type, an application of the resultant time-optimal inputs to the system causes the magnitude of the output to be less than or equal to $|y_d|$ in the interval (t_0, t_f^*), where t_0 is the initial time.

Although many techniques can be used to solve the preceding time-optimal problem, Kranc and Sarachik's approach [1] is probably best because it does not require the solution of a two-point boundary value problem. In fact, a particularly simple and rapid determination of t_f^* and the optimal control is possible when their technique is used. To briefly summarize their results (for single output plants), consider an r-input single-output nth-order linear time-varying plant characterized by its impulse response matrix $\boldsymbol{h}(t,\tau)$ (an rth-order row vector). If the system input vector is constrained as follows:

$$\|\boldsymbol{u}\|_p \triangleq \left[\int_{t_0}^{t_f} \sum_{i=1}^{r} |u_i(t)|^p \, dt\right]^{1/p} \leq L \qquad (1)$$

then the inputs which satisfy this constraint[1] and drive the output $y(t)$ from 0 to y_d in the least time are

$$u_i(t) = L^q \left|\frac{h_i(t_f^*,t)}{y_d}\right|^{q-1} \operatorname{sgn}\left[\frac{h_i(t_f^*,t)}{y_d}\right], \qquad i = 1,2,\cdots,r \qquad (2)$$

where L is a given positive number, h_i is the ith element of $\boldsymbol{h}$,

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad (3)$$

and t_f^* is the smallest value of t_f satisfying the equation

$$\|\boldsymbol{h}\|_q \triangleq \left[\int_{t_0}^{t_f^*} \sum_{i=1}^{r} |h_i(t_f^*,\tau)|^q \, d\tau\right]^{1/q} = \frac{|y_d|}{L}. \qquad (4)$$

It is observed that the solution of (4) only requires that a real root of a transcendental equation be found, a routine calculation. We are now in a position to prove the following theorem.

Theorem

If the time-optimal inputs in (2) are applied to a linear system whose impulse response is $\boldsymbol{h}(t,\tau)$, then

$$|y(t)| \leq |y_d|, \quad t_0 \leq t \leq t_f^*.$$

Proof: If the inputs to the plant are given by (2), the output at any time can be found by using the superposition integral. Thus

$$y(t) = \int_{t_0}^{t} \sum_{i=1}^{r} h_i(t,\tau) u_i(\tau) \, d\tau$$
$$= L^q \int_{t_0}^{t} \sum_{i=1}^{r} h_i(t,\tau) \left|\frac{h_i(t_f^*,\tau)}{y_d}\right|^{q-1} \times \operatorname{sgn}\left[\frac{h_i(t_f^*,\tau)}{y_d}\right] d\tau.$$

Taking magnitudes of both sides of this equation and assuming that the elements of $h(t,\tau)$ are all nondecreasing functions of τ, we find

$$|y(t)| \leq L^q \int_{t_0}^{t} \sum_{i=1}^{r} |h_i(t,\tau)| \left|\frac{h_i(t_f^*,\tau)}{y_d}\right|^{q-1} d\tau$$
$$\leq \frac{L^q}{|y_d|^{q-1}} \int_{t_0}^{t_f^*} \sum_{i=1}^{r} |h_i(t_f^*,\tau)|^q \, d\tau$$

or

$$|y(t)| \leq \frac{L^q}{|y_d|^{q-1}} [\|\boldsymbol{h}\|_q]^q. \qquad (5)$$

Making use of (4), (5) becomes

$$|y(t)| \leq \frac{L^q}{|y_d|^{q-1}} \frac{|y_d|^q}{L^q} = |y_d|.$$

Therefore,

$$|y(t)| \leq |y_d|. \qquad (6)$$

Even if the elements of $\boldsymbol{h}(t,\tau)$ are not nondecreasing functions of τ, (6) is still valid. This is easily seen by noting that the requirement that (4) be satisfied for the smallest value of t_f is identical to stating that the process terminates the first time $y(t) = y_d$. This completes the proof.

We have shown that when the time-optimal inputs of a linear system satisfy a constraint of the L_p type, the plant output cannot overshoot the final specified value. This result has been used to obtain a lower bound on t_f^* for problems which involve inequality constraints on the output. [2]

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[1] When $p = 1$ the total input area (fuel) is constrained; if $p = 2$, the total quadratic content (i.e., energy) of u is limited; and if $p = \infty$ the magnitude of all of the elements of u is bounded.

On the Stability of a Linear Time-Varying System

***Abstract*—A single-loop linear system with a time-invariant stable block G in the direct path and a time-varying gain in the feedback path is analyzed for asymptotic stability in the Popov framework by way of admitting noncausal "multipliers" in the stability criterion. It is shown that an autocorrelation bound, analogous to O'Shea's cross-correlation bounds [1], results in a constraint on dk/dt more restrictive than that of Gruber and Willems [2].**

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Formulation and Solution of the Problem

The linear feedback system of Fig. 1 is governed by

$$\frac{dx}{dt} = Ax - bk(t)\sigma$$
$$\sigma = c'x \qquad (1)$$

where A is an $n \times n$ stable matrix; $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{x}$ (state) are $n \times 1$ vectors; and σ, the output of the system, is a scalar. $k(t) \in (0, \infty)$ is absolutely continuous so that its derivative exists almost everywhere. The transfer function of the system $G(s) = \boldsymbol{c}'(sI - A)^{-1}\boldsymbol{b}$. The system being assumed to be asymptotically stable for all constant gains $k(t) = K \in (0, \infty)$, it follows that $-\pi < \arg G(i\omega) < \pi$ for all real ω (Nyquist criterion). In other words, there exists a multiplier $M(s)$, not necessarily causal, such that, for all real ω,

$$\operatorname{Re} M(i\omega)G(i\omega) \geq 0$$
$$\operatorname{Re} M(i\omega) > 0.$$

Problem

Find sufficient conditions for asymptotic stability of the time-varying system (1) in multiplier form.

In preparation for stating a theorem which employs a noncausal function in its multiplier, define $z(t)$ to be a real function of time with a two-sided Fourier transform $Z(i\omega)$ such that

$$Z(s) = \alpha + \beta s + Y_1(s) + Y_2(s)$$

where α and β are real numbers and

$$y_1(t) = y'(t) + \sum_{i=1}^{u} \epsilon_i' \delta(t + \gamma_i')$$

$$y_2(t) = y''(t) + \sum_{j=1}^{v} \epsilon_j'' \delta(t - \gamma_j'')$$

with $y'(t)$ and $y''(t)$ piecewise continuous functions; $y'(t) = 0$ for $t > 0$ and $y''(t) = 0$, $t < 0$; $|y'(t)| \leq le^{mt}$ and $|y''(t)| \leq le^{-mt}$ where l, m are positive numbers; and γ_i' (>0), $\gamma_j''(>0)$, ϵ_i', and ϵ_j'' are real numbers for all i and j. Let $y(t) = y_1(t) + y_2(t)$.

Theorem

System (1) is asymptotically stable if, for some α, $\beta > 0$ and $Z(s)$ given above, the following conditions occur.

a) $\operatorname{Re} Z(i\omega)\, G(i\omega) \geq 0$ for all real ω

b)

$$\int_{-\infty}^{\infty} [y'(t) + y''(t)] \, dt + \sum_{i=1}^{u} \epsilon_i' + \sum_{j=1}^{v} \epsilon_j'' < -\theta, \quad \theta > 0$$

c) dk/dt is monotonically nonincreasing and/or $\leq 2(\alpha - \theta)k(t)/\beta$.

Proof is based on the following lemmas which are easy to establish.

Lemma 1

For all σ_1 and σ_2 and $k(t) \geq 0$,

$$(\sigma_1^2 \pm \sigma_2\sigma_1)k(t) \geq (\sigma_1^2 - \sigma_2^2)k(t)/2.$$

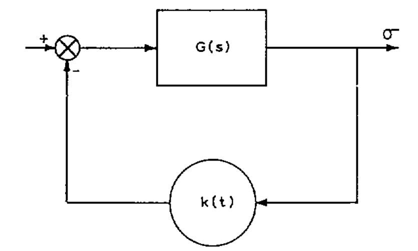

Fig. 1. Linear time-varying feedback system.

Lemma 2

If $k(t - \tau) - k(t) \geq 0$ for all τ, then for all τ and any $\sigma(t)$ in $L_2(-\infty, \infty)$,

$$\left| \int_{-\infty}^{\infty} k(t)\sigma(t)\sigma(t-\tau)\, dt \right| \leq \int_{-\infty}^{\infty} k(t)\sigma^2(t)dt.$$

Proof of Theorem: Set

$$\psi_T(\sigma, t) = \begin{cases} \psi(\sigma, t), & 0 \leq t \leq T \\ 0, & t > T \end{cases}$$

where $\psi(\sigma, t) = k(t)\,\sigma(t)$ and the subscript T denotes truncation (truncation guarantees the existence of Fourier transform). Define

$$\lambda_T(t) = \alpha\sigma_T(t) + \beta \frac{d\sigma_T}{dt} + \int_{-\infty}^{\infty} y(\tau)\sigma_T(t-\tau)\, d\tau \quad (2)$$

and consider

$$\rho(T) = \int_0^T \lambda_T(t)\psi_T(\sigma, t)\, dt. \quad (3)$$

Steps leading to the crucial inequality (4) are as follows:

1) obtain a bound on $\rho(T)$ using conditions b) and c) of the Theorem as well as Lemma 2,

2) replace $\sigma_T(t)$ by $\sigma_{Tp}(t) + \sigma_{TC}(t)$, where $\sigma_{Tp}(t)$ is the response of the system to the feedback signal $-\psi(\sigma, t)$ and $\sigma_{TC}(t)$ is that due to initial conditions,

3) invoke Parseval's theorem for the expression derived in 2) and use condition a) of the theorem,

4) combine the results of 1) and 3).

Finally, one obtains

$$\epsilon_1 \int_0^T k(t)\sigma_T^2(t)\, dt + \beta k(T)\sigma_T^2(T)/2 \leq M_0 + M_1 \sup_{0 \leq t \leq T} |\psi_T(\sigma, t)| \quad (4)$$

where ϵ_1, M_0, and M_1 are positive numbers independent of T. Boundedness of $\sigma(T)$ and its asymptotic stability follow from (4) as, of course, with minor modifications, in Popov's proof [3] for an autonomous nonlinear system.

Conclusion

The requirement that dk/dt be nonpositive for the asymptotic stability of (1) is so trivial as to make the direct adoption of O'Shea's method of correlation bounds unattractive. The essential Popov approach, however, retains its power if the noncausal part of this integral of (2) is converted to a slightly different form. By this means, the author has derived asymptotic stability conditions more general than those of Gruber and Willems [2]. These will be reported elsewhere.

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Dual Input Discrete Describing Function

Abstract—**The definition and development of a modified discrete describing function, referred to as dual input discrete describing function, is considered. High-frequency signal is injected into the nonlinearity to quench the limit cycles in closed-loop autonomous relay-type sampled-data systems. An analytic procedure is described, which aims at deriving the dual input discrete describing function for a relay in the Z domain.**

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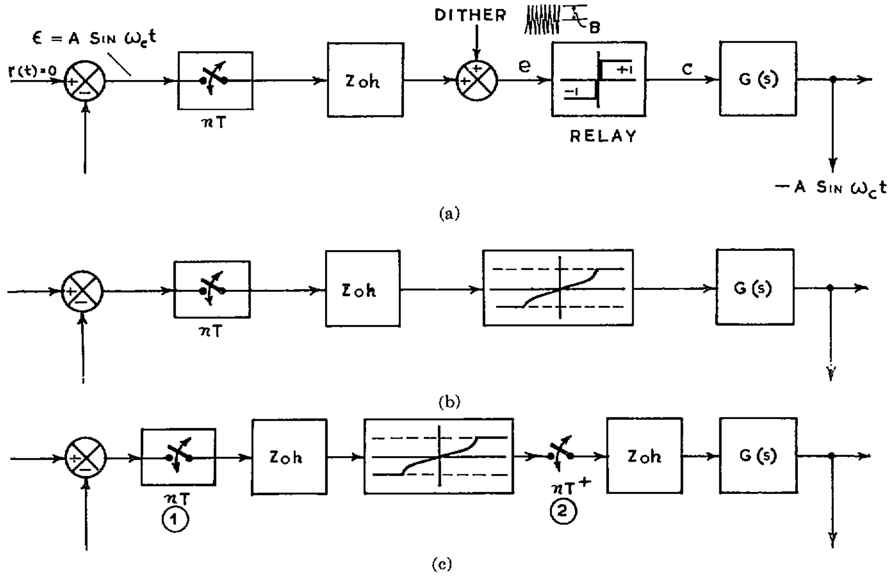

Fig. 1.

Introduction

This correspondence deals with the definition and development of the dual input discrete describing function (DIDDF). A high-frequency signal is injected into the nonlinearity to quench the limit cycles occurring in closed-loop autonomous relay-type sampled-data systems [1].

This method can be considered to be the counterpart of the conventional describing function developed and applied with success by Boyer [2]. Kuo [3] analyzes the problem of forced oscillations and the stabilizing effect of an external signal whose frequency is assumed to be an integral multiple of the sampling frequency. Furthermore, the external signal is to be applied at the input side of the control system. In this correspondence, however, such restrictions are removed to render the analysis more realistic.

Derivation of DIDDF

Let

$A \sin w_c t$	error signal
$B \sin w_d t$	dither signal
$w_d \gg w_c$	
$N(Z)$	describing function
T_c	period of oscillation of system
T	sampling period
ϕ	phase lead of error signal

and

$$Z = e^{jw_c T}.$$

Relay: $C = 1$; $e > 0$; $C = -1$; $e < 0$; $C = 0$; $e = 0$. $\overline{G(Z)} = G_{h0}\, G(Z)$, and for oscillations $-1/N(Z) = \overline{G(Z)}$.

Consider the system shown in Fig. 1(a). The nonlinearity with high-frequency dither can be replaced by the altered characteristic of the relay. Since the zero-order hold precedes the nonlinearity, the input to the nonlinearity is stepped. The output of the nonlinearity is also stepped. The introduction of a synchronized sampler and a zero-order hold on the output side, as shown, leaves the situation unaltered.