# Time-Domain Criteria for the $L_2$-Stability of Nonstationary Feedback Systems

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*Abstract*—Criteria for the $L_2$-stability of linear and nonlinear time-varying feedback systems are given. These are conditions in the time domain involving the solution of certain associated matrix Riccati equations and permitting the use of a very general class of $L_2$-operators as multipliers.

## I. Introduction

The problem of deriving criteria for the $L_2$-stability of systems containing a linear time-varying operator and a memoryless nonlinearity in cascade, in a negative feedback loop, is not amenable to simple treatment, owing to the difficulty in obtaining positivity conditions for time-varying operators. Recently, Willems [1], [2], as well as Estrada and Desoer [3], have obtained conditions for the positivity of systems with a state-space description, in terms of an associated matrix Riccati equation. Here, we follow this approach and give conditions for the positivity and stability of time-varying systems. The positivity condition obtained here is the same as that given in [3], but the method of derivation is felt to be simpler and more direct and has been inspired by the work of Willems [1], [2]. Although the analysis is concerned with finite-dimensional single-input single-output systems, the extension to the general case is straightforward.

## II. Problem Formulation

*Notation and Definitions:* Here, detailed definitions will be omitted as these can be found in an earlier paper by the authors [4]. Some new notation will be introduced.

Let $\vartheta_E$ denote the class of linear causal operators $H$ in $L_{2e}$ with an external (input–output) description, i.e., $H \in \vartheta_E \Rightarrow \exists$ a map $h$: $R^+ \times R^+ \to R^1$ such that $y_H(t) = Hu_H(t) = \int_0^\infty h(t,\tau)u_H(\tau)\,d\tau$, $\forall u_H(\cdot) \in L_{2e}$, where $u_H(\cdot): R^+ \to R$ is the input to $H$ and $y_H(\cdot)$: $R^+ \to R$ is the output of $H$ and $h(t,\tau) = 0$, $\forall \tau > t$.

$$\vartheta_{EA} \subset \vartheta_E \ni H \in \vartheta_{EA} \Rightarrow \int_0^\infty \int_0^\infty |h(t,\tau)|^2\,dt\,d\tau$$

is finite.[2] Note that $H \in \vartheta_{EA} \Rightarrow H: L_2 \to L_2$ [7].

Let $\vartheta_I$ denote the class of linear causal operators $H$ in $L_{2e}$ with an internal (state–space) description, i.e., $H \in \vartheta_I \Rightarrow \exists A_H(\cdot): R^+ \to R^n \times R^n$, $b_H(\cdot): R^+ \to R^n$, $c_H(\cdot): R^+ \to R^n$, and $d_H(\cdot): R^+ \to R$, such that $H$ is described by the dynamical equations

$$\dot{x}_H(t) = A_H(t)x_H(t) + b_H(t)u_H(t);\; y_H(t) = c_H'(t)x_H(t) + d_H(t)u_H(t) \quad (2.1)$$

where $x_H(\cdot): R^+ \to R^n$ is the state of $H$ and $u_H(\cdot)$, $y_H(\cdot)$ defined as above.

$\vartheta_{IM} \subset \vartheta_I \ni H \in \vartheta_{IM} \Rightarrow$ (2.1) is a minimal representation of $H$. It is well known [5] that $H$ is uniformly reachable and uniformly observable $\Rightarrow H \in \vartheta_{IM}$.

$H \in \vartheta_E$ is said to have finite gain if

$$\gamma(H) = \sup_{x(\cdot)\in L_{2e}, T\in R^+} \frac{\|(Hx(\cdot))_T\|}{\|x_T(\cdot)\|}, \qquad x_T(\cdot) \neq 0$$

is finite [$x_T(\cdot)$ is the truncation of $x(\cdot)$ defined by $x_T(t) = x(t)$,

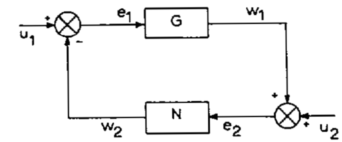


Fig. 1. The feedback system under consideration.

$\forall t \le T$, and zero otherwise; $x(\cdot) \in L_{2e} \Rightarrow x_T(\cdot) \in L_2$, $\forall T \in R^+$]. Note that $H \in \vartheta_{EA} \Rightarrow H$ has finite gain and $H_1, H_2 \in \vartheta_{EA} \Rightarrow H = H_1H_2$ has finite gain.

$H \in \vartheta_E \cap \vartheta_I$ is said to be positive ($e$) [strongly positive ($e$)] if the inequality $\langle u_{HT}(\cdot), y_{HT}(\cdot)\rangle \ge \epsilon \langle u_{HT}(\cdot), u_{HT}(\cdot)\rangle$, $\forall u_H(\cdot) \in L_{2e}$ and $\forall T \in R^+$ holds with $\epsilon = 0$ [$\epsilon > 0$].

*System:* The system (Fig. 1) is described by the input–output relations $e_1(\cdot) = u_1(\cdot) - w_2(\cdot)$, $e_2(\cdot) = u_2(\cdot) + w_1(\cdot)$ with $w_1(\cdot) = Ge_1(\cdot)$, $G \in \vartheta_E \cap \vartheta_I$ and $w_2(\cdot) = Ne_2(\cdot)$, $N \in \mathfrak{N}_{OM}$, i.e., $N: L_{2e} \to L_{2e} \ni Nx(\cdot) = n(x(\cdot))$, $0 \le x(\cdot)n(x(\cdot)) \le \eta x^2(\cdot)$, $\forall x(\cdot) \in L_{2e}$, and $n(\cdot)$ is odd and monotone nondecreasing. Note that $N = E$, the identity operator in $L_{2e}$, yields a linear system more general than that considered in [4].

*Problem:* Given that $u_1(\cdot)$, $u_2(\cdot) \in L_2$ and $e_1(\cdot)$, $e_2(\cdot) \in L_{2e}$, find conditions on $G$ which ensure $e_1(\cdot)$, $e_2(\cdot) \in L_2$.

The method of solution to this problem by factoring the open loop into two positive ($e$) operators, one of which is strongly positive ($e$) and has finite gain, is by now well established, and the introduction of "multipliers" to render flexibility to this approach is well known [4].

## III. Main Results

*Lemma:* An operator $H \in \vartheta_{IM}$ is positive ($e$) if: a)

$$d_H(t) > 0, \qquad \forall t \in R^+, \quad (3.1)$$

and b) there exists a real symmetric bounded nonnegative-definite $n \times n$ matrix $R_H(t)$ satisfying the Riccati equation

$$\dot{R}_H(t) = \tfrac{1}{2}d_H^{-1}(t)[c_H(t) - R_H(t)b_H(t)][b_H'(t)R_H(t) - c_H'(t)] - [R_H(t)A_H(t) + A_H'(t)R_H(t)]. \quad (3.2)$$

$H \in \vartheta_{IM}$ is strongly positive ($e$) if there exists an $\epsilon > 0$ such that (3.1) and (3.2) are satisfied with $d_H(t)$ replaced by $[d_H(t) - \epsilon]$.

*Proof:*

a) *Positivity* ($e$) *of* $H$: It is required to prove that

$$\langle u_{HT}(\cdot), y_{HT}(\cdot)\rangle \ge 0, \qquad \forall u_H(\cdot) \in L_{2e} \text{ and } \forall T \in R^+. \quad (3.3)$$

Since $H \in \vartheta_{IM}$, following Willems [1], [2], (3.3) is equivalent to the existence of a quadratic functon $S(x_H(t)) = \tfrac{1}{2}x_H'(t)R_H(t)x_H(t)$, satisfying

$$S(x_H(0)) + \langle u_{HT}(\cdot), y_{HT}(\cdot)\rangle \ge S(x_H(T)). \quad (3.4)$$

Consider

$$S(x_H(T)) - S(x_H(0)) - \langle u_{HT}(\cdot), y_{HT}(\cdot)\rangle = \int_0^T \left\{\frac{d}{dt}[\tfrac{1}{2}x_H'(t)R_H(t)x_H(t)] - u_H(t)y_H(t)\right\} dt$$
$$= \int_0^T \{\tfrac{1}{2}x_H'(t)[\dot{R}_H(t) + R_H(t)A_H(t) + A_H'(t)R_H(t)]x_H(t) + x_H'(t)R_H(t)b_H(t)u_H(t) - u_H(t)c_H'(t)x_H(t) - d_H(t)u_H^2(t)\}\,dt, \quad (3.5)$$

since $H \in \vartheta_{IM}$.

If the RHS $\le 0$, then (3.4) is satisfied. We will show this by proving that, if $R_H(t)$ is a solution of (3.2), the integrand on the RHS $\le 0$. Now, the integrand on the RHS of (3.5) is

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[1] $R$ denotes the real numbers; $R^+$, the nonnegative real numbers; and $R^n$, the $n$-dimensional Euclidean space.

[2] As an interesting aside, it may be noted that such operators are termed "Hilbert–Schmidt operators" in the mathematics literature [7, p. 54].

$$\leq \tfrac{1}{2}x_H'(t)[\dot{R}_H(t) + R_H(t)A_H(t) + A_H'(t)R_H(t)]x_H(t) + \sup_{u_H(\cdot)\in L_{2e}} [x_H'(t)R_H(t)b_H(t)u_H(t) - u_H(t)c_H'(t)x_H(t) - d_H(t)u_H^2(t)]. \quad (3.6)$$

Now evaluate the supremum on the RHS by differentiating this with respect to $u_H(t)$ and setting it equal to zero. Thus, on solving, we get

$$u_H(t) = \tfrac{1}{2}d_H^{-1}(t)[x_H'(t)R_H(t)b_H(t) - c_H'(t)x_H(t)] \quad (3.7)$$

and, because of (3.1), this $u_H(t)$, for which the supremum is attained, exists.

Substituting in (3.6) and simplifying, we have

$$\begin{aligned}\text{RHS of (3.6)} &= \tfrac{1}{2}x_H'(t)\{\dot{R}_H(t) + R_H(t)A_H(t) + A_H'(t)R_H(t) \\ &\quad + \tfrac{1}{2}d_H^{-1}(t)[R_H(t)b_H(t) - c_H(t)][b_H'(t)R_H(t) \\ &\quad - c_H'(t)]\}x_H(t) \\ &= 0,\end{aligned}$$

because of (3.2). Hence, the desired result follows.

b) *Strong positivity* ($\epsilon$) *of* $H$: It is required to prove that

$$\langle u_{HT}(\cdot), y_{HT}(\cdot)\rangle - \epsilon\langle u_{HT}(\cdot), u_{HT}(\cdot)\rangle \geq 0,$$

$$\forall u_H(\cdot) \in L_{2e}, \quad \forall\, T \in R^+, \quad \text{and for some} \quad \epsilon > 0. \quad (3.8)$$

$$\begin{aligned}\text{LHS} &= \langle u_{HT}(\cdot), c_H'(\cdot)x_{HT}(\cdot) + d_H(\cdot)u_{HT}(\cdot)\rangle - \epsilon\langle u_{HT}(\cdot), u_{HT}(\cdot)\rangle \\ &= \langle u_{HT}(\cdot), \bar{y}_{HT}(\cdot)\rangle,\end{aligned}$$

$$\text{where } \bar{y}_H(\cdot) = c_H'(\cdot)x_H(\cdot) + [d_H(\cdot) - \epsilon]u_H(\cdot).$$

Hence, (3.8) holds if $H$ is positive ($\epsilon$) with respect to the new output $\bar{y}_H(\cdot)$, i.e., with $d_H(\cdot)$ replaced by $[d_H(\cdot) - \epsilon]$. Q.E.D.

Using this lemma, the proofs of the following stability theorems are straightforward.

*Theorem 1 (Linear System):* If there exists an operator $M \in \vartheta_{EA} \cap \vartheta_{IM}$ such that $M^{-1} \in \vartheta_{EA}$, the composition $L = MG \in \vartheta_{IM}$ and if the following conditions:

a)

$$d_L(t) > \epsilon > 0, \qquad \forall\, t \in R^+, \quad d_M(t) > 0, \quad \forall\, t \in R^+; \quad (3.9)$$

b) there exist real symmetric bounded nonnegative-definite matrices $R_L(t)$ and $R_M(t)$, solutions of the Riccati equations,

$$\dot{R}_L(t) = \tfrac{1}{2}[d_L(t) - \epsilon]^{-1}[c_L(t) - R_L(t)b_L(t)][b_L'(t)R_L(t) - c_L'(t)] - [R_L(t)A_L(t) + A_L'(t)R_L(t)] \quad (3.10)$$

and

$$\dot{R}_M(t) = \tfrac{1}{2}d_M^{-1}(t)[c_M(t) - R_M(t)b_M(t)][b_M'(t)R_M(t) - c_M'(t)] - [R_M(t)A_M(t) + A_M'(t)R_M(t)] \quad (3.11)$$

are satisfied, then the system described by Fig. 1 is $L_2$-stable for all $G \in \vartheta_{EA} \cap \vartheta_{IM}$ and $N = E$.

*Theorem 2 (Nonlinear System):* If there exists an operator $M \in \vartheta_{EA} \cap \vartheta_{IM}$ such that $M^{-1} \in \vartheta_{EA}$, $L = MG \in \vartheta_{IM}$, and the following conditions:

a) $d_L(t) > \epsilon > 0$, $\forall\, t \in R^+$;

b) there exists a real symmetric bounded nonnegative-definite matrix $R_L(t)$, solution of (3.10); and

c) $M = \rho E + Z$, $\rho \in R^+$, $Z \in \vartheta_{EA} \ni z(t,\tau) = z(t - \tau)$, $\int_0^\infty |z(v)|\, dv < \rho$,

are satisfied, then the system described by Fig. 1 is $L_2$-stable for all $G \in \vartheta_{EA} \cap \vartheta_{IM}$ and $N \in \mathfrak{N}_{OM}$.

*Remark:* The condition of the theorems that $L = MG \in \vartheta_{IM}$ needs some explanation. Although $M,G \in \vartheta_I \Rightarrow L = MG \in \vartheta_I$,[3] it does not follow, in general, that $M,G \in \vartheta_{IM} \Rightarrow L \in \vartheta_{IM}$. But this does not pose a serious problem, since, $L$ being a linear operator, starting with an arbitrary realization of $L$(i.e., $L \in \vartheta_I$), it is possible to arrive at a minimal realization through well-established computational algorithms [6]. However, if one were to follow this approach, certain precautionary measures need be taken. Note that, since a minimal realization would result in certain extra states to be removed, which would not, consequently, influence the stability conditions, it is necessary to ensure *a priori* that these states do not contribute, by themselves, to instability. One sufficient condition guaranteeing this, for example, would be to require that $M$ and $G$ are globally asymptotically stable, which implies $\lim_{t\to\infty} x_M(t) = 0$ and $\lim_{t\to\infty} x_G(t) = 0$. It is to be emphasized that the stability theorems should be applied only after reducing $L$ to a minimal form.

[3] A simple realization of $L$ may be obtained by defining $x_L(t) = [x_G(t)\; x_M(t)]'$, $y_G(t) = u_M(t)$, $u_L(t) = u_G(t)$, and $y_L(t) = y_M(t)$.

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## Comments on "A Simplified Irreducible Realization Algorithm"

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In the above paper,[1] Chen and Mital have provided a theorem that enables a reduction in the size of the matrices used in Ho's algorithm for minimal realization. This is achieved by utilizing the additional information contained in the degrees of the least common denominators of the rows and columns of the transfer function matrix $G(s)$. Theorem 1[1] is shown here to be a simple consequence of known results for the realization of single-input multi-output and single-output multi-input systems. The proof given here is based on Lemma 4 [1], which is restated as the following.

*Lemma:* If $G(s)$ is a strictly proper rational matrix and the monic polynomial $\gamma(s)$ is the least common denominator of the entries, $g_{ij}(s)$, in $G(s)$, then $\gamma(s)$ is the minimal polynomial of the minimal realization of $G(s)$.

Consider a single-output multi-input system. The transfer function matrix $G(s)$ can then be written as (using the authors'[1] numbering of equations where possible)

$$G(s) = [g_1(s), g_2(s), \cdots, g_p] = \sum_{i=1}^{\infty} M_{i-1}s^{-i} \quad (1)$$

where

$$M_k = [m_{1k}m_{2k}\cdots m_{pk}] \quad (22)$$

and the $m_{ij}$ are defined by

$$g_i(s) = \sum_{j=1}^{\infty} {}_i m_{j-1}s^{-j}. \quad (3)$$

The usual Hankel matrix used in Ho's algorithm is defined by

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