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Exact computation of a matrix symmetrizer using p-adic arithmetic

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Abstract

A symmetric solution X of the matrix equation $XA = A^t X$ is called a symmetrizer of (an arbitrary matrix) A. A symmetrizer reduces an apparently nonsymmetric matrix eigenvalue problem into a symmetric one. Here we present an implementation of p-adic arithmetic on the Datta method for computing a symmetrizer exactly.

Key words : Matrix symmetrizer, p-adic arithmetic, Hessenberg matrix.

1. Introduction

A symmetrizer[1-3] of an arbitrary square matrix A is a symmetric solution X of the matrix equation $XA = A^t X$, where t indicates the transpose. Symmetrizers are useful in the (nonsymmetric) matrix eigenvalue problems and in the stability problems of Control Theory. Computation of eigenvalues is easier for a symmetric matrix than for a nonsymmetric matrix. A symmetrizer reduces an apparently nonsymmetric eigenvalue problem to a symmetric one as given below.

1.1. *A symmetrizer for transformation into a symmetric eigenvalue problem*

Let A be a given nonsymmetric matrix. If there exists a positive definite symmetrizer X for A then the nonsymmetric eigenvalue problem

$$Ay = \lambda y \tag{1}$$

can be transformed into a symmetric eigenvalue problem

$$Bx = \lambda x \tag{2}$$

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as shown below:

From (1)

$$Cy = \lambda Xy \tag{3}$$

where $C = XA$. Now $X = S^t DS$ where $S^t = S^{-1}$ and D is diagonal.

Let $G = SCS^t$ and $F = Sy$. Then

$$GF = SCS^t Sy = SCy = \lambda DF. \tag{4}$$

If $d_{ii} > 0$ for all i then we have $D = D_1^2$ for some D_1. Thus we have, from (4),

$$GF = \lambda D_1 D_1 F \text{ or } D_1^{-1} GF = \lambda D_1 F \text{ or } D_1^{-1} GD_1^{-1} D_1 F = \lambda D_1 F.$$

Let $D_1^{-1} GD_1^{-1} = B$ and $D_1 F = x$. Then (1) becomes

$$Bx = \lambda x \tag{5}$$

where B is symmetric and real.

1.2. *Symmetrizer for computing zeros of a polynomial*

A symmetrizer can be used to obtain the roots of a polynomial equation[4,5]. This is achieved by symmetrizing the companion matrix[3] of the polynomial and then computing its eigenvalues by a matrix method.

1.3. *Existence of a symmetrizer*

There always exists a symmetrizer[1] for a nonderogatory matrix A. An $n \times n$ matrix is nonderogatory if and only if its characteristic polynomial (which is of degree n) is the same as its minimum polynomial[3].

1.4. *Transformation into Hessenberg form*

An arbitrary nonsymmetric matrix A can be transformed into a lower Hessenberg form[3,6] B (A and B having identical eigenvalues since the transformation is of similarity type). It is assumed that B has nonzero codiagonal. This assumption does not reduce the generality.

1.5. *Algorithm and arithmetic used*

Howland and Farrell[5] were the first to propose a numerical procedure (though not stable in general) for computing a symmetrizer of an arbitrary matrix. Here we consider an algorithm due to Datta[2] for computing a symmetrizer of a real matrix (lower Hessenberg).

It is important to obtain an error-free symmetrizer. Often numerical instability of an algorithm used and errors involved for a given matrix are beyond tolerance. Here we employ p-adic arithmetic[7,8] for error-free computation of a symmetrizer and describe it for convenience. A criterion for the selection of a prime base p and a number of digits r to be involved in representing a quantity (in p-adic form) is described.

We also describe the computation of the global denominator. Its knowledge is necessary for converting p-adic output to rational output. It is interesting to note that in the Datta method of symmetrization of an integer (Hessenberg) matrix the global denominator undergoes no modification even during p-adic multiplication/division.

2. Steps for exact computation using p-adic arithmetic

Let there be a numerical problem that involves a finite number of arithmetic operations. Since a digital computer works with finite length numbers in n-ary scale where n is any positive integer (say 10) called the radix or base, our input data are always rational numbers. We assume the rational input data to be exact.

Step 1. *Choice of prime base p and number of digits r.* Choose a prime base p and a positive integer r such that the largest magnitude number (that arises while executing a numerical algorithm for solving the said numerical problem) is less than or equal to $\sqrt{((p^r - 1)/2)}$. The condition is sufficient.

Remarks : (i) This allows a negative number to be represented in complement form uniquely.

(ii) Choice of p and r depends on the given numerical problem and the algorithm to solve it.

(iii) Step 1 requires a method to be devised to obtain a small (preferably the smallest) number greater than or equal to the largest magnitude number (which the algorithm produces).

Step 2. Store the p-adic from of (a positive integral multiple of) the LCM of the denominators (positive) of all the input rational numbers in a location called Global Denominator ' GLODEN '. Encode all the input rational numbers (both negative and positive, assuming a negative sign is attached to a numerator only) to p-adic numbers.

Remark : Since the denominator is a positive integer, its p-adic representation has an exponent zero.

Step 3. Execute (using the p-adic arithmetic) the numerical algorithm whose input data are: (i) the p-adic numbers and (ii) the GLODEN. Update the GLODEN, if necessary, whenever there occurs a p-adic multiplication/division.

Remarks : (i) In a p-adic addition/substraction GLODEN remains unchanged.

(ii) For the modification of the GLODEN a knowledge of the denominators of the operands is essential. Since the operations are performed in p-adic arithmetic, the denominators are not explicitly known. Therefore, whenever a modification of the GLODEN is required, the operands must be converted to their rational forms using the current GLODEN.

Step 4. Decode the p-adic output to rational form using the GLODEN.

2.1. *Encoding rational input to p-adic input (Step 2)*

Let p and r be known from Step 1. Also, let the usual symbols (like $+, -, \cdot, /$) of real arithmetic be the symbols for p-adic operations (add, subtract, multiply, divide) as long as operands are p-adic numbers.

Substep 1. Compute the LCM of the denominators of all rational input data.

Substep 2. GLODEN $\leftarrow$ p-adic form of the LCM. [Let the LCM be q_0. Then its p-adic form is $.s_1 s_2 \ldots s_r$ where s_i = positive remainder of $q_{i-1} \div p$, $i = 1(1)r$; $(0 \leqslant s_i < p)$. q_i = positive quotient of $q_{i-1} \div p$, $i = 1(1)r - 1$.]

Substep 3. Convert the numerator and the denominator of each rational input into p-adic form in the same way as in Substep 2. If the numerator has a negative sign then consider, for the numerator, its complement, *viz.*, "p^r — numerator" before conversion. Divide the p-adic numerator by the p-adic denominator using p-adic division (see p-adic arithmetic Sec. 2.3).

Note : The input data are thus (i) the p-adic form of all rational input data, and (ii) the GLODEN.

2.2. *Updating global denominator GLODEN (Step 3)*

p-adic addition/subtraction $(+/-)$ do not alter the Global Denominator GLODEN. p-adic multiplication/division, however, may alter the GLODEN.

For each integer a, where $0 \leqslant a \leqslant p^r - 1$, define

$$v(a) = a \text{ if } 0 \leqslant a \leqslant \frac{p^r - 1}{2}, \quad v(a) = a - p^r \text{ if } \frac{p^r - 1}{2} < a \leqslant p^r - 1. \tag{6}$$

Also, for each mantissa part $m_\alpha = .s_1 s_2 \ldots s_r$ of a p-adic number $\alpha = m_\alpha E e_\alpha$ define

$$I(m_\alpha) = \sum_{i=1}^{r} s_i p^i. \tag{7}$$

(i) *GLODEN after a multiplication* : Let $\alpha = m_\alpha E e_\alpha$ and $\beta = m_\beta E e_\beta$ be two p-adic numbers to be multiplied. Compute

$$\begin{aligned} \tilde{\alpha} &= [v(I(m_\alpha \cdot \text{GLODEN}))] \frac{1}{p^{e_\alpha}} \\ \tilde{\beta} &= [v(I(m_\beta \cdot \text{GLODEN}))] \frac{1}{p^{e_\beta}}. \\ \overline{\text{GLODEN}} &= I(\text{GLODEN}). \end{aligned} \tag{8}$$

Compute

$$\begin{aligned} \alpha^+ &= GCD(\tilde{\alpha}, \overline{\text{GLODEN}}) \\ \beta^+ &= GCD(\tilde{\beta}, \overline{\text{GLODEN}}) \\ G &= GCD(\overline{\text{GLODEN}}/\alpha^+, \overline{\text{GLODEN}}/\beta^+). \end{aligned} \tag{9}$$

If $G = 1$ then GLODEN does not alter. Otherwise, GLODEN $\leftarrow$ GLODEN $\times$ p-adic form of G.

Remarks : (*a*) $\overline{\text{GLODEN}}/\alpha^+$ is the denominator of α.

(*b*) $\overline{\text{GLODEN}}/\beta^+$ is the denominator of β.

(ii) *GLODEN after a division* : Let $\alpha = m_\alpha E e_\alpha$ and $\beta = m_\beta E e_\beta$ be two p-adic numbers. α is to be divided by β. Here

$$\text{GLODEN} \leftarrow p\text{-adic form of LCM } (\overline{\text{GLODEN}},\ \bar{\beta}/\beta^+) \tag{10}$$

where $\overline{\text{GLODEN}}$, $\bar{\beta}$, β^+ are as defined in (i) of Sec. 2.2.

2.3. *p-adic arithmetic* (*Steps* 2 *and* 3)

Let usual symbols like $+$, $-$, $.$, $/$ represent respectively the p-adic add, subtract, multiply, and divide operations, as long as the operands are p-adic numbers. Let

$$\alpha = m_\alpha E e_\alpha,\ \beta = m_\beta E e_\beta,\ \gamma = m_\gamma E e_\gamma \tag{11}$$

be three p-adic numbers where m_α, m_β and m_γ are mantissas and e_α, e_β, e_γ are exponents. Also, let p be the prime base and r be the number of digits retained in a mantissa.

(i) *p-adic addition* ($\gamma = \alpha + \beta$) : Let

$$m_\alpha = \cdot s_1 s_2 \cdots s_r$$
$$m_\beta = \cdot t_1 t_2 \cdots t_r$$
$$e_\alpha \geqslant e_\beta. \tag{12}$$

Shift s_i's by $e_\gamma = e_\alpha - e_\beta$ places to the left. Then carry out p-adic addition on these mantissas in the same way as in the finite field arithmetic with prime base p. The p-adic point will then be shifted before the leftmost digit of the result. Let this result now be called m_γ. Any carry beyond r digits is neglected.

Example : Let $\alpha = \cdot 30000000$ E1, $\beta = \cdot 21313131$ E0. Compute $\gamma = \alpha + \beta$.

Here $p = 5$, $r = 8$, $m_\alpha = \cdot 30000000$, $m_\beta = \cdot 21313131$, $e_\alpha = 1$, $e_\beta = 0$

$$\begin{aligned} \text{shifted } m_\alpha &= 3\cdot 0000000 \\ m_\beta &= \ \ \cdot 2131313 \\ \hline m_\gamma &= \cdot 32131313 \\ e_\gamma &= e_\alpha - e_\beta = 1 - 0 = 1. \end{aligned}$$

Hence $\gamma = \alpha + \beta = \cdot 32131313$ E1.

(ii) *p-adic subtraction* ($\gamma = \alpha - \beta$): The process of p-adic subtraction is similar to that of p-adic addition.

Example : Let $\alpha = \cdot 30000000$ E1 $\beta = \cdot 33131313$ E0. Compute $\gamma = \alpha - \beta$.

Here $p = 5,\ r = 8,\ m_\alpha = \cdot 30000000,\ e_\alpha = 1,\ m_\beta = \cdot 21313131,\ e_\beta = 0$

$$\begin{aligned} \text{shifted } m_\alpha &= 3\cdot 0000000 \\ m_\beta &= \cdot 3313131 \\ \hline m_\gamma &= \cdot 32131313 \\ e_\gamma &= e_\alpha - e_\beta = 1 - 0 = 1. \end{aligned}$$

Hence $\gamma = \alpha - \beta = \cdot 32131313$ E1.

(iii) *p-adic multiplication* ($\gamma = \alpha \cdot \beta$) : Here exponents are added as in the floating point multiplication (real arithmetic). So $e_\gamma = e_\alpha + e_\beta$. Mantissas m_α and m_β are multiplied in almost the same way as in the finite field arithmetic. The p-adic point is shifted before the leftmost digit of the result. This gives m_γ.

Example : Let $\alpha = \cdot 32222222$ E0, $\beta = \cdot 32131313$ E1. Compute $\gamma = \alpha \cdot \beta$.

Here $p = 5, r = 8,\ m_\alpha = \cdot 32222222,\ e_\alpha = 0,\ m_\beta = \cdot 32131313,\ e_\beta = 1.$

$$\begin{aligned} m_\alpha &= \cdot 32222222 \\ m_\beta &= \cdot 32131313 \\ \hline & 42222222 \\ & 1000000 \\ & 322222 \\ & 42222 \\ & 3222 \\ & 422 \\ & 32 \\ & 4 \\ \hline m_\gamma &= \cdot 43040404 \\ e_\gamma &= e_\alpha + e_\beta = 0 + 1 = 1. \end{aligned}$$

Hence $\gamma = \alpha \cdot \beta = \cdot 43040404$ E1.

Note : In p-adic multiplication/division we cannot write $\cdot 43040404$ E1 $= 4\cdot 3040404$ like a floating point number.

(iv) *p-adic division* ($\gamma = \alpha/\beta$) : Here exponents are subtracted as in the floating point division (real arithmetic). So $e_\gamma = e_\alpha - e_\beta$. m_α is divided by m_β in almost the same way as in the finite field arithmetic.

If m_β has k zeros at the leftmost side then m_α is divided by the p-adic number which is 'm_β without the k zeros'. k zeros are then appended before the leftmost digit of the quotient and after the p-adic point. This gives m_γ.

Examples : Let $\alpha = \cdot 23431013$ E0, $\beta = \cdot 32131313$ E1. Compute α/β.

Here $p = 5$, $r = 8$, $m_\alpha = \cdot 23431013$, $e_\alpha = 0$, $m_\beta = \cdot 32131313$, $e_\beta = 1$.

```
   mβ          mα          mγ
·32131313)·23431013 (·41232340
          20131313
          ---------
           3300244
           3213131
           -------
            141013
            103131
            ------
             43321
             42444
             -----
              1421
              1031
              ----
               444
               424
               ---
                20
                20
                --
                 0
                 0
```

$$e_\gamma = e_\alpha - e_\beta = 0 - 1 = -1.$$

Hence $\gamma = \alpha/\beta = \cdot 41232340\ E - 1$.

Let $\alpha = \cdot 331313\ \mathrm{E}0$, $\beta = \cdot 030313\ \mathrm{E}0$. Compute α/β.

Here $p = 5$, $r = 6$, $m_\alpha = \cdot 331313$, $e_\alpha = 0$, $m_\beta = \cdot 030313$, $e_\beta = 0$.

```
   mβ     mα     mγ
·30313)·33131 (·11111
       30313
       -----
        3313
        3031
        ----
         331
         303
         ---
          33
          30
          --
           3
           3
```

$$e_\gamma = e_\alpha - e_\beta = 0 - 0 = 0.$$

Hence $\gamma = a/\beta = \cdot 011111\ \mathrm{E}0$.

Note : When there are five digits in the divisor, there is no need to consider more than 5 digits of the dividend.

2.4. *Decoding p-adic output to rational form* (Step 4)

Let $a = m_a E e_a$ with GLODEN g be a p-adic output. Then

$$\text{rational form of } a = \frac{[v\,(I\,(m_a \cdot g))]\,(1/p^{e_a})}{\bar{g}/a^+} \tag{13}$$

(also, see Sec. 2.2) where, for each integer a,

$$v\,(a) = a \text{ if } 0 \leqslant a \leqslant \frac{p^r - 1}{2}\,,\ v\,(a) = a - p^r \text{ if } (p^r - 1)/2 < a \leqslant p^r - 1. \tag{14}$$

For $m_a = \cdot s_1 s_2 \cdots s_r$ of a

$$I\,(m_a) = \sum_{i=1}^{r} s_i p^i \tag{15}$$

$$\bar{g} = I\,(g)$$

$$a^+ = GCD\,(\bar{a}, \bar{g}). \tag{16}$$

Note : $\bar{g}/a^+$ is the denominator of a (in rational form).

3. The Datta algorithm

Let $B = (b_{ij})$ be the given $n \times n$ lower Hessenberg matrix with nonzero codiagonal. Let $x_1, x_2, \ldots, x_n$ be the rows of a symmetrizer X to be computed. Then

Step 1 : Choose $x_n \neq 0$ arbitrarily. (To ensure nonsingularity of X, choose an x_n of the form $(c\,0\,0\ \cdots 0)$ where $c \neq 0$).

Step 2 : Compute $x_{n-1}, x_{n-2}, \ldots, x_1$ recursively from

$$x_i = \frac{1}{b_{i,\,i+1}}\,(x_{i+1} B - b_{i+1,\,i+1} x_{i+1} - b_{i+2,\,i+1} x_{i+2} - \cdots - b_{n,\,i+1} x_n) \tag{17}$$

$$i = n - 1,\ n - 2, \ldots, 2, 1.$$

Step 3 : Print $x_1, x_2, \ldots, x_n$ and stop.

4. Choice of prime base p and number of digits r

Let the lower Hessenberg matrix B have (negative and/or positive) integer elements with nonzero codiagonal. This does not reduce the generality. From the Datta algorithm (Sec. 3).

$$x_{n-1} = \frac{1}{b_{n-1,\,n}}\,(x_n B - b_{nn} x_n).$$

Choose $x_n = (1\ 0\ 0 \ldots 0)$. L_1-norm[3] of x_{n-1}, *viz.*,

$$\| x_{n-1} \|_1 \leqslant 2\,(\max_j \sum_{i=1}^{n} |\, b_{ij} \,|)/b_{n-1,\,n} \text{ since } |\text{ every element of } x_n \,| \leqslant 1. \tag{18}$$

Thus the largest magnitude element in x_{n-1} is less than or equal to the right hand side expression in (18). The factor 2 is due to the addition or subtraction of two numbers, *viz.*, $x_n B$ and $b_{nn} x_n$.

Again from the Datta algorithm

$$x_{n-2} = \frac{1}{b_{n-2,\, n-1}} (x_{n-1} B - b_{n-1,\, n-1} x_{n-1} - b_{n,\, n-1} x_n).$$

Hence,

$$\| x_{n-2} \|_1 \leqslant 3 (\max_j \sum_{i=1}^{n} | b_{ij} |)^2 / (b_{n-1,\, n} b_{n-2,\, n-1}). \tag{19}$$

Similarly,

$$\| x_{n-3} \|_1 \leqslant 4 (\max_j \sum_{i=1}^{n} | b_{ij} |)^3 / (b_{n-1,\, n} b_{n-2,\, n-1} b_{n-3,\, n-2}) \tag{20}$$

Hence, choose p and r such that

$$\sqrt{((p^r - 1)/2)} \geqslant n (\max_j \sum_{i=1}^{n} | b_{ij} |)^{n-1}. \tag{21}$$

Note : The choice of p and r is not unique. However, p and r are so chosen that $\sqrt{((p^r - 1)/2)}$ is the smallest positive real number satisfying (21).

5. Invariance of global denominator

In the Datta algorithm a global denominator GLODEN (for an integer matrix) can be computed in the beginning once and for all :

$$\text{GLODEN} \leftarrow p\text{-adic form of } \prod_{i=1}^{n-1} b_{i,\, i+1}. \tag{22}$$

This results in considerable reduction in computation and complexity.

Note : In fact, we require here the computation of

$$\overline{\text{GLODEN}} \leftarrow \prod_{i=1}^{n-1} b_{i,\, i+1}$$

only, although in the numerical example we actually compute GLODEN and then $\overline{\text{GLODEN}}$ as a matter of illustration.

6. Numerical example

Compute an exact symmetrizer for the matrix (lower Hessenberg)

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Since the elements of B are integers, it is clear from the algorithm that

$$\text{GLODEN} = p\text{-adic form of } \prod_{i=1}^{n-1} b_{i,\,i+1}$$

and that during p-adic multiplication/division it undergoes no modification.

Note : If the elements of B are (nonintegral) rational numbers then multiply B by the scalar matrix whose (i, i)-th elements are the LCM of the denominators of all the rational elements. This results in an integer matrix. Compute a symmetrizer of this integer matrix. Multiply this symmetrizer by the scalar matrix to obtain a symmetrizer of B.

Let max. col. sum $= \max_j \sum_{i=1}^{n} |b_{ij}| = 6.$

$$n.(\text{max. col. sum})^{n-1} = 3 \times 6^2 = 108.$$

Therefore $p = 5$ and $r = 7$ will serve our purpose since

$$\sqrt{((p^r - 1)/2)} = \sqrt{((5^7 - 1)/2)} \geqslant 108.$$

Choose $x_3 = (1\ 0\ 0)$.

(i) *Encoding the input data into p-adic form* : B, in p-adic form, is

$$B' = \begin{bmatrix} \cdot 2000000\ \text{E0} & \cdot 1000000\ \text{E0} & \cdot 0000000\ \text{E0} \\ \cdot 3000000\ \text{E0} & \cdot 1000000\ \text{E0} & \cdot 2000000\ \text{E0} \\ \cdot 1000000\ \text{E0} & \cdot 2000000\ \text{E0} & \cdot 1000000\ \text{E0} \end{bmatrix}$$

x_3, in p-adic form, is

$$x_3' = (\cdot 1000000\ \text{E0} \quad \cdot 0000000\ \text{E0} \quad \cdot 0000000\ \text{E0})$$

The global denominator GLODEN is given by

$$\text{GLODEN} = p\text{-adic form of } \prod_{i=1}^{n-1} b_{i,\,i+1}$$

$$= p\text{-adic form of } 1 \times 2 = \cdot 2000000.$$

Note : The global denominator is always a positive number as we attach the negative sign (if any) to the numerator only. Thus GLODEN has always an exponent 0 (*viz.*, E0) and we do not write E0 at all.

(ii) *Executing the Datta Algorithm using p-adic arithmetic* :

$$x_2' = \frac{1}{b_{23}'}(x_3' B' - b_{33}' x_3')$$

$$= (\cdot 3222222\ \text{E0} \quad \cdot 3222222\ \text{E0} \quad \cdot 0000000\ \text{E0})$$

$$x_1' = \frac{1}{b_{12}'}(x_2' B' - b_{22}' x_2' - b_{32}' x_3')$$

$$= (\cdot 0000000\ \text{E0} \quad \cdot 3222222\ \text{E0} \quad \cdot 1000000\ \text{E0}).$$

(iii) *Converting p-adic output data to rational form* : Converting the p-adic output data x_1', x_2', x_3' into rational form we obtain x_1, x_2, x_3 as follows :

$$x_{11}' = \cdot 0000000 \text{ E}0 = m_\alpha E e_\alpha, \quad \text{GLODEN} = \cdot 2000000$$

$$m_\gamma = m_\alpha \times \text{GLODEN} = \cdot 0000000 \times \cdot 200000 = \cdot 0000000$$

$$\overline{\text{GLODEN}} = 2 \times 5^0 + 0 \times 5^1 + 0 \times 5^2 + 0 \times 5^3 + 0 \times 5^4 + 0 \times 5^5 + 0 \times 5^6$$
$$= 2.$$

$$v(\overline{\text{GLODEN}}) = 2 \text{ since } \overline{\text{GLODEN}} \leqslant (p^r - 1)/2 = (5^7 - 1)/2$$

(This is always the case).

$$I(m_\gamma) = 0 \times 5^0 + 0 \times 5^1 + 0 \times 5^2 + \ldots + 0 \times 5^6 = 0$$

$$v(I(m_\gamma)) = 0 \text{ since } m_\gamma \leqslant (p^r - 1)/2 = (5^7 - 1)/2$$

Since $e_\alpha = 0$, $p^{e_\alpha} = 5^0 = 1$.

Therefore,

$$x_{11} = \frac{1}{p^{e_\alpha}} \cdot \frac{v(I(m_\gamma))}{v(\overline{\text{GLODEN}})} = \frac{1}{1} \cdot \frac{0}{2} = 0.$$

$$x_{12}' = \cdot 3222222 \text{ E}0 = m_\alpha E e_\alpha, \quad \text{GLODEN} = \cdot 2000000$$

$$m_\gamma = m_\alpha \times \text{GLODEN} = \cdot 3222222 \times \cdot 2000000 = \cdot 1000000$$

As before, $v(\overline{\text{GLODEN}}) = 2$.

$$I(m_\gamma) = 1 \times 5^0 + 0 \times 5^1 + \ldots + 0 \times 5^6 = 1$$

$$v(I(m_\gamma)) = 1 \text{ since } m_\gamma \leqslant (p^r - 1)/2 = (5^7 - 1)/2$$

Since $e^\alpha = 0$, $p^{e_\alpha} = 5^0 = 1$

$$x_{12} = \frac{1}{p^{e_\alpha}} \cdot \frac{v(I(m_\gamma))}{v(\overline{\text{GLODEN}})} = \frac{1}{1} \cdot \frac{1}{2} = \frac{1}{2}.$$

Similarly we can obtain x_{13}, and x_2, x_3. Hence, the symmetrizer

$$X = (x_1\ x_2\ x_3) = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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