Schrödinger equation and the oscillatory semigroup for the Hermite operator

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Abstract

We discuss the regularity of the oscillatory semigroup e^{itH}, where $H = -\Delta + |x|^2$ is the n-dimensional Hermite operator. The main result is a Strichartz-type estimate for the oscillatory semigroup e^{itH} in terms of the mixed L^p spaces. The result can be interpreted as the regularity of solution to the Schrödinger equation with potential $V(x) = |x|^2$.

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1. Introduction

Associated to any self-adjoint differential operator L on $\mathbb{R}^n$, one can formally define an oscillatory semigroup e^{-itL}, using the spectral theory for L. Assume that L has the spectral representation

$$Lf = \int_E \lambda \, dP_\lambda(f), \quad f \in L^2(\mathbb{R}^n),$$

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where P_λ is a projection valued measure supported on the spectrum E of L. Then the operator e^{-itL} can be defined by

$$e^{-itL} f = \int_E e^{-it\lambda}\, dP_\lambda(f)$$

for $f \in L^2(\mathbb{R}^n)$. Notice that the spectrum E may be continuous, discrete or a combination of both.

Consider the differential operator $i\frac{\partial}{\partial t} - L$ and the associated initial value problem for the Schrödinger equation for L:

$$i\partial_t u(x,t) - Lu(x,t) = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{1.1}$$

$$u(x,0) = f(x). \tag{1.2}$$

Assuming $f \in L^2(\mathbb{R}^n)$, the solution u can be represented by

$$u(x,t) = e^{-itL} f(x). \tag{1.3}$$

We thus call e^{-itL}, the Schrödinger semigroup for L. The special case when $L = -\Delta$, the Laplacian on $\mathbb{R}^n$, is the usual Schrödinger equation, which has been studied extensively by many authors. An important feature of the solution operator $e^{it\Delta}$ is that it is unitary on $L^2(\mathbb{R}^n)$ and hence $(-\Delta)^s u$ cannot be in $L^2(\mathbb{R}^n)$ for any $s > 0$. A fortiori the solution fails to have any regularity in terms of H^s Sobolev spaces on $\mathbb{R}^n$.

Having failed to be in any H^s, one can look for the solution to be in higher order L^p spaces. In this context, the following theorem of Strichrtz [St] is significant. Let $u(x,t)$ be the solution to the inhomogeneous equation

$$i\partial_t u(x,t) + \Delta u(x,t) = g(x,t), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R},$$

$$u(x,0) = f(x).$$

Theorem (*Strichartz*). *Let $f \in L^2(\mathbb{R}^n)$, $g \in L^{\frac{2(n+2)}{n+4}}(\mathbb{R}^n \times \mathbb{R})$, and u be the solution to the above equation. Then $u \in L^{\frac{2(n+2)}{n}}(\mathbb{R}^n \times \mathbb{R})$ and satisfies the inequality*

$$\left(\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |u(x,t)|^{\frac{2(n+2)}{n}}\, dx\, dt\right)^{\frac{n}{2(n+2)}} \leqslant C(\|f\|_2 + \|g\|_{\frac{2(n+2)}{n+4}}).$$

This result may be interpreted as follows: If $g \in L^{\frac{2(n+2)}{n+4}}(\mathbb{R}^n \times \mathbb{R})$ and $f \in L^2(\mathbb{R}^n)$, then for almost all $t \in \mathbb{R}$ the solution $u(\cdot,t)$ lies in a higher L^p space, namely for $p = \frac{2(n+2)}{n}$. This may be thought of as a regularity for the solution $u(\cdot,t)$.

In the same paper, he has also posed the problem of regularity of the Schrödinger equation of the form

$$i\partial_t u(x,t) + \Delta_x u(x,t) - V(x)u(x,t) = 0, \tag{1.4}$$

$$u(x,0) = f(x)$$

for a general potential $V(x)$.

For the case of bounded potential the problem has been studied extensively by many authors. See for instance [JSS,SS]. In [JSS], the authors have studied the dispersive nature of the solution for the Schrödinger equation for $L = -\Delta + V$, where the potential V is assumed to be pointwise bounded and satisfies certain decay condition near infinity. They also prove an analogue of Strichartz's theorem in this set up. Where as in [SS], the authors considered smooth potentials having bounded derivatives of all orders. In this case, they studied the local regularity of solutions, in terms of mixed sobolev spaces on $\mathbb{R}^n \times \mathbb{R}$ for initial data from some sobolev space $W^{2,s}(\mathbb{R}^n)$.

Traditionally, the initial value problems with general potential were attempted using Fourier analysis, i.e., the spectral theory of Laplacian. Instead we propose the idea of looking at the equation as the Schrödinger equation for the operator $-\Delta + V(x)$ and exploit the spectral analysis of $-\Delta + V(x)$. This approach becomes significant, especially when the potential $V(x)$ is unbounded near infinity, say for instance when $V(x)$ is a polynomial in x. Spectral properties of $-\Delta + V$ for such potentials have been studied by Titchmarsh [Ti], see also [RS, Theorem XIII.81].

Of specific interest is the case when $V(x) = |x|^2$. In this case, the operator $-\Delta + V(x)$ is the Hermite operator and the harmonic analysis of Hermite functions comes into play. Thus when $V(x) = |x|^2$, the initial value problem (1.4) reduces to an initial value problem for the Schrödinger equation, associated to the Hermite operator $H = -\Delta + |x|^2$:

$$i\partial_t u(x,t) - Hu(x,t) = 0 \quad x \in \mathbb{R}^n, \ \ t > 0, \tag{1.5}$$

$$u(x,0) = f(x). \tag{1.6}$$

For higher order elliptic differential operators P of constant coefficients, the regularity properties of e^{itP} has been studied by Kenig et al. [KPV]. They gave an improvement of the Strichartz result by establishing an inequality in terms of the mixed L^p spaces given by

$$L_{p,q}(\mathbb{R}^n \times \mathbb{R}) = \{u : u \text{ is measurable on } \mathbb{R}^n \times \mathbb{R}, \|u\|_{L^q(\mathbb{R};L^p(\mathbb{R}^n))} < \infty\},$$

where $\|u\|_{L^q(\mathbb{R};L^p(\mathbb{R}^n))} = (\int_{\mathbb{R}} \|u(\cdot,t)\|_p^q \, dt)^{\frac{1}{q}}$ is the norm in $L_{p,q}(\mathbb{R}^n \times \mathbb{R})$. In fact they considered more general operators of the form $P([-\Delta]^{\frac{1}{2}})$, where P is an element from a class of functions of a real variable that includes polynomials. In this case they showed that the solution u even possesses some space derivatives lying in $L_{p,q}(\mathbb{R}^n \times \mathbb{R})$ for "suitable" p and q when the degree of P is greater than 2, see [KPV].

A natural way to measure regularity of e^{itH} is in terms of the Hermite–Sobolev spaces

$$W_H^{2,s}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : H^s f \in L^2(\mathbb{R}^n)\}.$$

Analogous to the property, that the Schrödinger semigroup $e^{it\Delta}$ is unitary on L^2, one can easily see, using the spectral theory of H, that the semigroup e^{itH} is also unitary on $L^2(\mathbb{R}^n)$. Consequently, $H^s e^{itH} f$ fails to be in $L^2(\mathbb{R}^n)$ when $f \in L^2$, for any $s > 0$. In other words, we cannot expect any better regularity for $e^{-itH} f(x)$ as a function of x, in terms of $W_H^{2,s}$ Sobolev spaces. This is a general feature of the oscillatory semigroups. We now state our main regularity result in terms of higher L^p spaces.

Theorem 1.1. *Let $f \in L^2(\mathbb{R}^n)$ and let $u(x,t) = e^{itH} f(x)$ be the solution of the initial value problem* (1.5), (1.6). *Then u is periodic in t and $u \in L^q([-\pi,\pi]; L^p(\mathbb{R}^n))$, $1 < q < \infty, 2 \leqslant p < \Lambda$, where $\Lambda = \infty$ for $n = 1$ and $\Lambda = \frac{2n}{n-2}$ for $n \geqslant 2$. Further u satisfies the inequality*

$$\|u\|_{L^q([-\pi,\pi];\, L^p(\mathbb{R}^n))} \leqslant C_n \|f\|_2 \tag{1.7}$$

for $1 < q < \infty, 2 \leqslant p < \Lambda$.

We remark that L^2 is the natural space to consider for the initial data, as the wave function in quantum mechanics, defining the probability density of finding a particle in a given region, is supposed to be an L^2 function.

An interesting contrast between the regularity of solution for the usual Schrödinger equation and the Schrödinger equation associated to Hermite operator is in order. It is observed in [St], that for initial data $f \in L^2(\mathbb{R}^n)$, the solution $u(\cdot,\cdot) = e^{it\Delta} f \in L^p(\mathbb{R}^n \times \mathbb{R})$ for $p = \frac{2(n+2)}{n}$ and hence for almost all $t \in \mathbb{R}$, $u(\cdot,t) \in L^{\frac{2(n+2)}{n}}(\mathbb{R}^n)$. Since $u(\cdot,t)$ is also in $L^2(\mathbb{R}^n)$ for each t, it follows that for almost all $t \in \mathbb{R}$, $u(\cdot,t) \in L^p(\mathbb{R}^n)$ for $2 \leqslant p \leqslant \frac{2(n+2)}{n}$.

However for the case of the Schrödinger equation for the Hermite operator, our result shows that for almost all $t \in \mathbb{R}$, the solution $u(\cdot,t) \in L^p(\mathbb{R}^n)$ for a wider range of p, namely $2 \leqslant p < \Lambda$, where Λ is as in Theorem 1.1. Notice that $\Lambda > \frac{2(n+2)}{n}$, the index in Strichartz estimate. This also indicates extra regularity in this situation.

Further, the solution exhibits periodicity in the time variable with period 2π (see Section 2), a feature solely a consequence of the discreteness of the spectrum of the Hermite operator. So the natural space to consider in this situation is the mixed L^p space $L^q(\mathbb{S}^1; L^p(\mathbb{R}^n))$, where $\mathbb{S}^1$ is the unit circle.

We also consider the inhomogeneous problem with a nonzero function $g(x,t)$ thrown in on the RHS of (1.5). In this case we show that the solution has the same regularity i.e., $u \in L^q(\mathbb{S}^1; L^p(\mathbb{R}^n)$, if $f \in L^2(\mathbb{R}^n)$ and $g \in L^{q'}(\mathbb{S}^1; L^{p'}(\mathbb{R}^n))$, $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$, $1<q<\infty$, $2 \leqslant p < \Lambda$.

Our approach relies on a simple regularization technique. We establish the above theorem first for regular initial data and then deduce the same result for the general L^2 function by suitable limiting arguments.

The layout of the paper is as follows. In Section 2, we recall the spectral theory of the Hermite operator. We represent the solution as an integral operator with an appropriate Kernel using Mehler's formula. An estimate for the kernel is provided in Section 3, where we also prove our main theorem as well as the regularity result for the nonhomogeneous equation.

A good reference for a general discussion of Schrödinger operators is the book by Reed and Simon [RS]. Also a somewhat recent survey of results on Schrödinger operators can be seen in [Si].

2. Spectral theory for the Hermite operator and the kernel estimate

Recall that for each nonnegative integer k, the Hermite polynomials $H_k(x)$ on $\mathbb{R}$ are defined by $H_k(x)e^{-x^2} = (-1)^k \frac{d^k}{dx^k}(e^{-x^2})$. Then the functions h_k given by

$$h_k(x) = \frac{(-1)^k}{\sqrt{2^k k! \sqrt{\pi}}} \left(\frac{d^k}{dx^k} e^{-x^2} \right) e^{\frac{x^2}{2}}$$

are in $L^2(\mathbb{R}^n)$ and $\|h_k\|_2 = 1$ for $k = 0, 1, 2, \ldots$. Now for each multiindex $\alpha = (\alpha_1, \ldots, \alpha_1)$, $\alpha_i \geqslant 0$, we define the n-dimensional Hermite functions by tensor product : $h_\alpha(x) = \Pi_{i=1}^n h_{\alpha_i}(x_i)$. Then the functions $h_\alpha, \alpha \in (\mathbb{Z}_+ \cup \{0\})^n$ are eigenfunctions for the Hermite operator with eigenvalue $2|\alpha| + n$ and they form a complete orthonormal system in $L^2(\mathbb{R}^n)$.

Thus every $f \in L^2(\mathbb{R}^n)$ has the Hermite expansion

$$f = \sum_\alpha \langle f, h_\alpha \rangle h_\alpha = \sum_{k=0}^{\infty} P_k f,$$

where P_k denotes the Hermite projection operator given by

$$P_k f(x) = \sum_{|\alpha|=k} \langle f, h_\alpha \rangle h_\alpha(x).$$

Setting $\Phi_k(x, y) = \sum_{|\alpha|=k} h_\alpha(x) h_\alpha(y)$, the Hermite projection may be written as

$$P_k f(x) = \int_{\mathbb{R}^n} \Phi_k(x, y) f(y)\, dy.$$

The function $\Phi_k(x, y)$ can be obtained by the following generating function identity:

$$\sum_{k=0}^{\infty} \omega^k \Phi_k(x, y) = \pi^{-\frac{n}{2}}(1-\omega^2)^{-\frac{n}{2}} e^{-\frac{1}{2}\frac{1+\omega^2}{1-\omega^2}(|x|^2+|y|^2)+\frac{2\omega}{1-\omega^2}x\cdot y}, \tag{2.1}$$

for $|\omega| < 1$. This identity is known as the Mehler's formula (see [T, p. 2]) for the n-dimensional Hermite functions h_α and is the key identity in our analysis. For more details about the harmonic analysis of the Hermite functions, we refer the reader to the beautiful monograph by Thangavelu [T].

The solution to the initial value problem (1.5), (1.6), is given by

$$u(x, t) = e^{-itH} f(x) = \sum_{k=0}^{\infty} e^{-it(2k+n)} P_k f(x). \tag{2.2}$$

Clearly $u(x, t)$ given by (2.2) converges in $L^2(\mathbb{R}^n)$ for all fixed $t \in \mathbb{R}$ whenever $f \in L^2(\mathbb{R}^n)$. It is also easy to see that $u(x, t) \to f(x)$ in $L^2(\mathbb{R}^n)$ as $t \to 0$. This follows from a dominated convergence argument since $|e^{-it(2k+n)} - 1| \leqslant 2$. Thus (2.2) gives the solution to the initial value problem (1.5), (1.6) in the L^2 sense.

The solution given by (2.2) can be formally expressed as an integral operator with kernel

$$K_t(x, y) = \sum_{k=0}^{\infty} e^{-it(2k+n)} \Phi_k(x, y) \tag{2.3}$$

which converges in the distribution sense. Clearly, $K_{t+2\pi}(x, y) = K_t(x, y)$ and hence the solution is periodic in t with period 2π.

For $z = r + it,\ r > 0,\ t \in \mathbb{R}$, we consider the kernel

$$K_z(x, y) = \sum_{k=0}^{\infty} e^{-z(2k+n)} \Phi_k(x, y) \tag{2.4}$$

which is the kernel associated to the semigroup e^{-zH}. Clearly, the semigroup e^{-zH} is also periodic in t with period 2π.

The above series can be re-written as

$$K_z(x, y) = e^{-n(r+it)} \sum_{k=0}^{\infty} \omega^k \Phi_k(x, y)$$

with $\omega = e^{-2(r+it)}$. Using Mehler's formula (2.1), we get

$$K_z(x, y) = \pi^{-\frac{n}{2}} e^{-n(r+it)} (1-\omega^2)^{-\frac{n}{2}} e^{-\frac{1}{2}\frac{1+\omega^2}{1-\omega^2}(|x|^2+|y|^2)+\frac{2\omega}{1-\omega^2}x\cdot y}. \tag{2.5}$$

Now we prove the following kernel estimate.

Lemma 2.1. *Let $K_z(x, y)$ be as before with $z = r + it$, $r > 0$, $0 < |t| \leqslant \pi$. Then*

$$|K_z(x, y)| \leqslant \frac{e^{-nr}}{|\sin 2t|^{n/2}}.$$

Proof. We use formula (2.5). Since $\omega = e^{-2(r+it)}$, we have

$$1 - \omega^2 = 2e^{-2(r+it)} \sinh[2(r + it)].$$

It follows that

$$e^{-n(r+it)}(1 - \omega^2)^{-\frac{n}{2}} = (2 \sinh[2(r + it)])^{-\frac{n}{2}}.$$

Also since

$$\sinh(2z) = \cos 2t \sinh 2r + i \sin 2t \cosh 2r,$$

we have

$$|\sinh 2z| \geqslant |\sin 2t \cosh 2r|.$$

Thus we get

$$|\sinh 2z|^{-\frac{n}{2}} \leqslant \frac{e^{-nr}}{|\sin 2t|^{\frac{n}{2}}}.$$

Thus to complete the proof of the lemma, it is enough to show that the last exponential term in (2.5) is bounded by 1. This amounts to showing that

$$\Re\left(-\frac{1}{2}\frac{1+\omega^2}{1-\omega^2}(|x|^2 + |y|^2) + \frac{2\omega}{1-\omega^2}x \cdot y\right) \leqslant 0, \tag{2.6}$$

where $\Re$ denotes the real part.

A simple computation shows that

$$\Re\left(\frac{1+\omega^2}{1-\omega^2}\right) = \frac{1-e^{-8r}}{1+e^{-8r}-2e^{-4r}\cos 4t}$$

and

$$\Re\left(\frac{\omega}{1-\omega^2}\right) = \frac{e^{-2r}(1-e^{-4r})\cos 2t}{1+e^{-8r}-2e^{-4r}\cos 4t}.$$

Using these and the fact that

$$x\cdot y\ \cos 2t \leqslant |x\cdot y| \leqslant \frac{|x|^2+|y|^2}{2},$$

we see that

$$\begin{aligned}
&\Re\left(-\frac{1}{2}\frac{1+\omega^2}{1-\omega^2}(|x|^2+|y|^2)+\frac{2\omega}{1-\omega^2}x\cdot y\right)\\
&\leqslant \frac{-\frac{1}{2}[(1-e^{-8r})-2e^{-2r}(1-e^{-4r})](|x|^2+|y|^2)}{1+e^{-8r}-2e^{-4r}\cos 4t}\\
&\leqslant -\frac{\frac{1}{2}(|x|^2+|y|^2)(1-e^{-4r})}{1+e^{-8r}-2e^{-4r}\cos 4t}[1-e^{-2r}]^2.
\end{aligned}$$

Since $r>0$ and the denominator is nonnegative, the above expression is nonpositive for $r>0$. This completes the proof. □

3. Schrödinger equation for the Hermite operator

We consider the initial value problem for the inhomogeneous equation

$$i\partial_t u(x,t) - Hu(x,t) = g(x,t), \quad x\in\mathbb{R}^n,\ \ 0<t\leqslant 2\pi, \tag{3.1}$$

$$u(x,0) = f(x). \tag{3.2}$$

We first consider the homogeneous case, namely $g(\cdot,\cdot)\equiv 0$. The solution is given by $u(x,t)=e^{-itH}f(x)$, which is an integral operator with kernel $K_t(x,y)$ given by (2.3). As the series (2.3) representing the kernel $K_t(x,y)$ does not have 'nice' convergence properties, for the given f, we first look at the regularized problem

$$i\partial_t u(x,t) - Hu(x,t) = 0, \quad x \in \mathbb{R}^n, \ 0 < t \leqslant 2\pi, \tag{3.3}$$

$$u(x,0) = f_r(x), \tag{3.4}$$

where $f_r = e^{-rH} f$. Notice that for a given $f \in L^2(\mathbb{R}^n)$ and $r > 0$, $f_r \in W_H^{2,s}(\mathbb{R}^n)$ for all $s \geqslant 0$. The solution in this case is given by

$$u_r(x,t) = e^{-itH} e^{-rH} f(x) = \sum_{k=0}^{\infty} e^{-(r+it)(2k+n)} P_k f(x).$$

The above solution u_r can be represented by the integral operator

$$u_r(x,t) = \int_{\mathbb{R}^n} K_z(x,y) f(y)\, dy, \quad z = r + it \tag{3.5}$$

with kernel $K_z(x,y)$ given by (2.4).

The following proposition is a simple application of the estimate of the previous Lemma 2.1 and the Riesz–Thorin interpolation theorem.

Proposition 3.1. *For each $r > 0$, the solution u_r to the regularized problem satisfies the inequality*

$$\|u_r(\cdot,t)\|_p \leqslant |\sin 2t|^{-n(\frac{1}{p'} - \frac{1}{2})} \|f\|_{p'},$$

for $|t| > 0$, where $1 \leqslant p' \leqslant 2$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Using Eq. (3.5) and the uniform estimate for $K_z(x,y)$ given by Lemma 2.1, we get the obvious $L^1 - L^\infty$ estimate

$$\|u_r(\cdot,t)\|_\infty \leqslant \frac{e^{-nr}}{|\sin 2t|^{n/2}} \|f\|_1$$

valid for every t such that $|t| > 0$. Also since e^{-zH} is a bounded on $L^2(\mathbb{R}^n)$ for $r > 0$, we have for each fixed t, $|t| > 0$,

$$\|u_r(\cdot,t)\|_2 < \|f\|_2.$$

Interpolating these two inequalities, we get

$$\|u_r(\cdot,t)\|_p \leqslant C_n(\theta) \|f\|_{p'}, \tag{3.6}$$

where $\frac{1}{p}=\frac{\theta}{\infty}+\frac{1-\theta}{2}$, $\frac{1}{p'}=\frac{\theta}{1}+\frac{1-\theta}{2}$ for $0<\theta<1$ and

$$C_n(\theta)=\left(\frac{e^{-nr}}{|\sin 2t|^{\frac{n}{2}}}\right)^{\theta}\leqslant\frac{1}{|\sin 2t|^{\frac{n\theta}{2}}}.$$

This completes the proof since $\frac{\theta}{2}=\frac{1}{p'}-\frac{1}{2}$. $\square$

Proposition 3.2. *Let* $h(\cdot,\cdot)\in L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n))$, $1\leqslant q'\leqslant\infty$, $\max(1,\frac{2n}{n+2})<p'\leqslant 2$. *Then*

$$\left\|\int_{\mathbb{S}^1}e^{-itH}h(\cdot,t)\,dt\right\|_2\leqslant C\|h\|_{L^{q'}(\mathbb{S}^1;L^{p'}(\mathbb{R}^n))}.$$

Proof. For each $\varepsilon>0$, let $h_\varepsilon(\cdot,t)=e^{-\varepsilon H}h(\cdot,t)$. Then $h_\varepsilon(\cdot,t)\in L^{p'}\cap L^2(\mathbb{R}^n)$, since $e^{-\varepsilon H}$ is bounded on $L^p(\mathbb{R}^n)$, for $1<p<\infty$ for $\varepsilon>0$, see [T, Theorem 4.2.1].

The proof basically relies on an idea of Tomas [To], originally used in the proof of the restriction theorem for the Fourier transform on $\mathbb{R}^n$, applied to $h_\varepsilon(\cdot,t)$. We have

$$\begin{aligned}\left\|\int_{\mathbb{S}^1}e^{-itH}h_\varepsilon(\cdot,t)\,dt\right\|_2^2&=\int_{\mathbb{R}^n}\left(\int_{\mathbb{S}^1}e^{-itH}h_\varepsilon(\cdot,t)\,dt\,\overline{\int_{\mathbb{S}^1}e^{-isH}h_\varepsilon(\cdot,s)\,ds}\right)dx\\&=\int_{\mathbb{R}^n}\int_{\mathbb{S}^1}\left(\overline{\int_{\mathbb{S}^1}e^{i(t-s)H}h_\varepsilon(\cdot,s)\,ds}\right)h_\varepsilon(\cdot,t)\,dt\,dx.\end{aligned}$$

Notice that the second equality follows, by simple manipulations using the expansions for $e^{-itH}h_\varepsilon(\cdot,t)$ and $e^{-isH}h_\varepsilon(\cdot,s)$. The double sum occurring there reduces to a single sum because of the orthogonality of the spectral projections. Now using the Hölder's inequality for the mixed L^p spaces we get

$$\begin{aligned}&\left\|\int_{\mathbb{S}^1}e^{-itH}h_\varepsilon(\cdot,t)\,dt\right\|_2^2\\&\quad\leqslant\|h_\varepsilon(x,t)\|_{L^{q'}(\mathbb{S}^1;\,L^{p'}(\mathbb{R}^n))}\left\|\int_{\mathbb{S}^1}e^{i(t-s)H}h_\varepsilon(\cdot,s)\,ds\right\|_{L^q(\mathbb{S}^1;\,L^p(\mathbb{R}^n))}.\end{aligned}\tag{3.7}$$

Now we establish the inequality

$$\left\|\int_{\mathbb{S}^1}e^{i(t-s)H}h_\varepsilon(\cdot,s)\,ds\right\|_{L^q(\mathbb{S}^1;\,L^p(\mathbb{R}^n))}\leqslant\|h(\cdot,\cdot)\|_{L^{q'}(\mathbb{S}^1;\,L^{p'}(\mathbb{R}^n))}.$$

By using Proposition 3.1, we see that for $2 \leqslant p \leqslant \infty$,

$$\left\| \int_{\mathbb{S}^1} e^{i(t-s)H} h_\varepsilon(\cdot, s)\, ds \right\|_p \leqslant \int_{-\pi}^{\pi} \|e^{i(t-s)H} h_\varepsilon(\cdot, s)\, ds\|_p\, ds$$
$$\leqslant \int_{-\pi}^{\pi} \frac{\|h(\cdot, s)\|_{p'}}{|\sin(2[t-s])|^{n(\frac{1}{p'}-\frac{1}{2})}}\, ds.$$

Notice that since $p' > \max(1, \frac{2n}{n+2})$, $n(\frac{1}{p'} - \frac{1}{2}) < 1$ and hence $|\sin 2s|^{-n(\frac{1}{p'}-\frac{1}{2})} \in L^1(\mathbb{S}^1\, ds)$. Since $\|h(\cdot, s)\|_{p'} \in L^{q'}(\mathbb{S}^1\, ds)$ for $1 \leqslant q' \leqslant \infty$, by Young's inequality the $L^{q'}(\mathbb{S}^1\, ds)$ norm of RHS of the above integral is at most a constant times

$$\left(\int_{-\pi}^{\pi} \|h(\cdot, s)\|_{p'}^{q'}\, ds \right)^{\frac{1}{q'}}$$

which is nothing but $\|h\|_{L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{R}^n))}$.

Since the operator $e^{-\varepsilon H}$ is bounded on $L^{p'}(\mathbb{R}^n)$, for $1 < p' < \infty$, for $\varepsilon > 0$, it follows that $e^{-\varepsilon H}$ is bounded on $L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{R}^n))$. Using this fact, from (3.7) we conclude that

$$\left\| \int_{\mathbb{S}^1} e^{-itH} h_\varepsilon(\cdot, t)\, dt \right\|_2 \leqslant C \|h\|_{L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{R}^n))}.$$

Since $h_\varepsilon(\cdot, t) \to h(\cdot, t)$ in $L^2(\mathbb{R}^n)$, as $\varepsilon \to 0$, for a.e., $t \in \mathbb{S}^1$, a simple dominated convergence argument will show that $\int_{\mathbb{S}^1} e^{-itH} h_\varepsilon(\cdot, t)\, dt$ converges to $\int_{\mathbb{S}^1} e^{-itH} h(\cdot, t)\, dt$ in $L^2(\mathbb{R}^n)$. Therefore letting $\varepsilon \to 0$ in the above inequality, we get the required estimate. $\square$

Remark. The condition $p' > \max\{1, \frac{2n}{n+2}\}$ distinguishes the two cases namely $n = 1$ and $n \geqslant 2$ arising in the main theorem.

Now we establish the estimate for the regularized problem.

Proposition 3.3. *Let $f \in L^2(\mathbb{R}^n)$ and let u_r be the solution to the regularized problem* (3.3), (3.4). *Then for each $r > 0$, u_r satisfies the inequality*

$$\|u_r\|_{L^q(\mathbb{S}^1;\, L^p(\mathbb{R}^n))} \leqslant C_n \|f_r\|_2 \tag{3.8}$$

for $1 < q < \infty$, $2 \leqslant p < \frac{2n}{n-2}$ with a constant C_n independent of r and f.

Proof. Let $h \in L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n))$. Since the operator e^{-itH} is unitary, we have

$$\int_{\mathbb{S}^1}\int_{\mathbb{R}^n}[e^{-itH} f_r](x)\overline{h(x,t)}\,dx\,dt = \int_{\mathbb{S}^1}\int_{\mathbb{R}^n} f_r(x)\overline{e^{itH}h(x,t)}\,dx\,dt.$$

An interchange of integrals followed by Cauchy–Schwarz inequality gives

$$\begin{aligned}\left|\int_{\mathbb{S}^1}\int_{\mathbb{R}^n}[e^{-itH} f_r](x)\overline{h(x,t)}\,dx\,dt\right| &= \left|\int_{\mathbb{R}^n} f_r(x)\left(\overline{\int_{\mathbb{S}^1} e^{itH}h(x,t)\,dt}\right)dx\right|\\ &\leqslant \|f_r\|_2\left\|\int_{\mathbb{S}^1} e^{itH}h(\cdot,t)\,dt\right\|_2.\end{aligned}$$

Now from the estimate for $\|\int_{\mathbb{S}^1} e^{itH}h(\cdot,t)\,dt\|_2$ given by Proposition 3.2, it follows that

$$\left|\int_{\mathbb{S}^1}\int_{\mathbb{R}^n} e^{-itH} f_r(x)\overline{h(x,t)}\,dx\,dt\right| \leqslant C\|f_r\|_2\|h\|_{L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{R}^n))}.$$

By density of $L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n))$ in $L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{R}^n))$ the required estimate follows by duality. $\square$

Now we deduce the regularity result for the original problem as stated in Theorem 1.1.

Proof of Theorem 1.1. For $r > 0$, e^{-rH} is a bounded operator on $L^2(\mathbb{R}^n)$, and we have $\|f_r\|_2 \leqslant \|f\|_2$. Therefore by previous proposition we get

$$\|u_r\|_{L^q(\mathbb{S}^1;\ L^p(\mathbb{R}^n))} \leqslant C_n\|f\|_2 \tag{3.9}$$

for some constant C_n independent of r. From this we deduce the corresponding inequality for u.

First we show that there is a subsequence u_{r_n} that converges to u almost everywhere. Notice that since $f \in L^2(\mathbb{R}^n)$, for each fixed $t > 0$ and $r > 0$, $u_r(\cdot,t) = e^{-rH}e^{-itH} f \in L^2(\mathbb{R}^n)$. Moreover

$$\begin{aligned}\|u_r(\cdot,t) - u(\cdot,t)\|_2^2 &= \left\|\sum [e^{-r(2k+n)} - 1]e^{it(2k+n)}P_k f\right\|_2^2\\ &= \sum_0^\infty [e^{-r(2k+n)} - 1]^2\|P_k f\|_2^2\end{aligned}$$

which converges to zero as $r \to 0$, by a dominated convergence argument applied to the summation. Thus $u_r(\cdot, t)$ converges to $u(\cdot, t)$ in $L^2(\mathbb{R}^n)$ for each fixed $t > 0$ as $r \to 0$. By periodicity of u in t, we need to consider only $0 < t \leqslant 2\pi$. Since $\|u_r(\cdot, t) - u(\cdot, t)\|_2^2 \to 0$ as $r \to 0$, and since

$$\|u_r(\cdot, t) - u(\cdot, t)\|_2^2 \leqslant (2\,\|u(\cdot, t)\|_2)^2 = 4\|f\|_2^2$$

by dominated convergence theorem, it follows that

$$\lim_{r \to 0} \int_0^{2\pi} \|u_r(\cdot, t) - u(\cdot, t)\|_2^2 \, dt \to 0.$$

Thus u_r converges to u in $L^2(\mathbb{R}^n \times \mathbb{S}^1)$. Hence we get a subsequence u_{r_m} that converges to u for $a.e.(x, t) \in \mathbb{R}^n \times \mathbb{S}^1$ as $r_m \to 0$.

Now by Fatou's lemma

$$\|u(\cdot, t)\|_p^p = \int_{\mathbb{R}^n} |u(x, t)|^p \, dx \leqslant \liminf_{r_m \to 0} \int_{\mathbb{R}^n} |u_{r_m}(x, t)|^p \, dx.$$

Applying Fatou's lemma once again, we get

$$\begin{aligned} \int_0^{2\pi} \|u(\cdot, t)\|_p^q \, dt &\leqslant \int_0^{2\pi} \left(\liminf_{r_m \to 0} \int_{\mathbb{R}^n} |u_{r_m}(x, t)|^p \, dx \right)^{q/p} dt \\ &\leqslant \liminf_{r_m \to 0} \int_0^{2\pi} \left(\int_{\mathbb{R}^n} |u_{r_m}(x, t)|^p \, dx \right)^{q/p} dt \\ &\leqslant (C_n \|f\|_2)^q, \end{aligned}$$

where the last inequality follows from (3.9) since the constant C_n is independent of r_m. Taking the qth root on both sides, we get the required inequality. □

Now we consider the inhomogeneous equation (3.1) and (3.2). By Duhamel's principle, the solution can be written as

$$u(x, t) = e^{itH} f(x) + \int_0^t e^{i(t-s)H} g(x, s) \, ds. \tag{3.10}$$

In this case the solution need not be periodic in the t variable, unless g is periodic in the t variable. For the inhomogeneous equation in the periodic case we prove the following:

Theorem 3.4. *Let* $f \in L^2(\mathbb{R}^n)$ *and* $g \in L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{R}^n))$ *then the solution* $u(x,t)$ *to the problem* (3.1), (3.2) *lies in* $L^q(\mathbb{S}^1;\, L^p(\mathbb{R}^n))$, *for* $1 < q < \infty,\ 2 \leqslant p < \Lambda,\ \frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$ *and satisfies the inequality*

$$\|u\|_{L^q(\mathbb{S}^1;\, L^p(\mathbb{R}^n))} \leqslant C_n \left(\|f\|_2 + \|g\|_{L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{R}^n))} \right),$$

where $\Lambda = \infty$ *for* $n = 1$ *and* $\Lambda = \frac{2n}{n-2}$ *for* $n > 1$.

Proof. By previous theorem, we have $\|e^{itH} f(x)\|_{L^q(\mathbb{S}^1;\, L^p(\mathbb{R}^n))} \leqslant C\|f\|_2$. Therefore by (3.10), it is enough to show that

$$\left\| \int_0^t e^{i(t-s)H} g(\cdot, s)\, ds \right\|_{L^q(\mathbb{S}^1;\, L^p(\mathbb{R}^n))} \leqslant C_n \|g\|_{L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{R}^n))}.$$

We have

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)H} g(\cdot, s)\, ds \right\|_p &\leqslant \int_0^t \left\| e^{i(t-s)H} g(\cdot, s) \right\|_p ds \\ &\leqslant C_n \int_0^{2\pi} \frac{\|g(\cdot, s)\|_{p'}}{|\sin 2(t-s)|^{n(\frac{1}{p'} - \frac{1}{2})}}\, ds \end{aligned}$$

in view of Proposition 3.1. Since $g(\cdot,\cdot) \in L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{R}^n))$, we have $\|g(\cdot, s)\|_{p'} \in L^{q'}(\mathbb{S}^1)$ as a function of s. Now by an application of the Young's inequality as before, we see that the $L^{q'}(\mathbb{S}\, ds)$ norm of the above is at most a constant times

$$\left(\int_{\mathbb{S}^1} \|g(\cdot, s)\|_{p'}^{q'}\, ds \right)^{\frac{1}{q'}} = \|g\|_{L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{R}^n))}.$$

This completes the proof of the theorem. □

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