On a Modified Lur'e Problem

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Abstract—This paper considers the problem of asymptotic stability in the large of an autonomous system containing a single nonlinearity. The nonlinear function is assumed to belong to several subclasses of monotonically increasing functions in the sector $(0, K)$, and the stability criterion is shown to be of the form

$$\operatorname{Re} Z(j\omega)\left[G(j\omega)+\frac{1}{K}\right]-\frac{\delta'}{K}\geq 0$$

where the constant δ' is equal to $Z(\infty)-Z_p(\infty)$ and $Z_p(s)$ is a Popov multiplier. The multiplier $Z(s)$ can, in general, have complex conjugate poles and zeros and is thus more general than the type of multipliers obtained in previous results. The nonlinear functions considered are odd monotonic functions, functions with a power law restriction, and a new class of functions with restricted asymmetry having the property

$$\left|\frac{f(\theta)}{f(-\theta)}\right|\leq c$$

for all θ. Unlike in certain earlier publications, no upper bound is placed on the derivative of the nonlinearity here.

The results obtained can be used to establish stability in some cases even when the Nyquist plot of the linear part transfer function lies in all four quadrants and the nonlinearity is not necessarily odd. Furthermore, it is shown that the conditions on the multiplier and, consequently, those on the linear part can be relaxed as the feedback function approaches linearity.

Manuscript received April 1, 1966; revised July 11, 1966, January 16, 1967, May 23, 1967, and August 11, 1967.

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I. INTRODUCTION

THE PROBLEM of stability of an autonomous system containing a single instantaneous nonlinearity has attracted considerable attention. Interest in this problem was aroused by a conjecture of Aizerman which states that if the linear system obtained by global linearization is asymptotically stable, the nonlinear system is asymptotically stable in the large (ASIL). However, in this form the conjecture was found to be false, and this led to the study of the problem formulated by Lur'e as one of finding the conditions on the linear part which ensure stability with any nonlinearity confined to first and third qudrants. In a very original contribution, Popov[1] gave an elegant frequency-domain solution to the problem. With this criterion the restriction on the linear part turns out to be quite severe, and hence stability can be proved only for a small class of systems. Attempts were made by Zames,[2] Brockett and Willems,[3] and Narendra and Neuman[4] to impose more restrictions on the nonlinearity and relax conditions on the linear part transfer function.

It has been shown[3] that sufficient conditions for the stability of such a system can be stated in the form of positive realness of the product of a multiplier function $Z(s)$ and the linear part transfer function. The multiplier can be any positive real function for a linear system whereas it is restricted to the form $(\alpha+\beta s)$ in the Popov case where the nonlinearity is almost unrestricted. Thus the conditions on the linear part can be relaxed by admitting the multiplier to be a more general positive real function.

For an odd monotonic nonlinearity, the results derived earlier[3],[4] restrict the multiplier to have either real poles or real zeros. In the present paper a more general type of multiplier having complex conjugate poles and zeros is shown to be admissible utilizing the approach of Narendra and Neuman.[4] In addition, a new class of nonlinearities with a restricted odd property is introduced, and it is shown that complex conjugate poles and zeros can be permitted in the multiplier under somewhat more restrictive conditions. Criteria valid for nonlinearities with a power law restriction are also derived.

The possibility of using such a form of multiplier is mentioned in a recent paper by O'Shea[5] which appeared after the present paper had been submitted. However, no concrete examples are given and, furthermore, the nonlinearity is assumed to be odd, a property which is not necessary for some of the results presented.

II. Statement of the Problem

The dynamic system considered in this paper (Fig. 1) is described by the vector matrix equations

$$\dot{x} = Ax - bf(\sigma)$$
$$\sigma = h^T x \qquad (1)$$

where x, b, and h are n vectors, A is an $n\times n$ stable matrix, and σ is a real scalar time function.

It is assumed that $f(\sigma)$ belongs to the class $M_\infty \cap A_K$,[3] i.e., $f(\theta)$ is continuous, $f(0)=0$,

$$0 < \theta f(\theta) < K\theta^2, \quad \theta \neq 0 \qquad (2)$$
$$0 \leq (\theta_1 - \theta_2)[f(\theta_1) - f(\theta_2)] \qquad (3)$$

for all θ, θ_1, and θ_2.

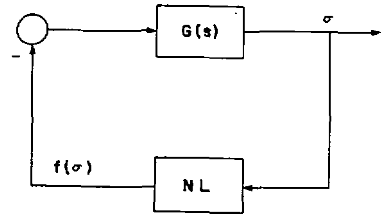

Fig. 1. The nonlinear system.

The problem is to find sufficient conditions for ASIL of (1) for all $f(\sigma)$ belonging to certain subclasses (defined in the sequel) of $M_\infty \cap A_K$.

III. Properties of the Linear Part

It can easily be derived from (1) that the transfer function $G(s)$ relating the output σ to the input $-f(\sigma)$ is given by

$$G(s) = h^T(sI - A)^{-1}b. \qquad (4)$$

The main properties of $G(s)$ are stated through the following lemma. The proof follows that in Narendra and Neuman.[4]

Lemma I

Let $s=-\lambda \pm j\mu$ be a pair of zeros of $G(s)+1/K$. Let B be a complex $n\times n$ matrix defined by

$$B = \frac{q}{\beta}[(\lambda - j\mu)I + A]^{-1}$$

where q is a complex scalar and β is a real positive number. Then,

$$h^T Bb = \frac{q}{\beta K} \qquad (5)$$

$$h^T B(sI - A)^{-1}b = \frac{q}{\beta}\frac{G(s) + 1/K}{s + (\lambda - j\mu)}. \qquad (6)$$

Also if B is defined as above and if $h^T Bb = q/\beta K$, then $-\lambda + j\mu$ is a zero of $G(s)+1/K$.

Corollary: Let $R = B + B^*$ and $\mathcal{J} = j(B - B^*)$. Then,

$$h^T Rb = \frac{2 \operatorname{Re} q}{\beta K}$$
$$h^T \mathcal{J} b = \frac{2 \operatorname{Im} q}{\beta K} \qquad (7)$$

$$\operatorname{Re} h^T R(sI - A)^{-1}b = \operatorname{Re}\left[\frac{q/\beta}{s + \lambda - j\mu} + \frac{q^*/\beta}{s + \lambda + j\mu}\right][G(s) + 1/K]$$

$$\operatorname{Re} h^T \mathcal{J}(sI - A)^{-1}b = \operatorname{Re}\left[\frac{jq/\beta}{s + \lambda - j\mu} - \frac{jq^*/\beta}{s + \lambda + j\mu}\right][G(s) + 1/K] \qquad (8)$$

where * denotes the complex conjugate and $2 \operatorname{Re} q = q + q^*$ and $2 \operatorname{Im} q = j(q - q^*)$.

IV. Odd Monotonic Nonlinearities

A. The Nonlinear Function

If the feedback function is odd in addition to being monotonically increasing, i.e., $f \in 0_\infty \cap A_K$, it can easily be shown that the following inequalities are satisfied in addition to (2) and (3)

$$(\theta_1 - \theta_2)[f(\theta_1) + f(\theta_2)] - \frac{1}{K} f(\theta_1) f(\theta_2) \geq 0$$

$$(\theta_1 - \theta_2)[f(\theta_1) + f(\theta_2)] - \frac{1}{K} f(\theta_1) f(\theta_2) \leq 2\theta_1 f(\theta_1)$$

$$k_1\theta_1 f(\theta_2) - k_2\theta_2 f(\theta_1) \leq \max(k_1, k_2)[\theta_1 f(\theta_1) + \theta_2 f(\theta_2)] \quad (9)$$

where $k_1, k_2 \geq 0$.

B. The Liapunov Function

The Lur'e type Liapunov function has been modified by Narendra and Neuman[4] by introducing additional integrals of the nonlinear function. The same type of Liapunov function has been used by Brockett and Willems[3] in a somewhat different form. The Liapunov function in the former[4] yields a multiplier which has, as its poles, the real zeros of $G(s)+1/K$. Thus, even with odd monotonic feedback the result offers no improvement over the Popov criterion for cases in which $G(s)+1/K$ has only complex zeros. In order to deal with such situations, a new Liapunov function is introduced in this section, and a stability criterion which uses a multiplier with complex poles and zeros is derived.

In their Liapunov function, Narendra and Neuman have made use of integrals with upper limits of the form h^TBx where B is a real matrix dependent on the real zeros of $G(s)+1/K$. For complex zeros, the matrix B becomes complex, and hence it seems logical to introduce limits of the form h^TRx and $h^T\mathcal{J}x$ which depend on the real and imaginary parts of the B matrix.

Thus consider as a candidate for a Liapunov function

$$V(x) = \tfrac{1}{2}x^TPx + \beta_0 \int_0^{h^Tx} f(\theta)d\theta + \sum_i \left[\beta_i \int_0^{\theta_i} f(\theta)d\theta + \beta_i' \int_0^{\theta_i'} f(\theta)d\theta\right] + \sum_k \left[\beta_k \int_0^{\theta_k} f(\theta)d\theta + \beta_k' \int_0^{\theta_k'} f(\theta)d\theta\right] \quad (10)$$

where

$$P = P^T > 0.$$

β_0, β_i, pi', $\beta_k, \beta_k' \geq 0$ for all i, k.

$\theta_i = h^TR_ix$ and $\theta_i' = h^T\mathcal{J}_ix$.

R_i and $\mathcal{J}_i$ are matrices associated with a pair $\lambda_i \pm j\mu_i$ (which may be assumed to be zeros of $G(s)+1/K$ in the first instance) as defined in Section III.

Similarly for θ_k and θ_k'.

i takes on values from 1 to v_1 and k takes on values from (v_1+1) to v, where v_1 and a are arbitrary nonnegative integers.

The time derivative of V along the trajectories of (1) can be written as

$$\begin{aligned} \dot V = {} & \tfrac{1}{2}x^T(A^TP + PA)x \\ & - f(h^Tx)x^T(Pb - \beta_0A^Th) - \beta_0h^Tbf^2(h^Tx) \\ & + \sum_i [\beta_i f(\theta_i)h^TR_iAx + \beta_i' f(\theta_i')h^T\mathcal{J}_iAx] \\ & + \sum_k [\beta_k f(\theta_k)h^TR_kAx + \beta_k' f(\theta_k')h^T\mathcal{J}_kAx] \\ & - \sum_i [\beta_i f(\theta_i)f(h^Tx)h^TR_ib + \beta_i' f(\theta_i')f(h^Tx)h^T\mathcal{J}_ib] \\ & - \sum_k [\beta_k f(\theta_k)f(h^Tx)h^TR_kb + \beta_k' f(\theta_k')f(h^Tx)h^T\mathcal{J}_kb]. \end{aligned} \quad (11)$$

C. The Stability Criterion

For the system to be ASIL, $\dot V$ should be a negative semidefinite function not identically zero along the trajectories of the system. In order to ensure this, the following lemma due to Yakubovich,[5] Kalman,[6] and Meyer[7] is used.

Lemma 2

Let A be a real $n \times n$ matrix all the characteristic roots of which have negative real parts. Let γ be a real non-negative number, and let b and $\hat k$ be two real n vectors. If

$$\gamma + \operatorname{Re} \hat k^T(j\omega I - A)^{-1}b \geq 0$$

for all real w, then there exist two real $n \times n$ symmetric matrices P and D and a real n vector g such that

$PA + A^TP = -2gg^T - 2D$

$Pb - \hat k = 2\sqrt{\gamma}g$

D is positive semidefinite and P is positive definite

$\{x \in R^n, x^TDx = 0\} \cap \{x \in R^n, g^Te^{At}x \equiv 0\} = \{0\}$.

Sufficient conditions for the ASIL of the system (1) can be stated in the form of the following theorem.

Theorem 1

Consider the dynamic system (1) with $f \in 0_\infty \cap A_K$. The system is ASIL if there exist non-negative constants $\delta, \beta_0, \beta_i, \beta_i', \beta_k, \beta_k', \alpha_i, \alpha_i', \alpha_k, \alpha_k'$ and $\gamma_i, \gamma_i', \gamma_k, \gamma_k'$ such that

1) $$\operatorname{Re} Z(j\omega)[G(j\omega) + 1/K] - \delta'/K \geq 0 \quad (12)$$

for all real w, where

$$Z(s) = \delta + \beta_0 s - \sum_i \left[\frac{(\gamma_i - j\gamma_i')q_i/\beta_i}{s + \lambda_i - j\mu_i} + \frac{(\gamma_i + j\gamma_i')q_i^*/\beta_i}{s + \lambda_i + j\mu_i}\right] + \sum_k \left[\frac{(\gamma_k - j\gamma_k')q_k/\beta_k}{s + \lambda_k - j\mu_k} + \frac{(\gamma_k + j\gamma_k')q_k^*/\beta_k}{s + \lambda_k + j\mu_k}\right] \quad (13)$$

$$\frac{q_i}{\beta_i} = \frac{\gamma_i}{2\beta_i} - j\frac{\gamma_i'}{2\beta_i'}, \quad \frac{q_k}{\beta_k} = \frac{\gamma_k}{2\beta_k} - j\frac{\gamma_k'}{2\beta_k'}. \tag{14}$$

$$2)\quad \lambda_i = \frac{\gamma_i + \epsilon_i}{\beta_i} = \frac{\gamma_i' + \epsilon_i'}{\beta_i'}, \quad \mu_i = \frac{\alpha_i}{\beta_i} = \frac{\alpha_i'}{\beta_i'}$$

$$\lambda_k = \frac{\gamma_k + \epsilon_k}{\beta_k} = \frac{\gamma_k' + \epsilon_k'}{\beta_k'}, \quad \mu_k = \frac{\alpha_k}{\beta_k} = \frac{\alpha_k'}{\beta_k'}. \tag{15}$$

$$3)\quad \delta > \delta' = \sum_i (\gamma_i + \gamma_i') + \sum_k (\gamma_k + \gamma_k')$$

$$\epsilon_i, \epsilon_i' \geq \max(\alpha_i, \alpha_i')$$

$$\epsilon_k, \epsilon_k' \geq \max(\alpha_k, \alpha_k'). \tag{16}$$

D. Interpretation of the Criterion

The main condition (12) of Theorem 1 implies that the real part of the product of the multiplier and $G(s)+1/K$ must not be less than a certain positive constant for all real frequencies. This constant can easily be seen to be

$$\frac{\delta'}{K} = \frac{[Z(s) - Z_p(s)]}{K}\bigg|_{s=\infty}$$

where $Z_p(s) = (6-\delta') + \beta_0 s$ is the Popov multiplier component of $Z(s)$.

An observation of (15) and (16) indicates that $\lambda > \mu$. Hence the poles of the multiplier should lie within a $\pm 45°$ wedge around the negative real axis. The special case of multiplier with real poles[3],[4] can be realized by setting $\mu = 0$. Furthermore, the results for the case when the nonlinear function is in the infinite sector, i.e., $f \in 0_\infty$, can be obtained by letting K tend to ∞. It may be noted that in this case the constant δ'/K becomes zero.

By writing the multiplier as a constant plus a term in s plus a sum of biquadratic functions and by using (14) through (16), it can be shown that the multiplier is positive real and thus can be considered to be a more general RLC driving point impedance than the multipliers in earlier results which are constrained to have either real poles or real zeros. Hon-ever, a simple classification of biquadratic driving point impedances in terms of networks is not available, and hence, in general, it appears difficult to identify the present multiplier with any class of networks.

The conditions (14) through (16) on the parameters of the multiplier appear complicated. However, by setting $\beta = \beta'$, a choice which imposes minimum restrictions on the location of the poles of the multiplier, a simpler criterion can be stated as follows.

Theorem 1A (Alternative to Theorem 1)

The dynamic system (1) with $f \in 0_\infty \cap A_K$ is ASIL under the following conditions.

$$1)\quad \mathrm{Re}\, Z(j\omega)[G(j\omega) + 1/K] - \frac{\sum_p \delta_p}{K} \geq 0 \tag{17}$$

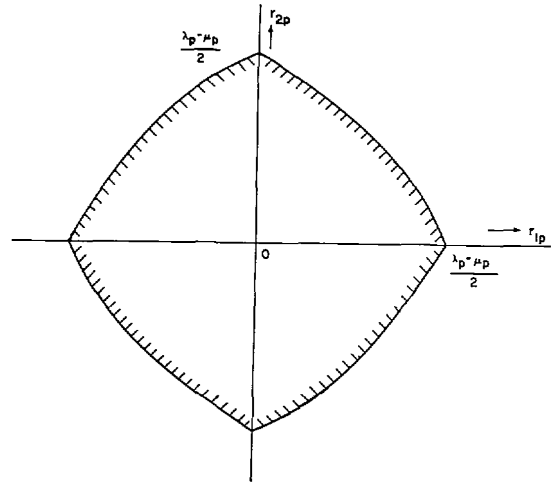

Fig. 2. Boundary for admissible values of r_{1p} and r_{2p} in the odd monotonic case.

for all real ω, where

$$Z(j\omega) = \alpha_0 + \beta_0 s + \sum \delta_p\left[1 + \frac{r_{1p} - jr_{2p}}{s + \lambda_p - j\mu_p} + \frac{r_{1p} + jr_{2p}}{s + \lambda_p + j\mu_p}\right]$$

$$\alpha_0 > 0; \quad \beta_0, \delta_p \geq 0 \tag{18}$$

and r_{1p}, r_{2p} are real

$$2)\quad (\lambda_p - \mu_p - |r_{1p}|)(\lambda_p - \mu_p - |r_{2p}|) \geq \frac{(\lambda_p - \mu_p)^2}{2}$$

$$\frac{\lambda_p - \mu_p}{2} \geq \max(|r_{1p}|, |r_{2p}|) \tag{19}$$

for all $p = 1, 2, \ldots$.

The proof is given in the Appendix.

The merit of Theorem 1A is that it readily enables one to determine whether a given multiplier is admissible or not. Furthermore, a simple geometric interpretation can be given for the inequality (19). This implies that in the r_{1p}, r_{2p} plane the allowable residues should be confined to a region around the origin bounded by arcs of rectangular hyperbolas (Fig. 2). This region is symmetrical about both the r_{1p} and r_{2p} axes.

V. Other Types of Nonlinearities

A. *Nonlinearities with Restricted Asymmetry*

The transition from the class M_∞ to the class 0_∞ is rather abrupt, and it seems desirable to introduce a new class of nonlinearities which has properties in between those of M_∞ and 0_∞. This is defined by N_∞^c, where all $f \in N_\infty^c$ satisfy the condition

$$\left|\frac{f(\theta)}{f(-\theta)}\right| \leq c \tag{20}$$

for all 8. M_∞ is a limiting case of N_∞^c with $c \to \infty$, and 0_∞ is another limiting case with $c = 1$.

The following inequalities can be easily derived for $f \in N_\infty^c \cap A_K$.

$$(\theta_1-\theta_2)[f(\theta_1)-f(\theta_2)]+\frac{1}{K}f(\theta_1)f(\theta_2)\geq 0$$

$$(\theta_1-\theta_2)[f(\theta_1)+f(\theta_2)]-\frac{1}{K}f(\theta_1)f(\theta_2)$$
$$\leq 2\theta_1 f(\theta_1)+(c-1)\theta_2 f(\theta_2)$$

$$k_1\theta_1 f(\theta_2)-k_2\theta_2 f(\theta_1)$$
$$\leq \max(k_1,k_2)\theta_1 f(\theta_1)+\max(k_1,ck_2)\theta_2 f(\theta_2) \quad (21)$$

where $k_1, k_2 \geq 0$.

Theorem 2

The dynamic system (1) with $f\in N_\infty{}^c\cap A_K$ is ASIL under the conditions stated in Theorem 1, with **(16)** replaced by

$$\delta>\delta' = \sum_i(\gamma_i+\gamma_i')+\sum_L(\gamma_k+\gamma_k')$$
$$\epsilon_i\geq \max(\alpha_i, c\alpha_i')$$
$$\epsilon_i'\geq \max(\alpha_i,\alpha_i')+(c-1)\gamma_i'$$
$$\epsilon_k\geq \max(\alpha_k, c\alpha_k')+(c-1)\gamma_k$$
$$\epsilon_k'\geq \max(\alpha_k,\alpha_k'). \quad (22)$$

The proof of Theorem 2 is similar to that of Theorem 1, except that inequalities (21) are used instead of (9).

From (15) and (22) it can be shown that the multiplier is limited to have poles which lie within a wedge defined by $\lambda>\sqrt{c}\mu$. This wedge degenerates into the negative real axis when $c\rightarrow\infty$, i.e., $f\in M_\infty$, and expands into the $\pm 45°$ wedge when $c\rightarrow 1$, i.e., $f\in 0_\infty$.

B. Nonlinearities with a Power Law Restriction

By imposing more restrictions on the nonlinearity, it is possible to relax conditions on the linear part transfer function. One way of doing this is to require that the nonlinear functions, in addition to being odd and monotonic, lie between tn-o power laws satisfying

$$\left|\frac{\theta_1}{\theta_2}\right|^{1/m}\leq\left|\frac{f(\theta_1)}{f(\theta_2)}\right|\leq\left|\frac{\theta_1}{\theta_2}\right|^m \quad (23)$$

for $|\theta_1|\geq|\theta_2|$, $m\geq 1$. Then f will be said to be in $P_\infty{}^m$.[3] The case $m=1$ corresponds to linear feedback, and as m approaches ∞, the feedback function approaches the class 0_∞.

If $f\in P_\infty{}^m\cap A_K$, the nonlinear function satisfies the following inequalities in addition to (2) and (3),

$$(\theta_1-\theta_2)[f(\theta_1)-f(\theta_2)]+\frac{1}{K}f(\theta_1)f(\theta_2)\geq 0$$

$$(\theta_1-\theta_2)[f(\theta_1)+f(\theta_2)]+\frac{1}{K}f(\theta_1)f(\theta_2)$$
$$\leq (1+c_1)\theta_1 f(\theta_1)+(c_1-1)\theta_2 f(\theta_2)$$
$$k_1\theta_1 f(\theta_2)-k_2\theta_2 f(\theta_1)\leq\{(c_2-c_1)\min(k_1,k_2)$$
$$+c_1\max(k_1,k_2)\}[\theta_1 f(\theta_1)+\theta_2 f(\theta_2)] \quad (24)$$

where $k_1, k_2\geq 0$ and

$$c_1=\max_{0<y<\infty}\frac{y^m}{1+y^{m+1}}=\frac{m}{m+1}\,\frac{1}{m^{1/(m+1)}}$$

$$c_2=\max_{0<y<\infty}\left|\frac{y^m-y}{1+y^{m+1}}\right|. \quad (25)$$

For linear feedback, i.e., $m=1$, the constant c_1 is 0.5 and $c_2=0$. As $m\rightarrow\infty$ the constants approach unity. For any m, $c_2\leq c_1$.

Theorem 3

The system **(1)** is ASIL for $f\in P_\infty{}^m\cap A_K$ under the conditions stated in Theorem 1 with (16), replaced by

$$\delta>\delta'=\sum_i(\gamma_i+\gamma_i'c_i)+\sum_k(c_1\gamma_k+\gamma_k')$$
$$\epsilon_i\geq(c_2-c_1)\min(\alpha_i,\alpha_i')+c_1\max(\alpha_i,\alpha_i')$$
$$\epsilon_i'\geq(c_2-c_1)\min(\alpha_i,\alpha_i')+c_1\max(\alpha_i,\alpha_i')-(1-c_1)\gamma_1'$$
$$\epsilon_k\geq(c_2-c_1)\min(\alpha_k,\alpha_k')+c_1\max(\alpha_k,\alpha_k')-(1-c_1)\gamma_k$$
$$\epsilon_k'\geq(c_2-c_1)\min(\alpha_k,\alpha_k')+c_1\max(\alpha_k,\alpha_k'). \quad (26)$$

The proof follows that of Theorem 1 except that inequalities (25) are used instead of **(9).**

Again, the results for the infinite sector case $f\in P_\infty{}^m$ can be obtained by letting $K\rightarrow\infty$. However, it can be shown that somewhat better results can be obtained here in that the constant in the multiplier is reduced and the inequalities (26) are relaxed. Thus (26) can be replaced by

$$\delta>\delta'=\sum_i(\gamma_i+c_2\gamma_i')+\sum_k(c_2\gamma_k+\gamma_k')$$
$$\epsilon_i\geq(c_2-c_1)\min(\alpha_i, \mathit{ail)})+c_1\max(\alpha_i,\alpha_i')$$
$$\epsilon_i'\geq(c_2-c_1)\min(\alpha_i, \mathit{ail)})+c_1\max(\alpha_i, \mathit{ail)})-(1-c_2)\gamma_i'$$
$$EL\geq(c_2-c_1)\min(\alpha_k,\alpha_k')+c_1\max(\alpha_k,\alpha_k')-(1-c_2)\gamma_k$$
$$\epsilon_k'\geq(c_2-c_1)\min(\alpha_k,\alpha_k')+c_1\max(\alpha_k,\alpha_k'). \quad (27)$$

It can be shown from **(15)** and (27) that a necessary condition to be satisfied is that

$$\lambda>c_2\mu$$

and hence the poles of the multiplier are constrained to lie in the wedge defined above. Thus for $m=\infty$ (odd monotonic feedback) the wedge forms an angle of $\pm$**45"** around the negative real axis. As $m\rightarrow 1$, the feedback function approaches linearity and the wedge can be widened to include the entire left half-plane. The poles of the multiplier can lie on the imaginary axis only in the case of linear feedback.

It is possible to restate Theorems 2 and **3** in a form similar to Theorem 1A. The equations for the boundaries of the admissible residues in the $r_{1p}r_{2p}$ plane can be readily calculated, though they turn out to be more

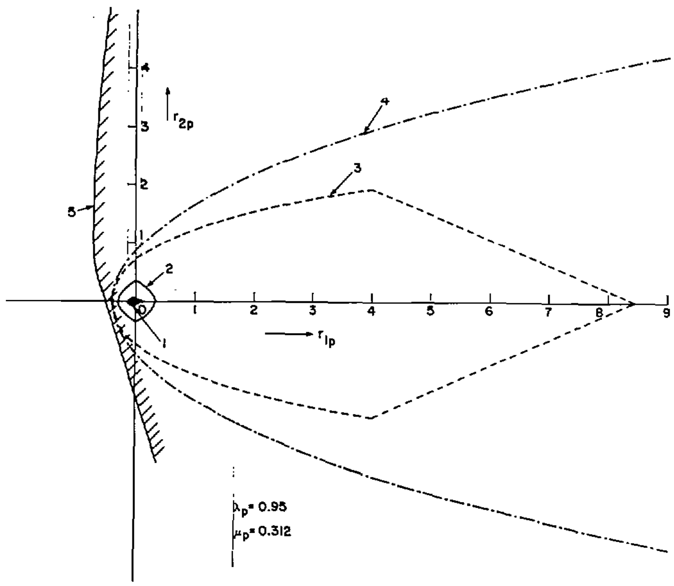

Fig. 3. Boundaries for admissible values of r_{1p} and r_{2p} in various cases: 1) N_∞^4; 2) 0_∞; 3) P_∞^2; 4) P_∞^1; 5) general positive real function.

complicated than (19) and are equations of more general hyperbolas. For the sake of comparison a sketch of these boundaries is shown in Fig. 3 for various types of nonlinearities for a particular choice of $\lambda_p \pm j\mu_p$. The limiting case of P_∞^1 or linear gain has a parabola as its boundary. It can readily be seen that the allowable zone for residues increases as more restrictions are imposed on the nonlinearity, and hence stability can be proved for a larger class of systems.

On the same figure is shown the boundary for admissible values of residues for a general biquadratic positive real function with poles at $-\lambda_p \pm j\mu_p$. It can be seen that this involves much less restrictions than even those for P_∞^1.

VI. Examples

Example 1: $f \in N_\infty^c$

Consider a system with forward transfer function

$$G(s) = \frac{s(s^2 + 1.9s + 1)}{(s+20)(s^2+2s+1.15)(s^2+s+100)}. \tag{28}$$

The system is stable for all negative linear feedback. The phase plot of $G(s)$ is shown in Fig. 4. The maximum positive phase is found to be 92.6°. It can readily be seen that the Nyquist plot lies in all the four quadrants and f is not odd, and so neither the Popov theorem nor the results of Brockett and Willems[3] and Narendra and Neuman[4] can prove stability for the entire linear stability range.

With some numerical work, it can be shown that under the conditions of Theorem 2, with $c = 2.885$, one can choose

$$Z(s) = \frac{(s+20)(s^2+2s+1.15)}{s^2+1.9s+1}$$

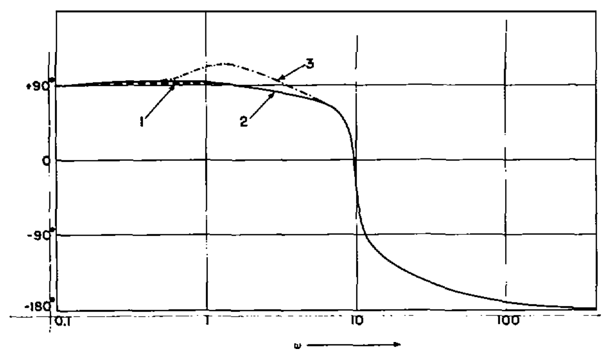

Fig. 4. Phase plots of $C(j\omega)$ in Examples 1–3.

so that

$$Z(s)G(s) = \frac{s}{s^2+s+100}$$

is positive real, and hence the system is ASIL for all $f \in N_\infty^{2.885}$.

Example 2: $f \in 0_\infty$

Consider a system with transfer function

$$G(s) = \frac{s(s^2+1.8s+1)}{(s+20)(s^2+2s+1.35)(s^2+s+100)}$$

which is stable for all linear negative feedback gains in $(0, \infty)$. The maximum positive phase is about 97°. Results of Popov[1] and Narendra and Neuman[4] are again not applicable. It is not readily apparent whether the results of Brockett and Willems[3] can give a suitable multiplier. With the results in the present paper, one can choose

$$Z(s) = \frac{(s^2+2s+1.35)(s+20)}{s^2+1.8s+1}$$

under the conditions of Theorem 1, so that

$$Z(s)G(s) = \frac{s}{s^2+s+100}$$

is positive real. Thus, the system is ASIL for all $f \in 0_\infty$.

Example 3: $f \in P_\infty^2$

The system with transfer function

$$G(s) = \frac{s(s^2+1.2s+1)}{(s+20)(s^2+2s+2)(s^2+s+100)}$$

is stable for all linear negative feedback gains in $(0, \infty)$ and has a maximum positive phase of 116.25°. As before, results of Popov[1] and Narendra and Neuman[4] are not applicable here, and it is difficult to check whether Brockett and Willems[3] can be useful. Nevertheless, Theorem 3 of the present paper enables one to choose

$$G(s) = \frac{(s + 20)(s^2 + 2s + 2)}{s^2 + 1.2s + 1}$$

so that $Z(s)G(s)$ is positive real and hence the system is ASIL for $f \in P_\infty^2$.

VII. Comparison with Earlier Results

It appears desirable to indicate the relationship of the results of this paper to those obtained in earlier publications. At the outset it must be noted that no upper bound is placed on the maximum slope of the nonlinearity in the present paper unlike in Brockett and Willems.[3] Hence a meaningful comparison with these results can only be made for the infinite sector case.

As is to be expected, the multiplier for nonlinearities with restricted asymmetry has properties intermediate to those for monotonic and odd monotonic nonlinearities. In this case the multiplier with real poles can be written in the form

$$Z(s) = 1 + \beta_0 s + \Sigma\gamma_i \frac{s + c_i\lambda_i}{s + \lambda_i} + \Sigma\gamma_k \frac{s + c_k\lambda_k}{s + \lambda_k}$$

where $0 \leq c_i \leq 1$ and $1 \leq c_k \leq 1 + 1/c$ as compared with $1 \leq c_k \leq 2$ for odd monotonic functions obtained in Brockett and Willems[3] and in Narendra and Neuman.[4] Thus many systems which were proved to be ASIL with odd monotonic nonlinearities, using earlier results, can now be proved to be ASIL with a more general class of nonlinearities, provided $c_k \leq 2$.

Results for $f \in P_\infty$ are available only in Brockett and Willems.[3] In this case one can show, with a little algebraic manipulation, that the form of multiplier with real poles obtained in the present paper is the same as in that paper. The relation[3] $\phi(u) = 1 + 1/c_2^2$ establishes the equivalence of the two results.

Example 1 clearly shows that the present method can take care of situations where the earlier methods fail. Examples 2 and 3 cannot be worked by Narendra and Neuman.[4] Though one cannot make a categorical statement about the success of criteria in Brockett and Willems[3] in the case of Examples 2 and 3, it is to be observed that even if a suitable multiplier with real poles or zeros exists, the product $Z(s)G(s)$ will be a high-order rational function whose positive realness is difficult to establish. The present results are easier to apply since by using a multiplier with complex poles and zeros, checking for positive realness can in many cases be made simple. Furthermore, the results obtained here are more general than earlier ones, as all the previous multipliers with real poles[1]–[4] can be obtained as special cases of the multiplier derived here.

The results obtained here also lend support to some of the qualitative statements made in Brockett and Willems.[3] From the examples it is seen that as the phase angle of the transfer function undergoes larger changes, stability can be proved only for a more restrictive class of nonlinearities.

It is significant that the multiplier cannot have poles on the imaginary axis as well as a pole at infinity unless feedback is linear. It appears that in cases where $Z(s)G(s)$ is positive real only when $Z(s)$ has the above property, the system is not ASIL with any nonlinearity spanning the entire linear stability range. The last example in Brockett and Willems, pt. I,[3] strengthens this conjecture.

VIII. Conclusions

It has been shown that, subject to certain conditions, the multiplier used to prove ASIL can have complex conjugate poles and zeros. Three subclasses of monotonically increasing nonlinearities are considered, and it has been shown that as the feedback function approaches linearity the conditions on the multiplier can be relaxed, and hence a larger class of linear transfer functions can be covered by the present method. The multiplier can be an RLC driving point impedance and is considerably more general than the ones obtained in previous results. Examples have been included to show that the present method can succeed where previous methods using the same approach fail.

Appendix

Proof of Theorem 1

Let $-\lambda_i \pm j\mu_i$ and $-\lambda_k \pm j\mu_k$ be assumed to be zeros of $G(s) + 1/K$ in the first instance. Then (8) and the result

$$h^T b + h^T A(sI - A)^{-1} b = s h^T (sI - A)^{-1} b$$

show that (12) and (13) are together equivalent to

$$\begin{aligned}
&\beta_0 h^T b + \frac{\delta - \delta'}{K} \\
&\quad + \mathrm{Re}\Big[\delta h^T + \beta_0 h^T A - \sum_i \gamma_i h^T R_i + \sum_i \gamma_i' h^T g_i \\
&\quad + \sum_k \gamma_k h^T R_K - \sum_k \gamma_k' h^T g_k\Big](j\omega I - A)^{-1} b \geq 0. \qquad (29)
\end{aligned}$$

The time derivative of the Liapunov function (10) along the trajectories of (1) is given by (11). Add and subtract the following terms from the right-hand side of (11)

$$\begin{aligned}
&+ \gamma_i[f(\theta_i) - f(h^T x)][\theta_i - h^T x] \\
&+ \gamma_i'[f(\theta_i') - f(h^T x)][\theta_i' - h^T x] \\
&+ \gamma_k[f(\theta_k) - f(h^T x)][\theta_k - h^T x] \\
&+ \gamma_k'[f(\theta_k') - f(h^T x)][\theta_k' - h^T x] \\
&- \alpha_i f(\theta_i)\theta_i' + \alpha_i' f(\theta_i')\theta_i \\
&- \alpha_k f(\theta_k)\theta_k' + \alpha_k' f(\theta_k')\theta_k \\
&+ \delta f(h^T x) h^T x - \frac{(\delta - \delta')}{K} f^2(h^T x).
\end{aligned}$$

Grouping the terms suitably and using the inequalities (9), set the matrices B_i and B_k as

$$B_i = \frac{q_i}{\beta_i}\left[\frac{\gamma_i + \epsilon_i - j\alpha_i}{\beta_i} I + A\right]^{-1} = \frac{q_i}{\beta_i}\left[\frac{\gamma_i' + \epsilon_i' - j\alpha_i'}{\beta_i'} I + A\right]^{-1}$$

$$B_k = \frac{q_k}{\beta_k}\left[\frac{\gamma_k + \epsilon_k - j\alpha_k}{\beta_k} I + A\right]^{-1} = \frac{q_k}{\beta_k}\left[\frac{\gamma_k' + \epsilon_k' - j\alpha_k'}{\beta_k'} I + A\right]^{-1}$$

where q_i and q_k are complex numbers defined in (14) and all Greek letters denote scalars. It can than be shown that

$$\begin{aligned} V \leq{} & \frac{1}{2} x^T(A^TP + PA)x + f^2(h^Tx)\left(\beta_0 h^Tb + \frac{\delta'}{K}\right) \\ & - f(h^Tx)x^T\Big[Pb - \beta_0 A^Th - \delta h + \sum_i \gamma_i R_i^Th \\ & - \sum_i \gamma_i' g_i^T h - \sum_k \gamma_k R_k^T h + \sum_k \gamma_k' g_k^T h\Big] \\ & - \Big[\delta - \sum_i (\gamma_i + \gamma_i') - \sum_k (\gamma_k + \gamma_k')\Big] f(h^Tx)h^Tx \\ & - \frac{\delta - \delta'}{K} f^2(h^Tx) - \sum_i [\epsilon_i - \max(\alpha_i, \alpha_i')] f(\theta_i)\theta_i \\ & - \sum_i [\epsilon_i' - \max(\alpha_i, \alpha_i')] f(\theta_i')\theta_i' \\ & - \sum_k [\epsilon_k - \max(\alpha_k, \alpha_k')] f(\theta_k)\theta_k \\ & - \sum_k [\epsilon_k' - \max(\alpha_k, \alpha_k')] f(\theta_k')\theta_k'. \end{aligned} \tag{30}$$

Let

$$\gamma = \beta_0 h^T b + \frac{\delta - \delta'}{K}.$$

Then (29) and Lemma 2 together imply the existence of a positive definite matrix P, a positive semidefinite matrix D, and an n vector g such that V is positive definite and

$$\dot V \leq -[g^Tx + \sqrt{\gamma} f(\sigma)]^2 - (\delta - \delta')f(\sigma)\left[\frac{\sigma - f(\sigma)}{K}\right] - x^TDx$$

plus terms that are negative semidefinite provided inequalities (16) are satisfied. (31)

Thus V is negative semidefinite. Furthermore, since $\dot V = 0$ implies that each of the terms in (31) is zero, one can show along the same lines as in Narendra and Neuman[4] that the only trajectory which remains in the set $\dot V = 0$ is $x(t) \equiv 0$. Furthermore, $V(x) \to \infty$ as $\|x\| \to \infty$. Thus the system (1) is ASIL.

The assumption that the poles of the multiplier be the zeros of $G(s)+1/K$ can be dispensed with as follows.

The system (1) consists of a linear part with transfer function

$$G(s) = \frac{q(s)}{p(s)} = h^T(sI - A)^{-1}b \tag{32}$$

in cascade with a nonlinearity $f(\sigma)$ in a negative feedback loop.

It is known[3] that if the system obtained by replacing $G(s)$ by $G'(s)$, where

$$G'(s) = \frac{q(s)n(s)}{p(s)n(s)} \tag{33}$$

is ASIL, then the system (1) is also ASIL provided all the zeros of $n(s)$ have negative real parts. Now any number $-\eta = -\lambda + j\mu$ with negative real part and its complex conjugate can be regarded as a pair of zeros of $n(s)$ and hence of $G'(s)+1/K$ by suitably choosing $n(s)$. However, $-\eta$ and $-\eta^*$ cannot be used as poles of the multiplier since the matrix B is undefined for these two numbers since $-\eta$ and $-\eta^*$ are also poles of $G'(s)+1/K$.

In order to overcome this difficulty, consider the system obtained by replacing $G(s)$ by $G''(s)$, where

$$G''(s) = \frac{q(s)n(s)}{p(s)[n(s) + \epsilon n'(s)]} \tag{34}$$

By choosing ϵ sufficiently small, the poles of $G''(s)$ can be made to lie arbitrarily close to but distinct from the zeros, and hence the matrix B will now be defined for a pair of zeros of $G''(s)+1/K$. It can be seen that the system with transfer function $G'(s)$ possesses structural stability[9] with respect to the poles of $G'(s)$, and hence has the same asymptotic stability property as the system with transfer function $G''(s)$ provided $|\epsilon|$ is sufficiently small. It follows that if the system with $G''(s)$ is XSIL, so is system (1).

Proceeding as in the earlier part of the **proof,** a stability criterion for the system with $G''(s)$, and hence for (1), can be stated in terms of the product $Z''(s)[G''(s)+1/K]$. Since the poles of $Z''(s)$ cancel with zeros of $G'(s)+1/K$, this product is equivalent to $Z(s)[G(s)+1/K]$, where $Z(s)$ has poles at $-(\eta+e)$ and $-(\eta^*+e)$, e being a small real number depending on the ϵ chosen. Equating $\eta+e$ to $\lambda+j\mu$, the stability criterion can be obtained in the form of Theorem 1. The conditions (15) need not be changed since e is arbitrarily small and can be of either sign.

Proof of Theorem 1A

The multiplier (13) with $\beta_i=\beta_i'$ and $\beta_k=\beta_k'$ can be written as

$$Z(s) = (\delta - \delta') + \beta_0 s + \sum_p (\gamma_p + \gamma_p') \cdot \left[1 + \frac{r_{1p} + jr_{2p}}{s + \lambda_p + j\mu_p} + \frac{r_{1p} - jr_{2p}}{s + \lambda_p - j\mu_p}\right] \quad (35)$$

where

$$p = 1, \cdots, v_1, v_1 + 1, \cdots v$$

$$r_{1p} = \pm \frac{\gamma_p - \gamma_p'}{\beta_p}$$

$$r_{2p} = \pm \frac{\gamma_p \gamma_p'}{\beta_p(\gamma_p + \gamma_p')} \quad (36)$$

where the plus sign in r_{1p} and r_{2p} refers to the k subscript terms and the minus sign to the i subscript terms. Identifying $\delta-\delta'=\alpha_0$ and $\gamma_p+\gamma_p'=\delta_p$, (35) can be put in the form (18). To show that the inequalities (16) in Theorem 1 result in a restriction on the residues in the form of (19) in Theorem 1A, proceed as follows.

Set $\gamma_p/\beta_p=u_p$ and $\gamma_p'/\beta_p'=v_p$. Then

$$r_{1p} = \pm 1/2(u_p - v_p)$$

$$r_{2p} = \pm \frac{u_p v_p}{u_p + v_p} \quad (37)$$

subject to the inequalities (16) which can be written as

$$0 \leq u_p \leq \lambda_p - \mu_p$$

$$0 \leq v_p \leq \lambda_p - \mu_p. \quad (38)$$

It is evident from (37) that

$$r_{1p\,\max} = \frac{1}{2} | u_p - v_p |_{\max} = \frac{\lambda_p - \mu_p}{2} \quad (39)$$

and hence r_{1p} can have any value in

$$\left[-\frac{\lambda_p - \mu_p}{2}, \frac{\lambda_p - \mu_p}{2}\right].$$

The inequalities (38) impose conditions on r_{2p} for any given r_{1p} in this interval. For example, if $r_{1p}, r_{2p} \geq 0$, it can be shown that for any r_{1p}

$$r_{2p} \leq \frac{(\lambda_p - \mu_p)(\lambda_p - \mu_p - 2r_{1p})}{2(\lambda_p - \mu_p - r_{1p})} \quad (40)$$

or

$$(\lambda_p - \mu_p - r_{1p})(\lambda_p - \mu_p - r_{2p}) \geq \frac{(\lambda_p - \mu_p)^2}{2}. \quad (41)$$

As the conditions in the theorem are symmetrical with reference to the signs of r_{1p} and r_{2p}, the inequalities for the other cases can be obtained by replacing r_{1p} and r_{2p} by their moduli.

The second inequality of (19) is necessary to eliminate the possibility of satisfaction of (41) when both terms in the left-hand side are negative.

Acknowledgment

The authors are indebted to the reviewers whose suggestions have considerably improved the value of this paper.

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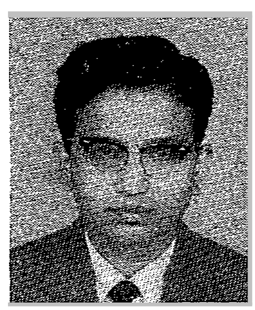

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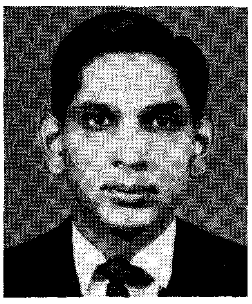

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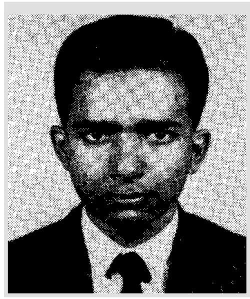

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