same form as RL_1 if

$$\alpha(q - p) = \pm k\ 180°, \quad k = 0, \pm 1, \pm 2, \cdots \quad (6)$$

where q is the number of zeros and p the number of poles. (This is a sufficient condition.)

Demonstration: It will be proved with property 3.

Property 3

If in the same conditions as indicated in Property 2, the angular condition is

$$\alpha(q - p) = \pm k\ 360° \quad (7)$$

then RL_2 will be equivalent to RL_1.

Note that the rotations referred to in Properties 2 and 3 must be compatible with the fundamental property of RL being symmetrical with respect to the real axis.

Demonstration: By introducing a rotation α to a BC, each critical frequency will rotate the same angle α with respect to the real axis. Therefore, the added contribution to the angular condition of the new RL will be $\alpha(q - p)$. Thus if $\alpha(q - p) = \pm k\ 180°$, each point of the primitive RL will belong to the new one after being rotated α degrees.

Note that the $\pm k\ 180°$ in the angular condition for $s \in RL$ is due to the fact that K now describes the interval $-\infty < K < \infty$.

If k is an odd number, the form is invariant but the branches of the positive K parameter become negative and vice versa. When k is an even number, the modulus of each factor of $P(s)$ and $Q(s)$ remains invariant with respect to α, and since $|K| = |P(s)/Q(s)|$, it is apparent that RL_2 will be equivalent to the primitive RL_1.

Note that Properties 2 and 3 are only sufficient conditions. This is why it is possible that some RL_2 preserve their form, their equivalence, or even remain identical to the primitive without fulfilling the angular condition of (6) or (7). For example, for a BC which has the zeros and poles in the vertices of a regular polygon with single or multiple critical frequencies in the geometrical center, it is easy to find compatible rotations which do not fulfill the conditions (6) or (7) and give identical BCs.

Property 4

If a BC_2 is generated by a parallel to the real axis translation of a BC_1, the RL_2 generated for BC_2 will be equivalent to the RL_1 as generated for BC_1.

Note that this important property (the demonstration of which is omitted because it is well known) has been used by many writers to prove intrinsic properties of RLs regardless of the position of the imaginary axis.

Property 5

Let BC_1 and BC_2 be two basic configurations such that where there is a zero in one, there is a pole in the other and vice versa (intercharge of poles and zeros). Then RL_1 and RL_2, the corresponding RL, are coincident and in each point the relation $K_1 = 1/K_2$ holds.

Demonstration: By interchange of poles and zeros, the RL equation $P(s) + K_1Q(s) = 0$ becomes $K_2P(s) + Q(s) = 0$, which equations are equal if a substitution of $K_1 = 1/K_2$ is made.

This known property gives another way of producing RLs of the same form, but, in general, BC_1 will not be associate with BC_2, which requires that $p = q$.

By using the particular case of $p = q$, it is easy to prove Property 1 for zeros shifting.

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Comments on "On the Inverses of Certain Matrices"

Abstract—It is shown here that the method suggested by Chidambara[1] (Barlett's equation) is a particular version of the general matrix modification formula[2]

$$[A + USV^T]^{-1} = A^{-1} - A^{-1}U \cdot [S^{-1} + V^TA^{-1}U]^{-1} \cdot V^TA^{-1}$$

and that the modification suggested by Chidambara is obtained from this general form.

The matrix modification formula

$$[A + USV^T]^{-1} = A^{-1} - A^{-1}U \cdot [S^{-1} + V^TA^{-1}U]^{-1} \cdot V^TA^{-1} \quad (1)$$

[where A is an $n \times n$ nonsingular matrix, U and V are $n \times m$ matrices, S is an $m \times m$ nonsingular matrix (another formula is given when S is singular), and m is less than n] can be used to invert a modified matrix using the inverse of the original matrix. This procedure requires 3 mn^2 (or $2mn^2$) mathematical operations as against n^3 operations when the matrix is inverted directly (or $n^3/3$ operations when inversion is achieved through triangulation).

To invert a matrix B, the inverse of a matrix A, whose elements are the same as B except for one column or one row, is used and the following equation obtained from (1) gives the inverse of B.

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[1] M. R. Chidambara, *IEEE Trans. Automatic Control (Correspondence)*, vol. AC-12, pp. 214–215, April 1967.

[2] J. Walsh, *Numerical Analysis, An Introduction.* New York: Academic Press, 1966.

$$B^{-1} = (A + UV^T)^{-1} = A^{-1} - \frac{A^{-1}UV^TA^{-1}}{1 + V^TA^{-1}U} \quad (2)$$

where U and V are column vectors. If B has one column [nth column as given in Chidambara[1]] different from A, then

$$U^T = [d_1, d_2, \cdots, d_n] \quad (3)$$

$$V^T = [0, 0, \cdots, 1] \quad (4)$$

$$d_i = B_{ij} - A_{ij}. \quad (5)$$

Similarly, for a difference in one row, V will be full.

Our purpose here is to use (2) to get the following equation[1] (Barlett's equation)

$$[A + D]^{-1} = [I + A^{-1}D]^{-1}A^{-1} \quad (6)$$

where

$$D = \begin{bmatrix} 0 & 0 \cdots d_1 \\ 0 & 0 \cdots d_2 \\ 0 & 0 \cdots d_3 \\ \cdot & \cdots\cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 \cdots d_n \end{bmatrix}. \quad (7)$$

Rewriting (6),

$$[A + D]^{-1} = \begin{bmatrix} 1 & 0 \cdots - g_1/1 + g_n \\ 0 & 1 \cdots - g_2/1 + g_n \\ 0 & 0 \cdots \quad 1/1 + g_n \end{bmatrix} A^{-1} \quad (8)$$

where

$$g_i = \sum_{j=1}^{n} \alpha_{ij}d_j \quad (9)$$

and α_{ij} is an element of matrix A^{-1}, i.e.,

$$[A + D]^{-1} = \begin{bmatrix} 1 & 0 \cdots & -g_1/1 + g_n \\ \cdots & & \\ \cdots & & \\ 0 & 0 & 1 - g_n/1 + g_n \end{bmatrix} A^{-1} \quad (10)$$

$$= \left[I - \frac{1}{1 + g_n} G\right] A^{-1} \quad (11)$$

where

$$G = \begin{bmatrix} 0 & 0 \cdots g_1 \\ 0 & 0 \cdots g_2 \\ 0 & 0 \cdots g_n \end{bmatrix}$$

$$[A + D]^{-1} = A^{-1} - \frac{1}{1 + g_n} GA^{-1}. \quad (12)$$

So it is necessary to show that (2) and (12) are identical.

$$V^TA^{-1}U = [0, 0, \cdots, 1] \begin{bmatrix} \alpha_{11} \cdots \alpha_{1n} \\ \cdots \\ \alpha_{n1} \cdots \alpha_{nn} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_n \end{bmatrix} \quad (13)$$

$$= \alpha_{n1}d_1 + \alpha_{n2}d_2 + \cdots + \alpha_{nn}d_n$$

$$= g_n \quad (14)$$

$$UV^T = \begin{bmatrix} 0 & 0\cdots d_1 \\ 0 & 0\cdots d_2 \\ 0 & 0\cdots d_n \end{bmatrix} \tag{15}$$

$$A^{-1}UV^T = \begin{bmatrix} \alpha_{11}\cdots\alpha_{1n} \\ \alpha_{21}\cdots\alpha_{2n} \\ \alpha_{n1}\cdots\alpha_{nn} \end{bmatrix} \begin{bmatrix} 0\cdots d_1 \\ 0\cdots d_2 \\ 0\cdots d_n \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} 0 & 0\cdots\Sigma\alpha_{1j}d_j \\ 0 & 0\cdots\Sigma\alpha_{2j}d_j \\ 0 & 0\cdots\Sigma\alpha_{nj}d_j \end{bmatrix} = G. \tag{17}$$

Hence (2) becomes

$$[A + UV^T]^{-1} = A^{-1} - \frac{A^{-1}UV^TA^{-1}}{1 + V^TA^{-1}U}$$

$$= A^{-1} - \frac{1}{1 + g_n} GA^{-1}. \tag{18}$$

Equations (12) and (18) are identical. Hence this shows that (6) is another form of (2) which is a particular form of (1).

Equation (1) can be very effectively utilized also when two columns or two rows are different. U and V become $n \times 2$ matrices and S becomes a unit matrix of order 2×2. Then (1) becomes

$$[A + USV^T]^{-1} = A^{-1} - A^{-1}U \cdot [I + V^TA^{-1}U]^{-1} \cdot V^TA^{-1}. \tag{19}$$

This can be used for obtaining the inverse of a matrix using the inverse of another matrix whose elements are different only in two columns or two rows from the corresponding elements of the matrix to be inverted.

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Author's Reply[3]

Subramanian and Ramachandra Rao prove an interesting equivalence between my proposed method and Barlett's equation. I agree, although whether or not the method suggested is a particular version of Barlett's formula is a question of opinion.

Dr. G. Krishna of Clarkson College of Technology, Potsdam, N. Y., has recently brought to my attention another method[4] which treats the same problem.

All three of these methods[1,2,4] being exact solutions to the same problem must be equivalent.

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[3] Manuscript received September 5, 1968.

[4] G. Hadley, *Linear Algebra.* Reading, Mass.: Addison-Wesley, 1961.

Comments on "A Graphical Method for Finding the Real Roots of *n*th-Order Polynomials"

Abstract—**These comments on a previous correspondence show that the method exposed is a trivial consequence of Lill's method. Furthermore, the Hurwitz restriction is removed from the polynomial.**

In a recent correspondence,[1] Eisenberg introduces a graphical method, based on Lill's method, to obtain the coefficients of the reduced polynomial when a real root has been determined.

However, the procedure is a trivial consequence of Lill's method. Lill's method contains the tools to derive it directly as follows.

1) It is well known that if x_i is a real root of

$$\sum_{j=0}^{j=n} a_j x_{n-j} = 0$$

the coefficients of the reduced equation are

$$a_0' = a_0$$
$$a_k' = a_{k-1}'x_i + a_k. \tag{1}$$

2) From Lill's method,[2] the length $\overline{A_{j-1}P_{j-1}}$ has the following value (see Fig. 1):

$$\overline{A_{j-1}P_{j-1}} = a_{j-1}'x_i. \tag{2}$$

From (2) it follows that

$$\overline{P_{j-1}P_j} \cos\theta = \overline{P_{j-1}A_j} = a_j + a_{j-1}'x_i.$$

Use of (1) yields

$$\overline{P_{j-1}P_j} \cos\theta = a_j'.$$

3) If $a_0 = 1$, then

$$\overline{OP_0} \cos\theta = a_0 = 1.$$

Normalization with respect to $a_0' = \overline{OP_0}$ yields

$$\frac{P_{j-1}P_j}{\overline{OP_0}} = \frac{a_j'}{\overline{OP_0}\cos\theta} = a_j'.$$

On the other hand, to restrict the application of the method to Hurwitz polynomials does not appear justified, provided that Lill's method may be applied to any polynomial of real coefficients. The rule to obtain the sign of the coefficients of the reduced polynomial is the same as for the original one.

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[1] L. Eisenberg, *IEEE Trans. Automatic Control (Correspondence)*, vol. AC-12, pp. 629–631, October 1967.

[2] J. Rey Pastor, *Lecciones de Algebra.* Madrid: 1947, pp. 117–119.

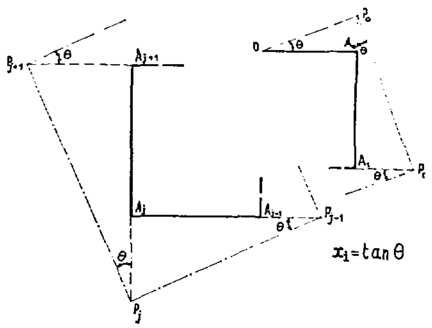

Fig. 1.

Author's Reply[3]

I would first like to thank Lage for pointing out that my original derivation need not be restricted to Hurwitz polynomials, but is true for all polynomials with real coefficients. Furthermore, the statement that "Lill's method contains the tools to derive it directly" is true as evidenced by the fact that both Lage and I utilized Lill's method as a starting point to further develop the technique. In fact, some of the equations in Lage's derivation are identical to mine except for the difference in notation. For example, his equation (2) is identical to (5) in Eisenberg[1] if one sets $j = 2$, $i = 1$,

$$A_{j-1} = C, \quad P_{j-1} = Q,$$

$$x_i = r_1, \quad a_{j-1}' = \sum_{i=2}^{n} r_i.$$

In conclusion, I would like to commend Lage for judiciously selecting a notation that yields a shorter derivation.

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[3] Manuscript received September 28, 1968.

Controllability of Stationary Linear Multivariable Systems Using a Frequency-Domain Criterion

Abstract—**The notion of truncated function is used as a new approach to establish controllability criteria for stationary linear multivariable systems. The method presented provides a simple technique to test the controllability of a system represented by its inputs and initial conditions transfer matrices. In addition, the formulation of the**

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