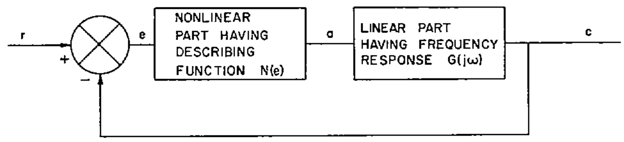

Fig. 1. Control system with one zero memory nonlinearity.

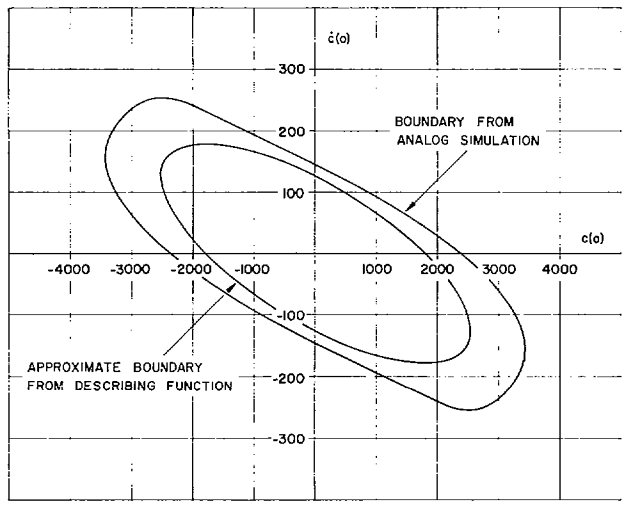

Fig. 2. Approximate and exact region of stable initial conditions for example problem.

With the assumption that $N(e)$ is replaced by a linear element as in the preceding paragraph, the response $e(t)$ can be calculated from knowledge of the initial conditions of the linear part; $e(t)$ will contain an oscillatory component caused by the two poles on the j axis. The steady amplitude of oscillation can be found from the evaluation of the residues of the two neutrally stable poles.

The peak of the steady-state response of e is now set equal to the limit cycle amplitude read along the $-N(e)$ locus at the $-N(e)$, $1/G(j\omega)$ crossing. An equation results which contains all of the initial conditions of the linear-part differential equations, and which expresses a surface in the initial condition space. Every point on the surface of initial conditions produces steady-state oscillation in the system (subject to the approximation inherent in the describing function method and the degree with which the variable e is represented by the linearized system). A new surface of initial conditions is obtained for each crossing of the $1/G(j\omega)$ and $-N(e)$ loci.

The following example illustrates the method. Let the nonlinear element be saturation (with saturation occurring at one unit of e), and let the linear part be expressed by

$$e = \ddot{c} + 0.1\dot{c} + 0.01c$$
$$\dddot{c} = -a. \tag{1}$$

Then

$$G(s) = \frac{-e}{a}(s) = \frac{s^2 + 0.1s + 0.01}{s^3}. \tag{2}$$

There is one crossing of $1/G(j\omega)$ and $-N(e)$ (which occurs on the negative real axis) with[2]

$$1/G(j\omega) = -N(e) = 0.1$$
$$e = 12.7$$
$$\omega = 0.1. \tag{3}$$

Now set

$$a = 0.1e = -0.1c \tag{4}$$

and substitute in (1) taking the Laplace transform

$$e(s) = \frac{0.01s^2c(0) + (0.1s^2 + 0.01s)\dot{c}(0)}{(s^2 + 0.01)(s + 0.1)} \tag{5}$$

where $\ddot{c}(0)$ has been assumed zero in the interest of simplicity. By evaluating the residue at $s = 0.1j$, the peak steady state $e(t)$ is found to be

$$\max\{e\} = \sqrt{\frac{1}{2}c^2(0)10^{-4} + \dot{c}^2(0)10^{-2} + c(0)\dot{c}(0)10^{-3}}. \tag{6}$$

By setting (6) equal to 12.7 [from (3)], the approximation to the boundary curve of stable initial conditions is obtained. A plot of the $\dot{c}(0)$ vs. $c(0)$ stability boundary approximation is shown in Fig. 2. The stability boundary as determined from analog simulation of (1) is shown for comparison.

H. N. Scofield
Astrionics Lab.
Marshall Space Flight Center
Huntsville, Ala.

[2] *Ibid.*, p. 572.

Forced Response of Linear Time-Varying Systems

The utility of error coefficients in studying the behavior of constant parameter linear systems has been well established [1]. This concept can be extended easily to time-varying linear systems in determining the response to any input, knowing $H(s, t)$, where $H(s, t)$ is Zadeh's system function [2]–[4].

The general input-output relation of a time-varying linear system can be written as

$$L(p, t)e_2(t) = k(p, t)e_1(t) \tag{1}$$

where $p = d/dt$, $e_1(t)$ is the input and $e_2(t)$, the output. In other words,

$$L(p, t)W(t, \tau) = k(p, t)\delta(\tau) \tag{2}$$

where $W(t, \tau)$ is the impulse response and $\delta(\tau)$ the delta function. The method suggested here consists in considering the output of the system as a superposition integral, given by (for signals applied to $t = 0$)

$$e_2(t) = \int_{-\infty}^{t} W(t, \tau)e_1(t - \tau)d\tau \tag{3}$$

and in expanding $e_1(t - \tau)$ in terms of Taylor's series so as to get (assuming that the first n-derivatives of $e_1(t)$ exist)

$$e_2(t) = +\int_{-\infty}^{t} W(t, \tau)\left[e_1(t) - \tau e_1'(t) + \frac{\tau^2}{2!}e_1''(t) \cdots\right]d\tau. \tag{4}$$

Since we are interested only in the forced response of the system, letting $t \to \infty$,

$$e_2(t) = \int_{-\infty}^{+\infty} W(t, \tau)\left[e_1(t) - \tau e_1'(t) + \frac{\tau^2}{2!}e_1''(t) \cdots\right]d\tau. \tag{5}$$

Let us now define

$$c_0 = \int_{-\infty}^{+\infty} W(t, \tau)d\tau$$
$$c_n = (-1)^n \int_{-\infty}^{+\infty} W(t, \tau)\tau^n d\tau. \tag{6a}$$

Substituting from (6a), (5) may be written as

$$e_2(t) = c_0e_1(t) + c_1e_1'(t) + \cdots \tag{6b}$$

which gives the forced response to any input $e_1(t)$. Also, by Zadeh's definition, the system function $H(s, t)$ is given by [2]–[4]

$$H(s, t) = \int_{-\infty}^{+\infty} W(t, \tau)e^{-s\tau}d\tau \tag{7}$$

$$\underset{s \to 0}{Lt}\, H(s, t) = \int_{-\infty}^{+\infty} W(t, \tau)d\tau = c_0 \tag{8}$$

$$\underset{s \to 0}{Lt}\, \frac{\partial}{\partial s}H(s, t) = -\int_{-\infty}^{+\infty} \tau W(t, \tau)d\tau = c_1 \tag{9}$$

or in general

Manuscript received June 1, 1965.

$$\underset{s\to 0}{Lt}\,\frac{\partial^n}{\partial s^n}H(s,t) = (-1)^n\int_{-\infty}^{+\infty}\tau^n W(t,\tau)d\tau$$
$$= c_n. \qquad (10)$$

Thus, given $H(s, t)$ we can find the response to any forcing function $e_1(t)$ using (6b) and (10). The coefficients $c_0, c_1, \cdots$ are all functions of time, and by comparison with their counterparts in the case of constant parameter linear systems, these may be called time-varying error coefficients. However, the use of these coefficients may not be very pronounced in the synthesis of linear time-varying systems.

Since $c_0, c_1, \cdots$ are all independent of $e_1(t)$, the forced response of any input may be quickly determined using (6b).

Examples

1) Let us consider a system, described by the differential equation,

$$\left[p^2 + \frac{3+8e^t}{1+2e^t}p + \frac{12e^t+12e^{2t}+2}{(1+2e^t)^2}\right]e_2 = f(t).$$

First we shall find the coefficients $c_0, c_1, \cdots$ of this system and then find the forced response when $f(t)$ is a unit step input, or a ramp input.

It can be readily checked that the $H(s, t)$ of this system is

$$H(s,t) = \frac{1}{1+2e^t}\left[\frac{1+e^t}{s+1} - \frac{1}{s+2} - \frac{e^t}{s+3}\right].$$

Hence

$$c_0 = \underset{s\to 0}{Lt}\,H(s,t) = \frac{3+4e^t}{6(1+2e^t)}$$
$$c_1 = \underset{s\to 0}{Lt}\,\frac{\partial}{\partial s}H(s,t) = -\frac{27+32e^t}{36(1+2e^t)}.$$

Thus $e_2(t)$ when $f(t)$ is a unit step would be

$$e_2(t) = \frac{3+4e^t}{6(1+2e^t)}$$

and $e_2(t)$ when $f(t)=t$ would be

$$e_2(t) = \frac{3+4e^t}{6(1+2e^t)} - \frac{27+32e^t}{36(1+2e^t)}.$$

The results may be checked by direct evaluation. The response to any input $f(t)$ may be determined in a similar manner.

2) For the system given by

$$[t^2p^2 + (3t^2+4t)p + (2t^2+6t+2)]e_2(t) = f(t)$$

it can be easily checked that

$$H(s,t) = \frac{1}{t^2(s+2)(s+1)}.$$

Hence

$$c_0 = \underset{s\to 0}{Lt}\,H(s,t) = \frac{1}{2t^2}$$
$$c_1 = \underset{s\to 0}{Lt}\,\frac{\partial}{\partial s}H(s,t) = \frac{-3}{4t^2}$$
$$c_2 = \underset{s\to 0}{Lt}\,\frac{\delta^2}{\partial s^2}H(s,t) = \frac{7}{4t^2}.$$

Thus when $f(t)=t$

$$e_2(t) = \frac{1}{2t} - \frac{3}{4t^2} \qquad \text{and for any other } f(t)$$
$$e_2(t) = c_0f(t) + c_1f'(t) + e_2f''(t) + \cdots.$$

ACKNOWLEDGMENT

The authors gratefully acknowledge the kind help of Dr. M. D. Srinath and M. A. L. Thathachar.

R. BALASUBRAMANIAN[1]
B. L. DEEKSHATULU
Dept. of Elec. Engrg.
Indian Institute of Science
Bangalore, India

REFERENCES

[1] B. C. Kuo, *Automatic Control Systems*. Englewood Cliffs, N. J.: Prentice-Hall, pp. 151–153, 1962.
[2] L. A. Zadeh, "Frequency analysis of variable networks," *Proc. IRE*, vol. 38, pp. 291–299, March 1950.
[3] ——, "Circuit analysis of linear varying parameter networks," *J. Appl. Phys.*, vol. 21, pp. 1171–1177, November 1950.
[4] W. Kaplan, *Operational Methods for Linear Systems*. Reading, Mass.: Addison-Wesley, 1962, pp. 515–525.

[1] R. Balasubramanian is now with the Dept. of Elec. Engrg., University of New Brunswick, Fredericton, N.B., Canada.

Tables of Laguerre Coefficients for Representation of Piecewise Linear Functions

This correspondence describes a method of evaluating the Laguerre coefficients of a continuous (or piecewise continuous) function which can be approximated by a piecewise linear function. The function is viewed as a sum of ramp functions. Table I can be used to find the contribution of each ramp function to the Laguerre coefficients.

TABLE I

A_k \ t_i	0.0	0.2	0.4	0.6	0.8	1.0
A_0	1.00000	0.81873	0.67032	0.54881	0.44933	0.36788
A_1	−1.00000	−0.98246	−0.93845	−0.87810	−0.80879	−0.73576
A_2	0.00000	0.01637	0.05363	0.09879	0.14379	0.18394
A_3	0.00000	−0.01528	−0.04648	−0.07910	0.01549	0.12263
A_4	0.00000	0.01439	0.04004	0.09393	0.14666	0.07664
A_5	0.00000	−0.17701	2.77646	0.30735	−0.09790	4.37469

GENERAL

In systems engineering it is sometimes advantageous to represent a functional relationship in terms of polynomials. However, one may encounter a functional relationship $c(t)$ defined for $0<t<\infty$ which cannot be conveniently represented by a polynomial. In such cases, it is possible to approximate the functions by the piecewise linear representation [1]

$$c(t) \cong \sum_{k=0}^{n} K_k(t-t_k)U(t-t_k) \qquad (1)$$

where $U(t-t_k)$ is the unit step function and K_k is a coefficient of the ramp function $(t-t_k)$.

It is clear that the function $c(t)$ is viewed as a sum of ramp functions. The approximation of a typical nonlinear characteristic is shown in Fig. 1. In Fig. 1, $c(t)$ is approximated by

$$c(t) = tU(t) - (t-0.2)U(t-0.2). \qquad (2)$$

One can convert such functions to standard polynomial form by having available a table of coefficients for approximating the function $(t-t_k)U(t-t_k)$. Such a table is used to find the contribution of each ramp function to the Laguerre coefficients.

If $A_0, A_1, A_2, \cdots, A_n$ represent the Laguerre spectra of $c(t)$, one may write

$$c(t) \cong \sum_{k=0}^{n} A_kL_k(t). \qquad (3)$$

$L_k(t)$ is the kth normalized Laguerre polynomial and can be expressed as [3]

$$L_k(t) = \frac{1}{k!}e^t\frac{d^k}{dt^k}(t^ke^{-t}). \qquad (4)$$

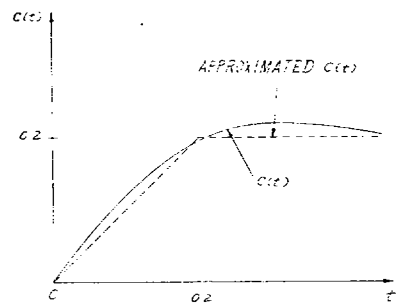

Fig. 1. The approximation of a nonlinear characteristic curve.

The characterizing coefficients A_k of the Laguerre spectrum of $c(t)$ are [3]

$$A_k = \int_0^\infty c(t)L_k(t)e^{-t}dt \qquad (5)$$

where e^{-t} is the weighting factor. $c(t)$ is a continuous (or piece-wise continuous) function such as (3):

An alternative expression for (4) is [2]

$$L_k(t) = \sum_{j=0}^{k}\frac{(-1)^j}{j!}\binom{k}{j}t^j$$
$$= 1 - kt + \frac{k(k-1)}{4}t^2 - \cdots$$
$$+ \frac{(-1)^k}{k!}t^k. \qquad (6)$$

The first few Laguerre polynomials are given as follows

Manuscript received August 20, 1964; revised March 22, 1965 and July 22, 1965. This work is a part of the research work under the Grant-in-Aid program supported by the Department of Electrical Engineering, Ohio State University, Columbus, Ohio.