not conclude that the system is asymptotically stable if $r(t)$ is not constant. However, it is stable and should tend to operate near $e = 0$, whereas the design using the Hermite matrix may not.

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Stability of Linear Time-Invariant Systems

Introduction

It is well known that the stability of a linear time-invariant system can be determined using a Liapunov function of the quadratic form. However, for establishing the asymptotic stability of such a system for all linear negative feedback gains in a given range, a Liapunov function which yields both necessary and sufficient conditions has not been available so far. This correspondence presents such a function and affirms a recent conjecture of Narendra.\(^1\) The interest in this problem is due to its being a limiting case of the Lur’e problem.

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Adding and subtracting the following terms,

1. \( K \beta s^2 \left( 1 - \frac{K}{K} \right) \)
2. \( \gamma \alpha K \left( H^T R x \right) \)
3. \( \alpha \beta K \left( H^T R x \right) \left( H^T \beta x \right) \),

\( \dot{V} \) takes the form

\[
\dot{V} = \frac{1}{2} x^T (A^T P + PA) x - \left( \beta \alpha \beta + \delta_0 \right) (K \alpha)^3
\]

\[
- K \alpha x^T \left[ P b - \beta A^T h - \delta h - \gamma \alpha \right]
\]

\[
\left( 1 - \frac{K}{K} \right) R s^T h - \sum_k \gamma s_k K R s^T h
\]

\[
- \sum_k K (H^T R) x^T
\]

\[
\left[ \beta R A + \gamma I + \alpha \beta \right] x
\]

\[
- \sum_k K (H^T \beta x) R x - \beta \alpha R A - \alpha \beta \alpha x
\]

\[
- K \beta s^2 \left( 1 - \frac{K}{K} \right).
\]

(10)

Setting \( \alpha = \beta \mu \delta \) and using relations (6),

\[
\dot{V} = \frac{1}{2} x^T (A^T P + PA) x - \left( \beta \alpha \beta + \delta_0 \right) (K \alpha)^3
\]

\[
- K \alpha x^T \left[ P b - \beta A^T h - \delta h - \gamma \alpha \right]
\]

\[
\left( 1 - \frac{K}{K} \right) R s^T h - \sum_k \gamma s_k K R s^T h
\]

\[
- K \beta s^2 \left( 1 - \frac{K}{K} \right).
\]

(11)

THE STABILITY CRITERION

A direct application of the Meyer-Kalman-Yakubovich lemma\(^2\) gives the criterion for asymptotic stability as

\[
\beta \alpha \beta + \delta_0 + \Re \left[ \delta + \beta \alpha \beta \right] + \sum_k \gamma s_k \left( 1 - \frac{K}{K} + \delta \alpha \right) H^T R x
\]

\[
\left[ j \alpha I - A \right] s^T \geq 0
\]

(12)

for all real \( \omega \), where \( \delta > 0 \).

Use of property (7) yields the criterion in the form

\[
\Re \left[ j \alpha I - A \right] s^T \geq 0
\]

(13)

for all real \( \omega \), where

\[
Z(s) = \delta + \beta \alpha \beta + \sum_k s_k \frac{C_k}{s^2 + \mu_k^2}
\]

(14)

Since \( d_0 \geq 0 \) and \( K < K \), one can always choose \( \gamma s \) and \( \delta_0 \) to get

\[
Z(s) = \delta + \beta \alpha \beta + \sum_k \frac{C_k}{s^2 + \mu_k^2}
\]

(15)

where \( \delta > 0 \) and \( \beta \alpha \) are any non-negative numbers. If only neutral stability is desired, \( \delta > 0 \).

INTERPRETATION OF THE CRITERION

Brockett and Willems\(^4\) have shown that a necessary and sufficient condition for a transfer function \( G(s) \) to be stable for all linear negative feedback gains in \( \left( 0, \frac{K}{K} \right) \) is that there should exist a multiplier \( Z(s) \) such that

\[
Z(s) \left( G(s) + \frac{1}{K} \right)
\]

is positive real, and

\[
Z(s) = \frac{1}{\sum_k \left( s^2 + \mu_k^2 \right)}
\]

where \( \lambda_k \) and \( \mu_k \) are frequencies at which

\[
\text{Arg} \left[ G(j\omega) + \frac{1}{K} \right] = 0
\]

and

\[
\frac{d}{d\omega} \left[ \text{Arg} \left[ G(j\omega) + \frac{1}{K} \right] \right] < 0 \text{ at } \omega = \lambda_k
\]

\[
\frac{d}{d\omega} \left[ \text{Arg} \left[ G(j\omega) + \frac{1}{K} \right] \right] > 0 \text{ at } \omega = \mu_k.
\]

This assumes that the first nonzero value of \( \text{Arg} \left[ G(j\omega) \right] \) is negative.

A slight extension of the result yields the form of multiplier for asymptotic stability as

\[
Z(s) = \delta + \sum_k \frac{s_k}{s^2 + \mu_k^2}
\]

(18)

where \( \delta > 0 \) can be arbitrarily small.

By expanding (18) in partial fractions, \( Z(s) \) can be put in the same form as (15).

Thus the criterion (13) developed using the Liapunov function (8) is equivalent to the criterion of Brockett and Willems and is hence necessary and sufficient.

If the phase angle of \( G(j\omega) \) starts in the positive direction, one can consider \( 1(G(j\omega) + 1/K) \) in the place of \( G(j\omega) + 1/K \) and proceed as before.

The results can be extended to the case where \( G(s) \) is the ratio of equal-order polynomials. Using a well-known transformation, the stability of such a system is equivalent to that of a system having a transfer function \( G(s) - G(\infty) \) with negative feedback gains in \( (0, K/(1 + KG(\infty))) \), and hence the results derived earlier can be applied.

CONCLUSION

It has been shown that a necessary and sufficient condition for the asymptotic stability of a linear time-invariant system for all linear negative feedback gains \( K \) in \( (0, K) \)

is the existence of a Liapunov function of the form

\[
V = x^T P x + K x^T M x
\]

where \( P \) is positive definite, and

\[
M = \beta \alpha \beta + \sum_k \beta_R R h f^T h f^T
\]

is positive semidefinite.

Thus the Liapunov function is the sum of a positive-definite quadratic form and a positive-semidefinite quadratic form depending linearly on \( K \) as conjectured by Narendra\(^1\).

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Author’s Reply\(^6\)

Thathachar and Srinath are to be congratulated on deriving a simple Liapunov function for proving the stability of a linear time-invariant feedback system in the entire range \( 0 < K < \infty \) (or \( K \)) of the feedback gain. The proof is based on the fact that a \( Z(s) \) function always exists so that \( G(s)Z(s) \) is positive real if the feedback system is stable for all feedback gains in the range \( 0 < K < \infty \).

The authors’ contribution lies in choosing the terms of the Liapunov function, particularly the matrices \( R \) and \( F \) in (8) so that the desired multiplier \( Z(s) \) can be obtained. Since the existence of the multiplier \( Z(s) \) is both necessary and sufficient to prove the stability of the feedback system, the existence of the Liapunov function of the form derived is both necessary and sufficient for stability in the range \( 0 < K < \infty \) (or \( K \)).

On the basis of the results obtained, one can proceed to determine, for the time-invariant case, a Liapunov function having the general form

\[
V(x) = x^T P x + K x^T Q x
\]

where \( P \) is a positive-definite matrix and \( Q \) a semidefinite matrix which can be expressed as

\[
Q = \sum_i \beta R R h f^T h f^T
\]

where \( H_f \) is an \((n \times n)\) matrix. The Liapunov function (8) derived in the correspondence can be used for the most restrictive case to obtain an \( LC \) function \( Z(s) \), for which \( G(s)Z(s) \) is positive real. In most cases, however, where \( Z(s) \), \( X(s) \), or \( Z(s) \) multipliers exist, simpler Liapunov functions of the general form (19) can be determined. In such cases, the Liapunov function for the linear time-invariant case can be suitably modified for different nonlinear situations.

The form of the \( Q \) matrix is closely related to the form of the multiplier \( Z(s) \) that is required to make \( G(s)Z(s) \) positive real and consequently to the class of nonlinear

\(^1\) Manuscript received February 15, 1967.
Explicit Stability Criteria for the Damped Mathieu Equation

Introduction

The purpose of this correspondence is to establish explicit stability criteria for the Mathieu equation with damping term $\delta > 0$:

$$\ddot{x} + \delta \dot{x} + (\omega^2 + \epsilon \cos 2t)x = 0. \quad (1)$$

Since the time-invariant part of (1) is uniformly asymptotically stable, it is thus known that for $\epsilon$ sufficiently small the perturbed equation will also be stable. An application of Liapunov's direct method enables one to obtain a bound on the parameter $\epsilon$ within which bound stability is assured. In particular, it will be shown that under the reasonable assumptions of large $\omega$ and small $\epsilon$, the condition for stability of (1) becomes a very simple inequality relationship involving only $\delta$ and $\omega$, i.e., independent of the frequency $\omega$.

Application of Liapunov's Direct Method

Equation (1), the damped Mathieu equation with damping term $\delta > 0$, frequency $\omega$, and small parameter $\epsilon$, is equivalent to the system

$$\begin{align*}
\dot{s}_1 &= x_2 \\
\dot{x}_2 &= -(\omega^2 + \epsilon \cos 2t)x_1 - \frac{\delta}{\omega^2}x_2.
\end{align*}$$

The matrix $B$ determined in this manner is the matrix of a positive definite quadratic form.

With $B$ now given by (4), one has

$$P'B + BP = -\begin{bmatrix} 1 & \omega^2 + \delta \\
\omega^2 + \delta & \frac{\omega^2 + \delta}{\epsilon} \cos 2t \end{bmatrix}$$

and thus

$$\dot{V}(x, t) = -x'Dx$$

where

$$D = \begin{bmatrix}
1 + \frac{\delta}{\omega^2} & \omega^2 + 1 \\
\omega^2 + 1 & \frac{\omega^2 + \delta}{\epsilon} \cos 2t
\end{bmatrix}.$$ 

In order that $V(x)$ be a Liapunov function, those values of $\epsilon$, $\delta$, and $\omega$ are sought such that $D$ is positive semidefinite. A sufficient condition that the real, symmetric, time-varying matrix $D$ be positive semidefinite is that its principal determinants be strictly greater than zero. This condition leads to the following constraints on $\epsilon$, $\delta$, and $\omega$:

$$1 + \frac{\delta}{\omega^2} > 0$$

and

$$\epsilon < \frac{\omega^2 + 1}{\delta} - 1 < 0.$$ 

The above inequalities are satisfied for all $t$ when

$$|\epsilon| < \omega^2.$$ 

(5)

and

$$-\epsilon < \epsilon < \epsilon.$$ 

(6a)

where

$$\epsilon = \frac{1}{2} \left( \frac{\omega^2}{\omega^2 + 1} \right)^{1/2}$$

and

$$1 + \frac{\delta}{\omega^2} \cos 2t > \gamma$$

$$\epsilon^2 \left( \frac{\omega^2}{\delta} + 1 \right)^{1/2} \cos 2t - \frac{\omega^2}{\delta} \cos 2t - 1 < -\gamma$$

where $\gamma$ is a positive constant independent of $x$ and $t$, then $V(x, t)$ would be negative definite, and the system (2) would therefore be asymptotically stable.

Simplification of the Stability Criteria

Consider the case when $\omega$ is large and $\delta$ relatively small, i.e.,

$$\frac{\delta}{\omega^2 + 1} < 1$$

(7)

and

$$\frac{\omega^2}{\omega^2 + 1} = 1.$$ 

(8)