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Who is afraid of Anomalies?

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At the outset let me thank Professor V. Radhakrishnan for inviting me to deliver a lecture at this symposium commemorating C. V. Raman. I join all of you in paying my homage to the memory of Professor Raman.

It may appear from its title that my talk has little to do with the theme of this symposium. But if the twin words defining this theme – Waves and Symmetry – are interpreted in their full modern sense then the subject matter of this lecture is directly encompassed by both these words. Our topic falls entirely within quantum field theory, and quantum field theory is nothing but the quantum theory of waves. Furthermore our discussion will be intimately involved with gauge symmetry. Gauge symmetry is one of the most important symmetries in physics at the fundamental level. All the basic forces of nature – the Electromagnetic, the Weak, the Strong and the Gravitational, are today described by theories with gauge symmetries.

Unfortunately, there are situations where gauge symmetry comes into unavoidable conflict with quantum theory. Such situations are examples of what are called "Anomalies" in quantum field theory. In these cases, although some form of gauge symmetry is present at the classical level, the process of quantisation necessarily destroys that symmetry. How to consistently treat such cases and obtain their novel features is the subject matter of this talk, and has been the theme of most of my research work in the last few years.

It should be pointed out that until these developments took place, the conventional wisdom in particle physics was that anomalous gauge theories (AGT) – i.e. theories of the kind I mentioned above – are all nonsensical for a variety of reasons alleged against them. What our work, along with that of several other people now working on this topic, has shown is that AGT are not, as a class, necessarily crazy or inconsistent. Specific examples of AGT, when treated appropriately, were shown to be quite consistent, unitary and Lorentz invariant – features it was once feared, none of them would possess.

I will describe in some detail the proper treatment of one such anomalous theory. Before I do that however, it will probably be useful, keeping in mind the composition of the audience, if I gave a brief introduction to anomalies in general.

Normally, quantities which are conserved for a given system at the classical level are also conserved at the quantum level. Thus, total momentum, energy and angular momentum which are conserved in any typical classical mechanical system continue to be conserved upon quantising that system. A physical reason behind this is that the symmetries associated with these three conservation laws, namely, space-time translational and rotational symmetry, continue to be symmetries of the quantum system as well.

Going on to field theory, generically this continues to hold. For instance consider that most successful of all field theories, electrodynamics in (3+1) dimensions. This theory is described by the Lagrangian density

$$L_{\text{QED}} = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - m)\Psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\Psi}\,\gamma^{\mu}\Psi A_{\mu}. \tag{1}$$

Classically, this system conserves total energy momentum, angular momentum and electric charge. The latter conservation is usually expressed in the form of a continuity equation $\partial_\mu j^\mu \equiv (\partial/\partial t)j^0 - \vec{\nabla}\cdot\vec{j} = 0$, where j^μ is the 4-vector electric current, $j^\mu \equiv e\bar{\Psi}\gamma^\mu\Psi$. This system is also gauge invariant, under the symmetry $\Psi \rightarrow \exp[-ie\Lambda(\vec{x},t)]\Psi$; $A_\mu \rightarrow A_\mu + \partial_\mu\Lambda$. When this theory is quantised (QED), all these conservation laws can be maintained. The associated symmetries, including the gauge symmetry mentioned above, continue to hold. (The preservation of these properties in QED is not trivially achieved. The quantum theory renders many quantities divergent, and the infinities in them have to be suitably removed. This proves to be possible in QED, furthermore in such a way that all the conservation laws and symmetries are preserved).

Consider however the special case of massless electrodynamics (Put $m = 0$ in $L_{\rm QED}$ above). Then the classical Lagrangian has the additional global symmetry under $\Psi \rightarrow \exp(i\gamma_5\Lambda)\Psi$. Associated with this is the conserved axial current $j_5^\mu = \bar{\Psi}\gamma^\mu\gamma_5\Psi$. But when the theory is quantised this current is no longer conserved. Instead we get

$$\partial_\mu j_5^\mu = \frac{e^2}{16\pi^2}\,\epsilon^{\lambda\mu\nu\rho}\,F_{\lambda\mu}\,F_{\nu\rho}$$

This is an "anomaly". It was discovered independently by Adler and by Bell and Jackiw. The reason why this anomaly occurs is not easy to understand in simple physical terms. The reason lies in the fact that the current j_5^μ is formally divergent upon quantisation, and the process of removing its infinite part necessarily destroys its conservation. (Of course, the vector (electromagnetic) current j^μ is also formally divergent. But it is possible to render it finite and at the same time conserved. Associated with this, it becomes also possible to maintain the gauge invariance of QED whether or not $m = 0$).

The anomaly in the example mentioned above is relatively harmless. It occurs in the axial current and not the vector (electric) current j^μ which couples to A_μ. The gauge invariance of the theory is also not affected. Hence massless QED, although it has an anomaly in $\partial_\mu j_5^\mu$, is not what I would call an anomalous gauge theory (AGT).

Consider instead another massless theory, where A_μ couples only to the right-chiral current $j_R^\mu \equiv (e/2)\bar{\Psi}\gamma^\mu(1-\gamma_5)\Psi$. The Lagrangian is

$$L_{ch} = \bar{\Psi}i\gamma^\mu\partial_\mu\Psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{e}{2}\bar{\Psi}\,\gamma^\mu(1-\gamma_5)\,\Psi A_\mu\,. \tag{2a}$$

This is gauge-invariant under

$$\Psi_L \equiv \tfrac{1}{2}(1+\gamma_5)\Psi \rightarrow \Psi_L\,,$$

$$\Psi_R \equiv \tfrac{1}{2}(1-\gamma_5)\Psi \rightarrow \exp[-ie\Lambda(\vec{x},t)]\Psi_R\,,$$

and $A_\mu \rightarrow A_\mu + \partial_\mu\Lambda$. (2b)

Now the anomaly appears in the gauge current j_R^μ. It can be shown that

$$\partial_\mu j_R^\mu = \frac{e^3}{32\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,F_{\mu\nu}\,F_{\rho\sigma}\,. \tag{2c}$$

The gauge invariance described above is also lost upon quantisation. The two phenomena are related, but we will not go into their connection here. This is a more serious form of anomaly and such a theory *is* an AGT.

As I mentioned earlier, AGT's were believed to be sick in a variety of ways. It was thought that they were

(a) non-renormalisable

(b) non-unitary

(c) Lorentz non-invariant and

(d) afflicted with mutually inconsistent field equations.

The last of these problems, that of internal inconsistency, was believed to arise because the field equation $D_\mu F^{\mu\nu} = j^\nu$ implies $D_\nu j^\nu = 0$ which appears to conflict with the anomaly equation (2c). The origin of the other allegations requires going into more detail than is possible in this talk.

Now, each of the four problems, if truly present, is a serious blow to AGT. Perhaps, the first, namely non-renormalisability, is the least serious. It is a perturbative notion and may or may not exist if non-perturbative solutions are someday found. Also, if we use the AGT in question purely as a low-energy theory, then renormalisability is not an issue. Further, some low-dimensional theories may be super-renormalisable in which case again this problem will not arise. But the remaining three problems in that list would render any theory quite useless.

We will now describe a two dimensional anomalous gauge theory called the Chiral Schwinger Model which can be shown to be quite healthy and free of all the above problems. In fact Jackiw and I solved this theory exactly and showed that it explicitly yielded the first counter-example to the belief that all AGT's are necessarily sick. The model happens to be super-renormalisable, so the first difficulty is absent rather trivially. But the remaining three alleged problems evaporate away in a rather interesting manner.

The Chiral Schwinger Model

This model is in 2 dimensions, and its classical action is

$$S[\Psi,\bar{\Psi},A_\mu] = \int d^2x[\bar{\Psi}i\gamma^\mu\partial_\mu\Psi - \frac{e}{2}\bar{\Psi}\gamma^\mu(1-\gamma_5)\Psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}], \tag{3}$$

where Ψ is a massless Dirac field, $\gamma_5 = \gamma^0\gamma^1$ and A_μ is a $U(1)$ gauge field. The classical action is gauge invariant under $\Psi_R \equiv \frac{1}{2}(1-\gamma_5)\Psi \to e^{-ie\Lambda}\Psi$ and $A_\mu \to A_\mu + \partial_\mu\Lambda$. When this system is quantised, two things happen. Firstly, the two-point function $\langle A_\mu(x)A_\nu(y)\rangle$ develops an ultraviolet divergence due to one-loop fermionic fluctuations. This is evident from power-counting in the one-fermion loop Feynman graph. Consequently, an $A_\mu A^\mu$ counter-term has to be added to remove this divergence, which as usual, leaves behind an arbitrary finite part. In short, the quantum theory is arbitrary up to a finite local term of the form $A_\mu A^\mu$ in the effective gauge field action. This effective action can be exactly evaluated for this system, using methods developed by Schwinger. The result is

$$W[A_\mu] = \int d^2x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha e^2}{8\pi}A_\mu A^\mu - \frac{e^2}{8\pi}A_\mu(g^{\mu\alpha}+\epsilon^{\mu\alpha})\frac{\partial_\alpha\partial_\beta}{\Box}(g^{\beta\nu}-\epsilon^{\beta\nu})A_\nu\right]$$

$$\equiv \tfrac{1}{2}\int A_\mu\, M^{\mu\nu}A_\nu\, d^2x \quad . \tag{4}$$

The second term, with the arbitrary real constant α represents the polynomial ambiguity in the effective action, referred to above. The situation is in fact very similar to the role of the ϕ^4 term in the Yukawa theory, where once again it is absent in the classical action, but is induced radiatively.

The second thing that happens upon quantisation is loss of gauge invariance. This is just another way of saying that the theory is anomalous. One can see that the effective action $W[A_\mu]$ in eq.(4) is not gauge invariant, even though the classical action (3) is. This is true for *any* value of α, including $\alpha = 0$. Hence gauge invariance does not dictate a unique value of α either. Let us therefore proceed keeping α arbitrary.

The non-local effective action can be analysed as it is. But it is convenient to make it local by introducing the chiral field $\phi(x)$. It can be verified that

$$\exp(iW[A_\mu]) = \int D\phi\, \exp[iS(\phi,A_\mu)], \tag{5}$$

where

$$S[\phi,A_\mu] = \int dt\, L(\phi,A_\mu),$$

$$L[\phi,A_\mu] = \int dx \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\alpha e^2}{8\pi} A_\mu A^\mu + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{e}{\sqrt{4\pi}}(g^{\mu\nu} - \epsilon^{\mu\nu})\partial_\mu\phi A_\nu \right]. \qquad (6)$$

This is nothing but the bosonised action corresponding to the original action (3). The constant α represents the regularisation ambiguity in bosonisation.

The action (6) is quadratic. The field equations are linear and the system is exactly soluble. The spectrum is easily obtained from the poles of the propagator $G_{\mu\nu}$ (k) gotten by inverting $M^{\mu\nu}$ in the quadratic action (4). The results depend on the value of the arbitrary constant α:

(i) $\alpha>1$: The propagator has a pole at $k^2 = m^2 = e^2\alpha^2/4\pi(\alpha-1)$ and a pole at $k^2 = 0$. Both poles have positive residue. Thus the theory is unitary and has relativistic excitations corresponding to masses m and zero. Evaluation of the energy-momentum tensor also confirms this.

(ii) $\alpha<1$: The pole in k^2 is now at negative m^2, and its residue also turns out to be negative. The theory is then non-unitary, carries tachyons and is clearly unphysical.

(iii) $\alpha=1$: This is a singular point. The theory exists unitarily even at this special point, but only the massless matter-field excitations survive.

A canonical Hamiltonian analysis starting from the bosonised action (6) reveals the altered constraint structure brought about by the anomaly. Note that the bosonised action (6) which is equivalent to the gauge field action $W[A_\mu]$ obtained by integrating over the fermionic fluctuations, already contains the anomaly. Thus, a *classical* Hamiltonian analysis of the bosonised system will already incorporate anomalous effects. Once this is done, quantisation can be carried out by replacing classical (Dirac) brackets by quantum commutators.

Let us denote by π_0, E and π the canonical momenta conjugate to A_0, A_1 and ϕ respectively. Early steps in the canonical analysis proceed as in anomaly-free gauge theories.

We have

$$\pi_0(x) = \frac{\partial L}{\partial(\partial_0 A_0)} = 0 \text{ as a constraint.} \qquad (7)$$

The Hamiltonian is

$$H = \int dx\, [E\mathring{A}_1 + \pi\mathring{\phi} - \mathcal{L} + \pi_0 v_0], \qquad (8)$$

where v_0 is an as yet undetermined velocity.

Then the consistency requirement

$$\begin{aligned} 0 &= \{\pi_0(x),H\}_{\text{P.B.}} \\ &= \partial_1 E + \frac{e}{\sqrt{4\pi}}(\pi + \partial_1\phi) + \frac{e^2}{4\pi}(A_1 + (1-\alpha)A_0) \\ &\equiv G(x) \end{aligned} \qquad (9)$$

yields a second constraint which is the analogue here of Gauss Law. However, as distinct from an anomaly free gauge theory, the constraints here are of the second class for all $\alpha\neq1$. (We will return to $\alpha = 1$ later).

$$\{\pi_0(x), G(y)\} = \frac{e^2}{4\pi}(1-\alpha)\ \partial(x-y). \qquad (10)$$

There are no first class constraints and no gauge freedom. The velocity field $v_0(x)$ in the Hamiltonian (8) is uniquely fixed by the consistency of the Gauss Law constraint, i.e. by $\{G(x),H\} = 0$. To put it in other words, the "gauge" for A_0 is fixed uniquely by the Gauss Law constraint (9). Next we just follow Dirac's procedure for treating second class constraints. Dirac brackets are constructed to replace Poisson brackets. This makes both constraints (7) and (9) hold strongly, and A_0 can be eliminated using these constraints. The Hamiltonian (8) becomes

$$H = \frac{1}{2}\int \mathrm{d}x \left[\left(\pi + \frac{eA_1}{\sqrt{4\pi}}\right)^2 + \left(\partial_1\phi + \frac{eA_1}{\sqrt{4\pi}}\right)^2 + E^2 + \frac{e^2}{4\pi}(\alpha - 1)\,(A_0^2 + A_1^2)\right]. \tag{11}$$

Clearly, this is real, unique *and* positive for $\alpha > 1$. It is quadratic and easily diagonalised to yield the spectrum mentioned earlier. Upon replacing *Dirac* brackets by commutators, quantisation is effected with no further complications, yielding a Hermitian, bounded-from-below Hamiltonian. The quantum theory is unitary, self consistent and Lorentz invariant.

Analysis of the $\alpha = 1$ case involves a few more steps, since now $\{\pi_0, G\} = 0$. As a result the time-preservation of $G(x) = 0$ forces two more constraints, $E = 0$ and $A_0 - A_1 = 0$. Collectively the four constraints are all of the second class. Upon using Dirac brackets, all gauge field variables A_0, A_1, π_0 and E can be eliminated. It can be checked that the remaining variables ϕ and π (the matter field) are governed by a free massless Hamiltonian:

$$H_{\alpha=1} = \tfrac{1}{2}\int \mathrm{d}x\ (\pi^2 + (\partial_1\phi)^2) \tag{12}$$

Notice that unlike the gauge-invariant Schwinger model, where again the gauge field can be eliminated, here the matter field remains massless.

Lessons from CSM about anomalous theories in general

The Chiral Schwinger Model has offered a valuable prototype example for demonstrating the fact that anomalous gauge theories (AGT), need not be inconsistent, violate unitarity and Lorentz invariance or in any other way be nonsensical. Being exactly soluble, it allows us to explicitly see how the different alleged problems of anomalous theories get resolved, except for the question of renormalisability which is absent as a serious problem in this 2 dimensional model.

From the study of CSM, one can abstract some lessons which may be expected to hold in more complicated AGT as well, even though the latter are not exactly soluble. Let us list a few such lessons.

(i) The pair of equations $D_\mu F^{\mu\nu} = j^\nu$ and $D_\nu j^\nu$ = anomaly $= R(A_\mu)$ are not necessarily mutually inconsistent, as was once believed. Of course these equations together imply that the anomaly $R(A_\mu)$ obeys $R(A_\mu) = 0$. This only means that the anomaly must vanish dynamically by virtue of the operator field equations, and not identically for arbitrary $A_\mu(x)$. Any solution of any given AGT will, for its consistency, satisfy $R(A_\mu) = 0$ for the Heisenberg field operator $A_\mu(x,t)$.

(ii) Despite the fact that they must satisfy the vanishing of the anomaly, the space of solutions of an AGT can be non-trivial. In CSM with $\alpha > 1$, we saw that any initial data for $A_1(x)$ and $\phi(x)$ leads to a solution. The spectrum consisted of one massive and one massless particle, whereas the anomaly-free Schwinger model contained only one massive particle. In general, other AGT's, if consistent, can be expected to have as large or a larger space of solutions (degrees of freedom) than the corresponding anomaly-free gauge theory.

(iii) Lorentz invariance need not be violated in an AGT. In the CSM, we explicitly found a relativistic spectrum corresponding to one massive and one massless species of particles. The Poincaré algebra has also been established for this model. On reason why some authors may have obtained non-Lorentz invariant results could be their use of the "Weyl ($A_0 = 0$) gauge". In an AGT, gauge invariance is broken by the anomaly. Consequently one does not have the freedom to fix any gauge condition. To impose an $A_0 = 0$ condition in the face of this is to explicitly break Lorentz invariance by hand. For instance in CSM, the field A_0 is fully determined by the constraint (9). It does not vanish identically. The resulting Lorentz invariant content of the theory would be destroyed if one required $A_0 = 0$. Of course, if one starts with some alternate gauge-invariant reformulation of an AGT (with some other action) then various gauges may be fixed, without violating Lorentz invariance.

(iv) However, one aspect of more complicated AGT's about which the analysis of CSM yields no clue at all is renormalisability.

Apart from renormalisability, one expects the other alleged problems of AGT's to disappear for the same reasons as they did in the Chiral Schwinger example. We explicitly showed this to be true for chiral $(\mathrm{QCD})_2$, even though the latter is not exactly soluble. Going to 4-dimensions, the absence of exact local bosonisation makes corresponding demonstrations difficult. But a great deal of work has been done by us and others in 4-dimensions using the so called Wess-Zumino-Witten action, which is the closest analogue in 4 dimensions to bosonisation. Unfortunately time does not permit me to present all this work.

Let me conclude by summarising the situation as follows. Our work on 2-dimensional AGT's shows that at least some AGT's can be quite consistent, unitary and Lorentz invariant. In four dimensions, the problem of renormalisability continues to remain unresolved even though the other alleged problems of AGT would probably go away. But unless renormalisability is established, conventional perturbation methods cannot be used, and one may have to hold on to the anomaly cancellation principle in 4 dimensions.

References

1. For a review of anomalies, see Jackiw R. in "Relativity, Groups and Topology II", (eds. De Witt B., and Stora R.) North-Holland, Amsterdam, 1984.
2. The analysis of the Chiral Schwinger model and Chiral $(QCD)_2$ were first done by Jackiw R. and Rajaraman R., *Phys. Rev. Lett.*, 1985, **54,** 1219, and Rajaraman R., *Phys. Lett.*, 1985, **154B,** 305; ibid 1985, **162B,** 148.
3. A long list of references of further work on AGT by us and by others is given in Percacci R. and Rajaraman R. *Int. J. Mod. Phys.*, 1989, **A4,** 4177.