal{D}_{dA} \cap \mathcal{D}_{LM}$ such that $M^{-1} \in \mathcal{D}_{dA}$, $L = MG \in \mathcal{D}_{LM}$, and if the following conditions:

a) $\frac{d(t)}{e(t)} > e(t) > 0 \quad \forall t \in R^+$;

c) $M = \rho E + Z$, $\rho \in R^+$, $Z \in \mathcal{D}_{dA}, z(t) = z(t - \tau), \int_0^\infty [z(t)] dt < \rho$.

satisfy the condition described in Fig. 1 is $L$-stable for all $G \in \mathcal{D}_{dA} \cap \mathcal{D}_{LM}$ and $N = \mathcal{D}_{LM}$.

Remark: The condition of the theorems that $L = MG \in \mathcal{D}_{LM}$ needs some explanation. Although $M, G \in \mathcal{D}_{LM}$ does not follow, in general, that $M, G \in \mathcal{D}_{LM} \Rightarrow L \in \mathcal{D}_{LM}$.

References


Comments on "A Simplified Irreducible Realization Algorithm"

R. D. GUPTA and F. W. FAIRMAN

In the above paper, Chen and Mital have provided a theorem that enables a reduction in the size of the matrices used in Ho's algorithm for minimal realization. This is achieved by utilizing the additional information contained in the degrees of the lowest common denominators of the rows and columns of the transfer function matrix $G(s)$. Theorem 1 is shown here to be a simple consequence of known results for the realization of single-input multi-output and single-output multi-input systems. The proof given here is based on Lemma 4 [1], which is restated as the following.

**Lemma:** If $G(s)$ is a strictly proper rational matrix and the monic polynomial $\gamma(s)$ is the least common denominator of the entries, $g_i(s)$, in $G(s)$, then $\gamma(s)$ is the minimal polynomial of the minimal realization of $G(s)$.

Consider a simple real-time multi-input system. The transfer function matrix $G(s)$ can then be written as (using the authors' numbering of equations where possible)

$$
G(s) = [g_1(s), g_2(s), \ldots, g_p(s)] = \sum_{i=1}^\infty M_{i-1}e^{-i}
$$

where

$$
M_i = [m_{i1}m_{i2} \ldots m_{ip}]
$$

and the $m_{ij}$ are defined by

$$
G_i(s) = \sum_{j=1}^\infty i_jm_{ij}s^{-i}.
$$

The usual Hankel matrix used in Ho's algorithm is defined by