Pure Proportional Navigation Against Time-Varying Target Maneuvers

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Capturability of the pure proportional navigation (PPN) guidance law against a target executing bounded piecewise continuous time-varying maneuvers is investigated. A qualitative analysis is carried out to obtain a set of sufficient conditions for capture defined on the engagement parameters and initial conditions. These conditions are significantly less restrictive than the ones obtained previously by others using the Liapunov method. It is shown that the actual capture region for time-varying target maneuvers, obtained by using the conditions derived in this work, is much larger than that obtained from the Liapunov technique. We also show that though a bounded time-varying target maneuver does change the constant target maneuver capture region to some extent, it does not reduce it drastically Further, we show that the worst case capture region is obtained when the target executes a constant maneuver equal to the bound on the maneuver level. Some bounds on the missile lateral acceleration are also obtained for certain regions in the engagement plane. These results are generalizations and extensions of existing results on the capturability of the PPN guidance law against targets executing constant or time-varying target maneuvers.

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I. INTRODUCTION

Proportional navigation (PN) and its variants are some of the most widely researched missile guidance laws in the literature [1–3]. These guidance laws are easy to implement and have shown good performance against nonmaneuvering and moderately maneuvering targets. For highly maneuvering targets the PN law has been extended to yield the augmented proportional navigation (APN) law [4] and guidance laws based on optimal control theory [2] and differential games [5]. Though these modern guidance laws have been derived using a rigorous mathematical theory, their underlying philosophy remains the same as that of PN.

Early research on PN guidance law concentrated mainly on the analysis of a linearized version of the engagement geometry [6]. Guelman [7] was the first to consider the exact nonlinear equations of motion with the missile using the pure proportional navigation (PPN) guidance law in which the missile latax is applied in a direction normal to the missile velocity vector. It was shown through a qualitative analysis that for a nonmaneuvering target, under reasonable assumptions on the missile capability, the missile can capture the target from almost all initial conditions. Subsequently, a similar qualitative analysis by Guelman [8, 9] led to the set of sufficient conditions for which the missile can capture a target executing a constant maneuver. Subsequently, a closed-form solution to the equations of motion for a nonmaneuvering target was obtained by Becker [10]. For maneuvering targets no such closed-form solution is available to date.

Though in [8, 9] the target was assumed to execute a constant maneuver, in reality this need not be, and is usually not, the case since the target (normally a piloted aircraft) is capable of changing its maneuver level within certain bounds. In fact, actual avoidance and escape maneuvers executed by fighter aircraft involves a combination of sharp turns and straight dashes which are executed by switching between positive, zero, and negative lateral accelerations. This kind of time-varying target maneuver was considered by Ha, et al. [11] for a planar engagement and by Song and Ha [12] for a 3-dimensional engagement geometry, using the Liapunov function technique. Although they were able to obtain certain sufficient conditions of capture, it turns out that these conditions are quite inadequate in the sense that they demarcate a very small portion of the actual capture region in the initial condition space. In the present paper we obtain far more representative conditions for capture which can demarcate almost the whole of the capture region for targets executing time-varying target maneuvers. The technique used is an extension of the qualitative analysis approach used by Guelman [8].

II. PROBLEM FORMULATION AND MOTIVATION

A. Engagement Model

The planar engagement geometry is described using point mass models of the target T and the missile M (see Fig. 1). The coordinate system is centered on the target, with T_X as the direction of the initial target velocity V_{T0}. The target is assumed to maneuver only laterally. The autopilot and seeker response is assumed to be instantaneous and the missile angle of attack is assumed to be negligibly small. The velocities V_M and V_T of the missile and the target are constants throughout the engagement. The target is assumed to execute piecewise continuous time-varying maneuvers with specified bounds on the maneuver level, i.e., for all t,

$$|A_T(t)| \le A_{T\max}. \tag{1}$$

The equations of motion are

$$V_\theta = r\dot\theta = V_m \sin\alpha + V_T \sin(\theta - \beta), \qquad V_\theta(0) = V_{\theta 0}, \qquad \theta(0) = \theta_0 \tag{2}$$

$$V_r = \dot r = V_M \cos\alpha - V_T \cos(\theta - \beta), \qquad V_r(0) = V_{r0} \tag{3}$$

$$\dot\beta = \frac{A_T(t)}{V_T}, \qquad \beta(0) = 0 \tag{4}$$

$$\dot\gamma = \frac{A_M}{V_M}, \qquad \gamma(0) = \gamma_0. \tag{5}$$

Here, $r,\theta,\beta,\alpha,\gamma,A_M,A_T$ are all time-varying quantities. The argument t is written explicitly in $A_T(t)$ to stress the fact that unlike the assumption of constant target maneuver level in [8, 9] here we assume the target latax to be a time-varying quantity rather than a constant. Further, V_r and V_θ are the relative velocities of the target with respect to the missile along the line of sight (LOS) and normal to the **LOS,** respectively. The closing velocity is given by

$$V_c = -V_r = -\dot r. \tag{6}$$

When the missile is guided by the PPN guidance law, we have

$$A_M = N V_M \dot\theta \tag{7}$$

where, N is the navigation constant. Define

$$k = N - 1; \qquad \nu = \frac{V_M}{V_T}. \tag{8}$$

Integrating (4), we get the net angular excursion of the target in **the** time **period** 0 to t **as**

$$\beta = \int_0^t \left\{ \frac{A_T(t)}{V_T} \right\} dt = \left(\frac{t}{V_T} \right) \hat A_T(t) = t \hat A_{\nu T}(t) \tag{9}$$

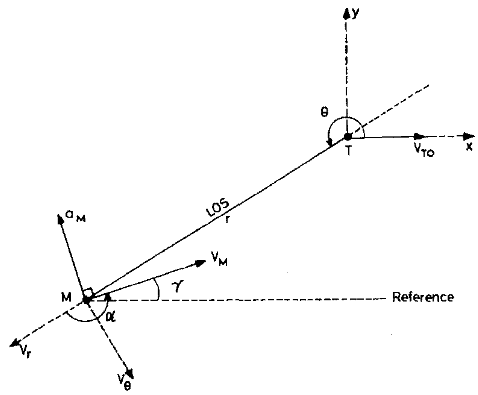

Fig. 1. Missile target engagement geometry.

where,

$$\hat A_T(t) = \left(\frac{1}{t} \right) \int_0^t A_T(t)\,dt \qquad \text{and} \qquad \hat A_{\nu T}(t) = \frac{\hat A_T(t)}{V_T}. \tag{10}$$

Here, $\hat A_T(t)$ is the average maneuver level (latax) of the target over the time period 0 to t and $\hat A_{\nu T}$ is the mean rate of change of the target orientation (or the mean turn rate of the target) over the time period 0 to t. Obviously, because of (1), $|\hat A_T(t)| \le A_{T\max}$ automatically holds. Note that when the target maneuver level is constant, i.e., $A_T(t) = A_T$ for all t, then $\hat A_T(t) = A_T$ and $\hat A_{\nu T}(t) = A_T/V_T$ for all t. This is the case considered by Guelman [8]. For time-varying target maneuvers both $\hat A_T$ and $\hat A_{\nu T}$ are functions of time, but for the sake of simplicity we drop the argument t in the subsequent analysis.

Substituting (7) in (5), integrating the resulting equation, and then using the relation $\alpha = \gamma - \theta + 2\pi$, (2) and **(3)** can be rewritten as

$$V_\theta(\theta,t) = V_M \sin(k\theta - \phi_0) + V_T \sin(\theta - \hat A_{\nu T} t) \tag{11}$$

$$V_r(\theta,t) = V_M \cos(k\theta - \phi_0) - V_T \cos(\theta - \hat A_{\nu T} t) \tag{12}$$

with $\phi_0 = k\theta_0 - \alpha_0$. Normalizing with respect to V_T, we get,

$$V_{\theta T} = \frac{V_\theta}{V_T} = \nu \sin(k\theta - \phi_0) + \sin(\theta - \hat A_{\nu T} t) \tag{13}$$

$$V_{rT} = \frac{V_r}{V_T} = \nu \cos(k\theta - \phi_0) - \cos(\theta - \hat A_{\nu T} t). \tag{14}$$

These are the fundamental equations on which a qualitative analysis is performed.

B. Capturability of PPN Guidance Law

Capturability of a guidance law is defined as its ability to ensure capture or intercept of a target

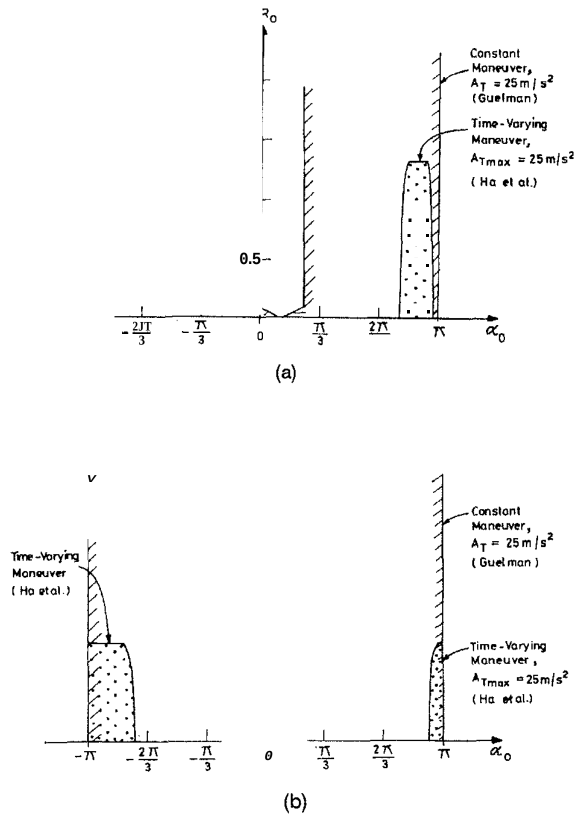

Fig. 2. **Comparison of guaranteed capture regions. (a)** $N = 4$, $V_M = 500$ m/s, $V_T = 350$ m/s, $\theta_0 = 210°$, $A_{T\max} = 25$ m/s^2. **(b)** $N = 5$, $V_M = 600$ m/s, $V_T = 300$ m/s, $\theta_0 = 150°$, $A_{T\max} = 25$ m/s^2.

by a missile. This is an important concept in the performance evaluation of missile guidance laws. The capture region is defined as the collection of all initial conditions from which a missile can intercept a target, and is a measure of the capturability performance of the guidance law.

For targets executing a constant maneuver, Guelman [8] showed that if $V_M > \sqrt{2}V_T$ and $NV_M > V_M + V_T$, then the missile intercepts the target from all initial conditions except from those for which the LOS rate is close to zero (i.e., $V_{\theta 0} \approx 0$) and the missile is initially moving away from the target (i.e., $V_{r0} > 0$). In Ha, et al. [11] some sufficient conditions of capture against an arbitrarily maneuvering target were obtained using the Liapunov technique. It turns out that the capture region which results from these conditions is very small when compared with that obtained in Guelman [8] for a constant target maneuver. To illustrate this, in Fig. 2 we show these capture regions for two different sets of initial conditions. The capture region for a constant maneuver is obtained by using the sufficient conditions given by Guelman [8, Theorems 1 and 2]. The capture region for a time-varying maneuver is obtained from the sufficient conditions given by Ha, et al. [11, Theorem 1]. The latter capture region is bounded by a maximum limit on the initial range, and is also dependent on the target velocity direction and the acceleration bounds. Also, from Fig. 2 it is immediately apparent that this capture region is insignificantly small compared with the constant maneuver capture region which is unbounded. Ha, et al. give another capture condition [11, Theorem 2] which attempts to define capture outside this region. Unfortunately, this condition cannot be translated in terms of initial conditions since it is required to be satisfied on the entire trajectory during the engagement period. However, even if we assume this condition to hold for the entire trajectory, the capture region so obtained still remains a very small subset of the constant maneuver capture region.

We show that the actual capture region for time-varying target maneuvers is much larger than is given by the capture conditions in Ha, et al. [11] (in fact, it is unbounded). It turns out that the seemingly drastic reduction in capture region shown by Ha, et al.'s results [11] is mainly due to the overly restrictive sufficient conditions imposed by the Liapunov technique and not because of the time-varying nature of the target maneuver. We also show that though a bounded time-varying target maneuver does change the capture region to some extent (as compared with the constant target maneuver case [S]), it does not reduce it drastically. Further, we show that the worst-case capture region is obtained when the target executes a constant maneuver equal to the bound on the maneuver level.

C. Target Maneuver Model

An intelligent target is expected to perform evasive maneuvers to increase its probability of escape. Since the target maneuvers considered here are restricted to the application of lateral acceleration normal to the velocity vector of the target, only the angular orientation of the target velocity vector, given by β, changes. Since the lateral acceleration A_T is the control used by the target to evade capture, it is reasonable to model this as a bounded piecewise continuous function in time (Ha, et al. [11]). The variation of A_T over a given interval of time is called a target maneuver profile and is denoted by **d**. The target flight direction (or angle) β varies according to the lateral acceleration applied by the maneuvering target. The angle β is a continuous function of time and, for a given target maneuver profile, is given by (9). Alternatively, one may also consider the instantaneous target angle β at a given instant in time to be a function of the average target acceleration $\hat{A}_T$, or the average turn rate $\hat{A}_{\nu T}$, until that time.

III. CAPTURABILITY CONDITIONS FOR TIME-VARYING TARGET MANEUVERS

A. Some Preliminary Results

Adopting the approach in Guelman [8], the following lemmas were used to carry out a qualitative analysis of (13) and (14). For this we denote the roots of $V_{\theta T} = 0$ as θ_θ and of $V_{rT} = 0$ as 8,.

LEMMA 1 **For** *any piecewise continuous target maneuver profile* ***A*** *and a constant time t, if $\nu > 1$ and $k\nu > 1$, then the roots of the equations $V_{rT} = 0$ and $V_{\theta T} = 0$ alternate along the θ axis.*

PROOF Let **us** assume $t = t_1$ (a constant). Further, let

$$\beta_1 = [\hat{A}_{\nu T} t]_{t=t_1} = \left[\frac{\hat{A}_T(t_1)}{V_T}\right]_{t=t_1}. \qquad (15)$$

Thus, β_1 is the net angular excursion of the target velocity vector until the time t_1. Consider a constant target maneuver profile **d** having a maneuver level $A_T = A_T(t_1)$. This will also cause a net angular excursion of β_1 at time t_1. Thus, $V_{\theta T}$ (and V_{rT}), plotted against 8 at $t = t_1$, will be the same for both $\mathcal{A}$ and $\tilde{\mathcal{A}}$. Since, according to [8, Lemma 1] (or [7, Lemma 1], if $\hat{A}_T(t_1) = 0$), the result is true for the constant maneuver profile $\tilde{\mathcal{A}}$, it must also be true for the piecewise continuous maneuver profile ***A*.** This completes the proof.

LEMMA 2 **For** *any piecewise continuous target maneuver profile* ***A*** *and a constant time t, if $\nu > 1$ and $k\nu > 1$ then,*

$$V_r(\theta_\theta, t)\frac{\partial V_\theta}{\partial \theta}(\theta_\theta, t) > 0$$

where θ_θ is a root of $V_{\theta T} = 0$.

PROOF Let $t = t_1$ (a constant) and define β_1 as in (15). Now, the proof follows a similar argument as Lemma 1 above but uses [7 and **8,** Lemma 2].

From Lemmas 1 and *2,* a general representation of V_{rT} and $V_{\theta T}$ profiles with respect to θ for any given time t can be obtained (see Fig. **3).** From (13) and (14) one can deduce that the roots of $V_{\theta T} = 0$ and of $V_{rT} = 0$ must satisfy the following necessary conditions

$$\theta_{n0} - \frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right) \le \theta_\theta \le \theta_{n0} + \frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right) \qquad (16)$$

and

$$\theta_{n0} + \frac{\pi}{2k} - \frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right) \le \theta_r \le \theta_{n0} + \frac{\pi}{2k} + \frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right) \qquad \textbf{(17)}$$

where,

$$8_{n0} = \theta_0 - \frac{\alpha_0}{k} - \frac{n\pi}{k}, \qquad n = 0, \pm 1, \pm 2, \ldots. \qquad (18)$$

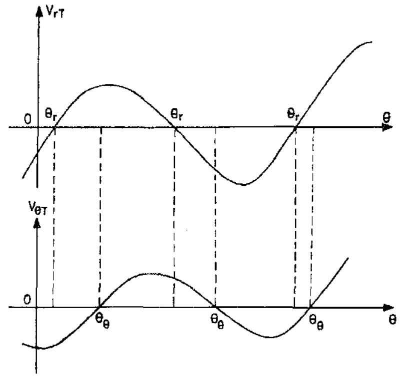

Fig. 3. V_{rT} **and** $V_{\theta T}$ **profiles** for **fixed** t.

Equation (16) gives the S_θ class of sectors, while (17) gives the S_r class of sectors defined in [8]. It is important to note that these S_θ and S_r sectors are independent of the target acceleration. However, unlike in [8], a θ belonging to one of these sectors is not a sufficient condition for it to be a root of the corresponding $V_{\theta T} = 0$ or $V_{rT} = 0$ equation for some t. In other words, for a given target maneuver profile $\mathcal{A}$, there may not exist a t for each $\theta \in S_\theta$ (or each $\theta \in S_r$) such that $V_{\theta T} = 0$ ($V_{rT} = 0$). This is in contrast to the situation in [8] where (16) and (17) were both necessary and sufficient conditions. In fact, this is the point of departure of our analysis from the analysis in [8].

For a given target maneuver profile ***A*,** let $\beta = \hat{A}_{\nu T} t$ be the net angular excursion by the target velocity vector. Then the roots of $V_{\theta T} = 0$ must satisfy,

$$\sin(\theta_\theta - \beta) = -\nu \sin(k\theta_\theta - \phi_0). \qquad (19)$$

Satisfaction of (16) ensures that the right-hand side (RHS) of **(19)** lies between **-1** and **1,** which makes the equation feasible. For a constant non-zero maneuver profile β goes on increasing (or decreasing) linearly with time, and hence for any $\theta_\theta \in S_\theta$, there is a time t for which the value of β is such that (19) is satisfied. But, for a time-varying target maneuver profile, this may not be the case. Now, consider such a maneuver profile ***A*** defined in the time interval $[0, T]$. Define,

$$\beta_{\max}(\mathcal{A}) = \max_{t \in [0,T]} \beta \qquad \textbf{(20a)}$$

$$\beta_{\min}(\mathcal{A}) = \min_{t \in [0,T]} \beta. \qquad \textbf{(20b)}$$

The existence of $\beta_{\max}(\mathcal{A})$ and $\beta_{\min}(\mathcal{A})$ is assured since β is a continuous function of t [13]. Obviously, $\beta_{\max}(\mathcal{A}) \ge 0$ and $\beta_{\min}(\mathcal{A}) \le 0$, for all ***A*.** The set $B(\mathcal{A}) = [\beta_{\min}(\mathcal{A}), \beta_{\max}(\mathcal{A})]$ is called the attainable set of the target flight direction for **a** given $\mathcal{A}$. **This** implies that for every $\beta \in B(\mathcal{A})$ there exists a time $t \in [0, T]$ such that $\hat{A}_{\nu T} t = \beta$. Let $N_\theta(\mathcal{A})$ and $N_r(\mathcal{A})$ be collections of the roots of $V_{\theta T} = 0$ and $V_{rT} = 0$, respectively, for some ***A*.** Thus, it is easy to see

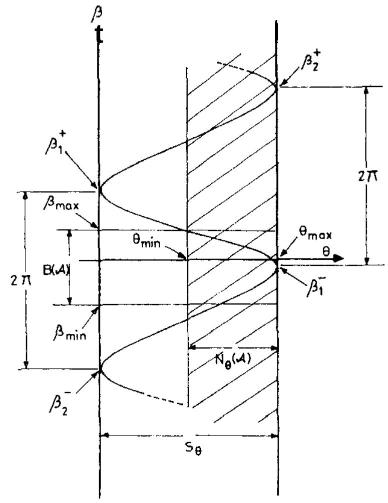

Fig. 4. β **versus** θ **in** S_θ sector.

that,

$$N_\theta(\mathcal{A}) = \{\theta \in S_0 : \exists \beta \in B(\mathcal{A}) \ni V_{\theta T} = 0\} \subseteq S_\theta \tag{21a}$$

$$N_r(\mathcal{A}) = \{\theta \in S_r : \exists \beta \in B(\mathcal{A}) \ni V_{rT} = 0\} \subseteq S_r \tag{21b}$$

and hence, using a result from [8], we conclude that for any $\boldsymbol{A}$, $N_\theta(\mathcal{A})$ and $N_r(\mathcal{A})$ are disjoint when $\nu > \sqrt{2}$ (i.e., $V_M > \sqrt{2}V_T$). Unlike the S_θ and S_r sectors the $N_\theta(\mathcal{A})$ and $N_r(\mathcal{A})$ sectors depend on the target maneuver profile and may vary in width in different S_θ and S_r sectors, respectively. Another point worth mentioning is that even for a known target maneuver profile (except the constant one) it is difficult to obtain an analytical expression for the boundaries of $N_\theta(\mathcal{A})$ and $N_r(\mathcal{A})$ sectors. However, it is fairly easy to compute these boundaries using the technique described below.

From (19) we obtain,

$$\frac{d\beta}{d\theta} = 1 + \frac{k\nu\cos(k\theta - \phi_0)}{\sqrt{1 - \nu^2\sin^2(k\theta - \phi_0)}} \tag{22}$$

It is easily proved that $d\beta/d\theta$ cannot attain the value zero for any value of $\theta \in S_\theta$ if $k\nu > 1$ and $\nu > 1$. Hence, $d\beta/d\theta$ **is** either positive or negative, i.e., β either strictly increases or strictly decreases with 6 in any S_θ sector (see Fig. 4). Let a function $\theta = f(\beta)$ be such that it satisfies (19). Obviously, f is continuous and, for a given target maneuver profile $\boldsymbol{A}$, is defined over a compact set $B(\mathcal{A})$. Thus, f will have both a minimum and a maximum. Let the values of $6 \in S_\theta$ for which these extrema occur be denoted by $\theta_{\min}$ and $\theta_{\max}$. Then it can be easily seen that

$$N_\theta(\mathcal{A}) = [\theta_{\min}, \theta_{\max}]. \tag{23}$$

This gives **us** a straightforward way of determining $N_\theta(\mathcal{A})$ without an exhaustive search over S_θ and $B(\mathcal{A})$ to obtain those (θ,β) pairs which satisfy (19). Let β_n^+ $(n = 1,2)$ be the nth nonnegative value of β for which $(\beta, f(\beta))$ touches the boundary of S_θ as we increase β from 0 to ∞. Similarly, let β_n^- $(n = 1,2)$ be the nth negative value of β for which $(\beta, f(\beta))$ touches the boundary of S_θ as we decrease β from 0 to $-\infty$. These are also shown in Fig. 4. To obtain these values we first compute the boundaries of S_θ given by (16). Substituting these in (19) and solving for β immediately gives us β_1^+ and β_1^-. Next we obtain β_2^+ and β_2^- from,

$$\beta_2^+ = \beta_1^- + 2\pi, \qquad \beta_2^- = \beta_1^+ - 2T. \tag{24}$$

The boundaries of $N_\theta(\mathcal{A})$ can then be obtained from the following lemma.

LEMMA 3 **For** *a given piecewise continuous target maneuver profile* $\boldsymbol{A}$, *and* $\nu > 1$, $k\nu > 1$, *if*

a) $\beta_{\max} \geq \beta_2^+$; *or* $\beta_{\min} \leq \beta_2^-$; *or* $\beta_{\max} \geq \beta_1^+$ *and* $\beta_{\min} \leq \beta_1^-$; *then* $N_\theta(\mathcal{A}) = S_0$.

b) $\beta_{\max} \geq \beta_1^+$ *and* $\beta_{\min} \geq \beta_1^-$; *then one of the boundaries of* $N_\theta(\mathcal{A})$ *is given by the boundary of* S_θ *which* $(\beta_1^+, f(\beta_1^+))$ *touches. The other boundary is given by solving for* θ *in the equation,*

$$\sin(\theta - \beta_{\min}) + \nu\sin(k\theta - \phi_0) = 0. \tag{25a}$$

c) $\beta_{\max} \leq \beta_1^+$ *and* $\beta_{\min} \leq \beta_1^-$; *then one of the boundaries of* $N_\theta(\mathcal{A})$ *is given by the boundary of* S_θ *which* $(\beta_1^-, f(\beta_1^-))$ *touches. The other boundary* ***is*** *given by solving* ***for*** θ *in the equation,*

$$\sin(\theta - \beta_{\max}) + \nu\sin(k\theta - \phi_0) = 0. \tag{25b}$$

d) $\beta_{\max} \leq \beta_1^+$ *and* $\beta_{\min} \geq \beta_1^-$; *then the boundaries of* $N_\theta(\mathcal{A})$ *are given by solving for 8 in* (25a) *and* (25b).

PROOF The proof is easily constructed using Fig. 4.

A similar result for determining $N_r(\mathcal{A})$ can be obtained. Note that for a constant non-zero maneuver profile, condition a) in Lemma ***3*** is automatically met for a sufficiently large time interval. In general, for each S_θ and S_r sector, both $N_\theta(\mathcal{A}) \subseteq S_\theta$ and $N_r(\mathcal{A}) \subseteq S_r$ are nonempty. This can be easily seen from the fact that B is always nonempty since $0 \in B(\mathcal{A})$ for all $\boldsymbol{A}$. Now, from [7] **we** know that there exists a θ in each S_θ such that (19) is satisfied for $\beta = 0$. Hence, this θ must belong to $N_\theta(\mathcal{A})$ for all $\boldsymbol{A}$. Similarly, we can show that for $\beta = 0$ there exists a $\theta \in S_r$ which also belongs to $N_r(\mathcal{A})$.

LEMMA 4 *Given any piecewise continuous maneuver* ***profile*** *A and a constant time* $t = t_1$; *if* $\nu > 1$ *and* $k\nu > 1$, *then*

a) *there exists one and only one value of* $\theta = \theta_\theta$ *in each* $N_\theta(\mathcal{A})$ *sector such that* $V_{\theta T}(\theta_\theta, t_1) = 0$;

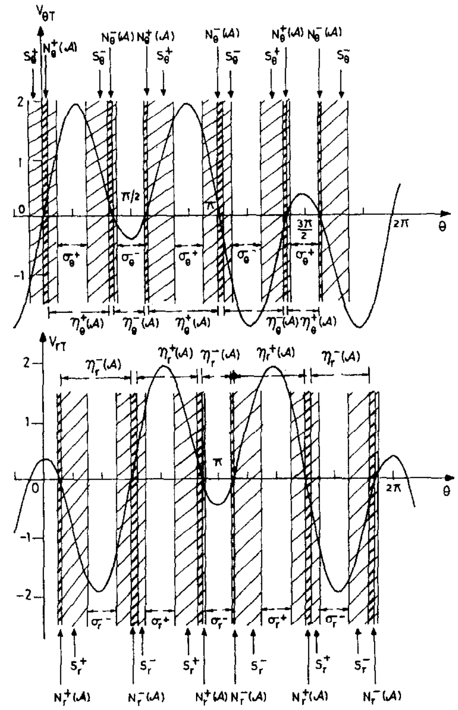

Fig. **5.** Sectors **in** $(V_{\theta T},\theta)$ and (V_{rT},θ) plane.

b) *there exists one and only one value of* $\theta = 8$, *in each* $N_r(\mathcal{A})$ *sector such that* $V_{rT}(\theta_r,t_1) = 0$.

PROOF Define β_1 as in (15). Consider a constant target maneuver profile **d** with a maneuver level $A_T = \hat{A}_T(t_1)$. According to [8, Lemma 3], the resulting function $V_{\theta T}(\theta,t_1)$ will have only one root $\theta_\theta \in S_\theta$. This particular θ_θ must satisfy (19) at $t = t_1$ and hence must belong to $N_\theta(\mathcal{A})$. Since there is only one such $\theta_\theta \in S_\theta$, there will be a single root inside $N_\theta(\mathcal{A})$ too. Similar arguments for $N_r(\mathcal{A})$ also hold. This completes the proof.

B. Capturability Conditions

For a given target maneuver profile ***A,*** the entire plane of pursuit in the target-centered coordinate system is divided into eight different sectors as follows:

$$
\begin{aligned}
N_\theta^+(\mathcal{A}) &= \{\theta : V_\theta(\theta,t) = 0,\ V_r(\theta,t) > 0, \text{ for some } t\}\\
N_\theta^-(\mathcal{A}) &= \{\theta : V_\theta(\theta,t) = 0,\ V_r(\theta,t) < 0, \text{ for some } t\}\\
N_r^+(\mathcal{A}) &= (6 : V_r(\theta,t) = 0,\ V_\theta(\theta,t) > 0, \text{ for some } t\}\\
N_r^-(\mathcal{A}) &= (6 : V_r(\theta,t) = 0,\ V_\theta(\theta,t) < 0, \text{ for some } t\}\\
\eta_\theta^+(\mathcal{A}) &= \{\theta : V_\theta(\theta,t) > 0, \text{ for all } t\}\\
\eta_\theta^-(\mathcal{A}) &= \{\theta : V_\theta(\theta,t) < 0, \text{ for all } t\}\\
\eta_r^+(\mathcal{A}) &= \{\theta : V_r(\theta,t) > 0, \text{ for all } t\}\\
\eta_r^-(\mathcal{A}) &= \{\theta : V_r(\theta,t) < 0, \text{ for all } t\}.
\end{aligned}
\tag{26}
$$

These sectors are shown in Fig. 5 in the $(V_{\theta T},\theta)$ and (V_{rT},θ) plane and in Fig. **6** in the target-centered polar plane for a constant t. It is easy to see that,

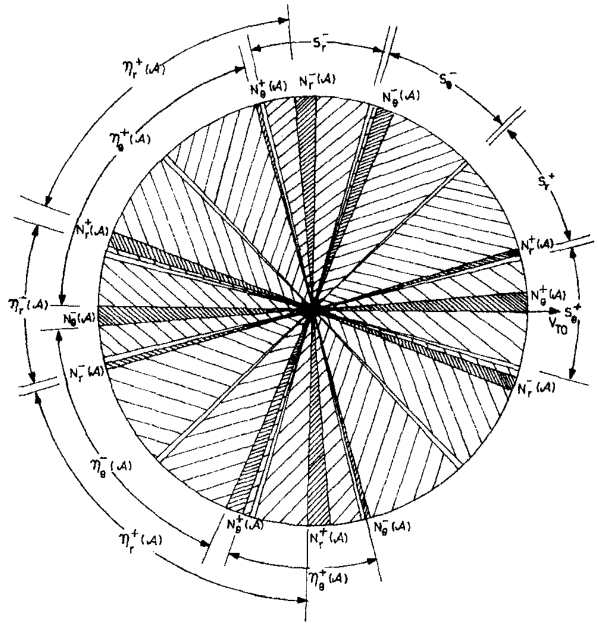

Fig. **6.** Sectors in target-centered polar plane.

$$
\sigma_\theta^+ \subseteq \eta_\theta^+(\mathcal{A}), \qquad \sigma_\theta^- \subseteq \eta_\theta^-(\mathcal{A}),
$$
$$
\sigma_r^+ \subseteq \eta_r^+(\mathcal{A}), \qquad \sigma_r^- \subseteq \eta_r^-(\mathcal{A})
$$

where, the σ sectors are as defined in [8] and are shown here in Fig. 5. Based on the above lemmas and the division of the plane of pursuit into sectors the following theorems are stated and proved.

THEOREM 1 *A missile M using a PPN guidance law and pursuing a target T maneuvering with a given bounded piecewise continuous target maneuver profile* **d** *is guaranteed to capture the target from any initial state not belonging to the sector* N_θ^+ **(d)**, *if* $V_M > \sqrt{2}V_T$ *and* $N > 1 + V_T/V_M$ *(i.e.,* $\nu > \sqrt{2}$ *and* $k\nu > 1$*). Also, the missile arrives at the target in the interior of a* N_θ^- **(d)** *sector*

PROOF Consider the initial state to be such that $\theta_0 \in \eta_\theta^+(\mathcal{A}) \cap \eta_r^+(\mathcal{A})$. From Fig. 5 we have $V_{r0} > 0$ and $V_{\theta 0} > 0$, i.e., both r and θ are increasing. It is easy to show that the state moves into $\eta_r^-(\mathcal{A})$ and then approaches and finally enters $N_\theta^-(\mathcal{A})$ in finite time. In this region $V_r < 0$ and so r goes on decreasing with time until the state at some time t reaches the value θ_θ for which $V_\theta(\theta_\theta,t) = 0$. This automatically leads to interception. Similarly, it can be shown that $\theta_0 \in \eta_\theta^-(\mathcal{A}) \cap \eta_r^+(\mathcal{A})$ also leads to interception. Finally, it **is** apparent that if $\theta_0 \in N_\theta^-(\mathcal{A})$ then θ remains in this sector until interception.

The above theorem provides a sufficient condition for capture for a given target maneuver profile. The capture region so demarcated contains the capture region for a constant target maneuver profile given in [8]. Hence, so far as the capture region outside the S_θ^+ sector is concerned, the capture region for a constant target maneuver is the same as that for an arbitrarily time-varying target maneuver. Also, any target maneuver profile $\mathcal{A}$ which satisfies condition a) in Lemma 3 will have the same capture region outside the $N_\theta^+(\mathcal{A})$ sector (which, for this case, is the same as the S_θ^+ sector) as the constant non-zero target maneuver. Hence, considering the worst case target behavior, subject to the limitation that the maneuver profile is piecewise continuous and bounded, we have the following corollary to Theorem 1.

COROLLARY 1 *For any bounded piecewise continuous target maneuver profile, capture is guaranteed if the initial state lies outside S_θ^+, and all the other conditions of Theorem 1 are satisfied.*

PROOF The proof follows from the fact that the largest $N_\theta^+(\mathcal{A})$ sector possible is the S_θ^+ sector (according to Lemma 3). Since Theorem 1 is true for any bounded piecewise continuous target maneuver profile, the proof is immediate.

Note that Corollary 1 is independent of the actual value of $A_{T\max}$. This is a generalization of [8, Theorem 1].

To express the capture region given in Theorem 1 in terms of a missile-centered coordinate system we may redefine the $N_\theta(\mathcal{A})$ sectors in terms of the missile flight direction angle α (defined with respect to the LOS in Fig. 1) and redenote them as $N_\alpha^+(\mathcal{A})$ and $N_\alpha^-(\mathcal{A})$ corresponding to $N_\theta^+(\mathcal{A})$ and $N_\theta^-(\mathcal{A})$, respectively. Thus, $\alpha_0 \in N_\alpha(\mathcal{A})$ if there exists a $\beta \in B(\mathcal{A})$ such that

$$\sin(\theta_0 - \beta) + \nu \sin\alpha_0 = 0. \tag{27}$$

Obviously, a necessary condition for an α_0 to satisfy (27) is

$$n\pi - \sin^{-1}(1/\nu) \le \alpha_0 \le n\pi + \sin^{-1}(1/\nu), \qquad n = 0 \text{ or } 1. \tag{28}$$

Hence, we may alternatively define $N_\alpha^+(\mathcal{A})$ and $N_\alpha^-(\mathcal{A})$ as

$$N_\alpha^+(\mathcal{A}) = \{\alpha : -\sin^{-1}(1/\nu) + \Delta_1 \le \alpha \le \sin^{-1}(1/\nu) - \Delta_2;\ \Delta_1 \ge 0,\ \Delta_2 \ge 0\} \tag{29a}$$

$$N_\alpha^-(\mathcal{A}) = \{\alpha : -\sin^{-1}(1/\nu) + \pi + \Delta_2 \le \alpha \le \sin^{-1}(1/\nu) + \pi - \Delta_2;\ \Delta_1 \ge 0,\ \Delta_2 \ge 0\} \tag{29b}$$

where Δ_1 and Δ_2 depend on $\mathcal{A}$ and θ_0. Hence, $N_\alpha(\mathcal{A})$ depends not only on the target maneuver profile but also on the flight direction of the target with respect to the LOS (i.e., θ_0). However, if the target maneuver profile is such that it satisfies condition a) in Lemma 3 then $\Delta_1 = \Delta_2 = 0$ and $N_\alpha^+(\mathcal{A})$ and $N_\alpha^-(\mathcal{A})$ become independent of $\mathcal{A}$ and θ_0. In fact, they become identical to S_α^+ and S_α^-, respectively, which are defined in [8] for a constant non-zero target maneuver profile.

Hence, according to Theorem 1, for a given target maneuver profile $\mathcal{A}$, if $\nu > \sqrt{2}$ and $k\nu > 1$, then the missile can capture the target if the initial missile velocity vector lies outside the $N_\alpha^+(\mathcal{A})$ sector. According to Corollary 1, for any bounded piecewise continuous target maneuver profile, the missile can capture the target if its initial velocity vector lies outside the S_α^+ sector.

C. Capturability in the $N_\theta^+(\mathcal{A})$ Sector

Now let us consider the only remaining sector $N_\theta^+(\mathcal{A})$. Let the initial state be such that $\theta_0 \in N_\theta^+(\mathcal{A})$. From (11) we get,

$$dV_\theta/dt = r\ddot{\theta} + \dot{r}\dot{\theta} = kV_M\cos(k\theta - \phi_0)\dot{\theta} + V_T\cos(\theta - \beta)(\dot{\theta} - \dot{\beta}). \tag{30}$$

Substituting (12) and (4) we obtain

$$r\ddot{\theta} = V_S\dot{\theta} - A_T(t)\cos(\theta - \beta) \tag{31}$$

where

$$V_S = (k-1)V_M\cos(k\theta - \phi_0) + 2V_T\cos(\theta - \beta). \tag{32}$$

Note that $A_T(t)$ is piecewise continuous and hence may have a finite number of points of discontinuity. However, at these points β is continuous though it is not differentiable in the conventional sense. Thus, in such cases we have to use the notion of a subgradient [14] of β. The value of the subgradient will naturally be given by the expression for $\dot{\beta}$ in (4).

LEMMA 5 *Consider a piecewise continuous target maneuver profile $\mathcal{A}$. Let*

$$V_e = (N-2)\sqrt{V_M^2 - V_T^2} - 2V_T. \tag{33}$$

a) *If $\theta \in N_\theta^+(\mathcal{A})$ then, $V_S \ge V_e$.*
b) *If $\theta \in N_\theta^-(\mathcal{A})$ then, $V_S \le -V_e$.*

PROOF If $\theta \in N_\theta^+(\mathcal{A})$ then it must also satisfy $|\sin(k\theta - \phi_0)| \le 1/\nu$ since $N_\theta^+(\mathcal{A}) \subseteq S_\theta^+$. Hence, we have

$$|\cos(k\theta - \phi_0)| \ge \frac{\sqrt{\nu^2 - 1}}{\nu}. \tag{34}$$

Also, since $\theta \in N_\theta^+(\mathcal{A})$,

$$V_{rT} > 0 \qquad \text{for all } t \tag{35}$$

i.e.,

$$\nu\cos(k\theta - \phi_0) > \cos(\theta - \beta) \qquad \text{for all } t. \tag{36}$$

We know that $N_\theta^+(\mathcal{A}) \subseteq S_\theta^+$ and S_θ^+ is invariant with respect to the target maneuver profile. Thus, every

$\theta \in S_\theta^+$ must satisfy (35) and (36) even for a constant non-zero maneuver level [8], which implies that,

$$\cos(k\theta - \phi_0) > 0 \tag{37}$$

must be true. From (34) and (37) we get

$$\cos(k\theta - \phi_0) \geq \frac{\sqrt{\nu^2 - 1}}{\nu}. \tag{38}$$

Also, $\cos(\theta - \beta) \geq -1$ always. Hence, from (32),

$$V_S \geq (k-1)V_M \frac{\sqrt{\nu^2-1}}{\nu} - 2V_T \tag{39}$$

i.e.,

$$V_S \geq V_e.$$

A similar proof for b) also holds.

THEOREM 2 ***For** a missile using a PPN guidance law and pursuing a target maneuvering with a bounded piecewise continuous maneuver profile $\mathcal{A}$, if the initial state is such that $\theta_0 \in N_\theta^+$* **(d)***and*

a) $V_M > \sqrt{2}V_T$ (i.e., $\nu > \sqrt{2}$)
b) $N > 2 + 2V_T/\sqrt{V_M^2 - V_T^2}$ (i.e., $V_e > 0$) (40)
c) $|\dot{\theta}_0| > A_{T\max}/V_e$

then the missile is guaranteed to capture the target.

PROOF Note that a) and b) above automatically ensure that $k\nu > 1$. From Lemma 5, and condition b) above we have $V_S \geq V_e > 0$. Hence,

$$-\frac{A_{T\max}}{V_e} \leq -\frac{|A_T(t)|}{V_e} \leq -\frac{|A_T(t)|}{V_S} \leq \frac{A_T(t)}{V_S}\cos(\theta - \beta)$$
$$\leq \frac{|A_T(t)|}{V_S} \leq \frac{|A_T(t)|}{V_e} \leq \frac{A_{T\max}}{V_e}. \tag{41}$$

From (31), and the fact that $V_S > 0$, $\dot{\theta}$ is an increasing function of time (i.e., $\ddot{\theta} > 0$) provided that

$$\dot{\theta} > \frac{A_T(t)}{V_S}\cos(\theta - \beta)$$

and a decreasing function of time if

$$\dot{\theta} < \frac{A_T(t)}{V_S}\cos(\theta - \beta).$$

Hence, from condition c) in the theorem if $\dot{\theta}_0 > A_{T\max}/V_e > 0$ then by the above arguments $\dot{\theta}$ goes on increasing until the state leaves the N_θ^+ **(d)**sector. Similarly, if $\dot{\theta}_0 < -A_{T\max}/V_e < 0$ then $\dot{\theta}$ ***goes*** on decreasing until the state leaves the $N_\theta^+(\mathcal{A})$ sector.

COROLLARY 2 ***For** any piecewise continuous time-varying target maneuver profile bounded by $A_{T\max}$ and satisfying all the conditions in Theorem 2, the missile is* ***guaranteed to capture the target if the initial*** *state lies in the S_θ^+ sector.*

PROOF Theorem 2 is valid for any arbitrary **d** with $\theta_0 \in N_\theta^+(\mathcal{A})$, and the largest such $N_\theta^+(\mathcal{A})$ sector is the S_θ^+ sector. Hence the theorem.

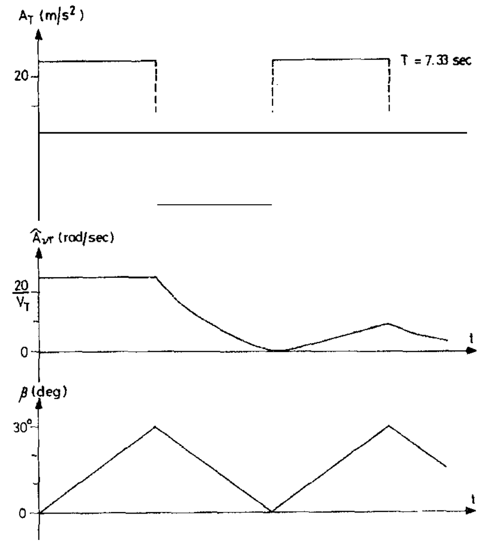

Fig. 7. Target maneuver profile in example.

Corollary 2 is a generalization of [8, Theorem 2]. Simulation results of planar missile-target engagements [15] with various bounded piecewise continuous time-varying maneuver profiles confirm the theoretical results presented here.

D. Example

Consider the engagement parameters and initial conditions assumed in Fig. 2(a), i.e., $N = 4$, $V_M = 500$ m/s, $V_T = 350$ m/s, $\theta_0 = 210°$, $A_{T\max} = 25$ m/s^2. But the target maneuver profile $\mathcal{A}$ is a periodic square wave switching between $+A_{T\max}$ and $-A_{T\max}$ (Fig. 7(a)). Note that the target maneuver profile is a piecewise continuous function with a finite number of discontinuities in any finite time interval. Fig. 7(b)–(c) show $\hat{A}_{\nu T}(t)$ and $\beta(t)$, respectively, with respect to time. From these data we obtain,

$$\beta_{\min}(\mathcal{A}) = 0°,$$
$$\beta_{\max}(\mathcal{A}) = 30°,$$
$$\beta \in B(\mathcal{A}) = [0°, 30°].$$

The corresponding relative velocity equations are,

$$V_{\theta T} = (\tfrac{10}{7})\sin(3\theta) + \sin(\theta - \beta)$$
$$V_{rT} = (\tfrac{10}{7})\cos(3\theta) - \cos(\theta - \beta).$$

Using the analysis presented in this work we compute the following sectors:

$$S_\theta^+ : [-14.8°, 14.8°], [105.2°, 134.8°], [225.2°, 254.8°]$$
$$S_\theta^- : [45.2°, 74.8°], [165.2°, 194.8°], [285.2°, 314.8°]$$
$$S_r^+ : [15.2°, 44.8°], [135.2°, 164.8°], [255.2°, 284.8°]$$

S_r^- : $[75.2°, 104.8°], [195.2°, 224.8°], [315.2°, 344.8°]$

$N_\theta^+(\mathcal{A})$: $[0°, 5.6°], [105.2°, 105.9°], [248.65°, 254.1°]$

$N_\theta^-(\mathcal{A})$: $[68.65°, 74.1°], [180°, 185.6°], [285.2°, 285.9°]$

$N_r^+(\mathcal{A})$: $[15.2°, 15.9°], [158.65°, 164.1°], [270°, 275.6°]$

$N_r^-(\mathcal{A})$: $[90°, 95.6"], [195.2°, 195.9°], [338.65°, 344.1°]$

σ_θ^+ : $(14.8°, 45.2°), (134.8°, 165.2°), (254.8°, 285.2°)$

σ_θ^- : $(74.8°, 105.2°), (194.8°, 225.2°), (314.8°, 345.2°)$

σ_r^+ : $(104.8°, 135.2°), (224.8°, 255.2°), (344.8°, 375.2°)$

σ_r^- : $(44.8°, 75.2°), (164.8°, 195.2°), (284.8°, 315.2°)$

$\eta_\theta^+(\mathcal{A})$: $(5.6°, 68.65°), (105.9°, 180°), (254.1°, 285.2°)$

$\eta_\theta^-(\mathcal{A})$: $(74.1°, 105.2°), (185.6°, 248.65°), (285.9°, 0°)$

$\eta_r^+(\mathcal{A})$: $(95.6°, 158.65°), (195.9°, 270°), (344.1°, 375.2°)$

$\eta_r^-(\mathcal{A})$: $(15.9°, 90°), (164.1°, 195.2°), (275.6°, 338.65")$.

Actually, in Figs. 5 and **6** we have shown these sectors and the variation of $V_{\theta T}$ and V_{rT} with respect to θ for the target maneuver profile **d** given in Fig. 7. Since $\theta_0 = 210°$, we see that $\theta_0 \notin N_\theta^+$ **(d)** sector and hence capture is guaranteed by Theorem 1. In fact, with this initial condition, the target will be captured irrespective of the maneuver profile it employs. Now, we obtain the capture region for this case with the initial missile flight direction angle (Y0 as a free variable. For this we need to obtain the $N_\alpha(\mathcal{A})$ sectors, i.e., those values of (Y which satisfy

$$\left(\tfrac{10}{7}\right)\sin\alpha + \sin(210° - \beta) = 0$$

with $\beta \in B(\mathcal{A}) = [0°, 30°]$. This yields,

$$N_\alpha^+ \textbf{(d)} = \{a : \alpha \in [0°, 20.48°]\}$$
$$N_\alpha^- \textbf{(d)} = \{\alpha : \text{(Y} \in [159.52°, 180°]\}.$$

These are shown in Fig. **8.** Now, using Theorem **1** we obtain the result that so long as the initial missile velocity vector does not belong to the $N_\alpha^+(\mathcal{A})$ sector, the missile can guarantee capture of the target, having a maneuver profile given in Fig. 7(a), irrespective of the initial range and the actual value of $A_{T\max}$. Further, from Corollary 1 we obtain the result that, if the missile velocity vector lies outside the S_α^+ sector then the missile can capture the target irrespective of initial range, target maneuver profile, initial target flight direction 8, and the actual value of $A_{T\max}$. Inside the $N_\alpha^+(\mathcal{A})$ sector guaranteed capture is governed by the sufficient conditions prescribed in Theorem 2. We note that conditions a>and b) in Theorem 2 are automatically satisfied by the values of N, V_M, and V_T chosen in this example. To satisfy condition a) in Theorem 2 the following has to be satisfied:

$$|V_{\theta 0}| > \frac{R_0 A_{T\max}}{V_e}$$

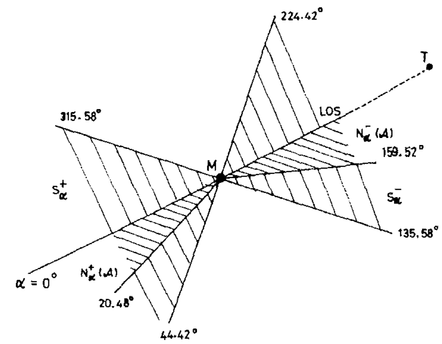

Fig. 8. $N_\alpha^+(\mathcal{A})$ and $N_\alpha^-(\mathcal{A})$ sectors.

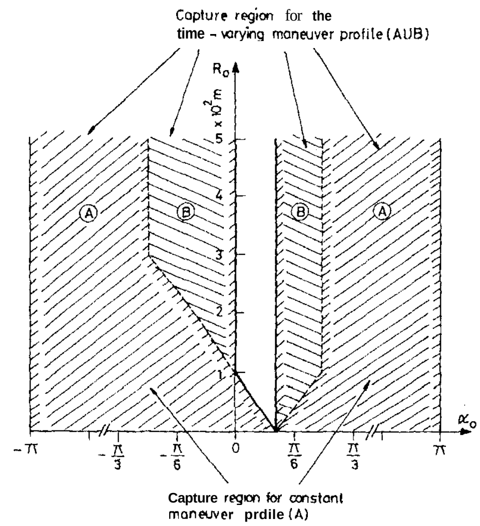

Fig. **9.** Guaranteed capture region for time-varying target maneuver.

i.e.,

$$|500\sin\alpha_0 + 350\sin(210°)| > 1.767 R_0, \qquad \text{(Y0} \in N_\alpha^+(\mathcal{A}).$$

Fig. **9** shows the complete capture region for the target maneuver profile $\mathcal{A}$. Note that the capture region for the constant target maneuver profile, which is shown here for comparison, is also the guaranteed capture region given by Corollary 2 irrespective of the target maneuver profile as long as it is bounded by $A_{T\max}$.

IV. LATERAL ACCELERATION BOUNDS

Performance of a guided missile is also limited by its lateral acceleration capability. The upper bound on the latax is mainly due to structural reasons. **It** would be useful to obtain conditions under which the missile latax remains within specified bounds. This is of special relevance here since it was shown in the previous section that for any bounded piecewise continuous target maneuver profile the missile can capture the target if the initial state lies outside the

S_θ^+ sector. No restriction on the actual value of $A_{T\max}$ was put. In practice, the maximum missile latax during an engagement does depend on $A_{T\max}$. Hence, results on missile latax bounds as functions of $A_{T\max}$ will serve as useful measure of the performance of a guidance law. In Guelman [8] upper bounds on the required latax was analytically obtained for certain initial conditions and for a target executing a constant maneuver. Below we present extensions of these results for a target executing bounded piecewise continuous time-varying maneuvers.

A. Bounds on Missile Latax in $N_\theta^-(\mathcal{A})$

THEOREM 3 *Given a bounded piecewise continuous target maneuver profile $\mathcal{A}$, if $\theta_0 \in N_\theta^-(\mathcal{A})$ and*

a) $V_M > \sqrt{2}V_T$ (*i.e.,* $\nu > \sqrt{2}$)
b) $N > 2 + 2V_T/\sqrt{V_M^2 - V_T^2}$ (*i.e.,* $V_e > 0$)

then the missile acceleration A_M is bounded by A_{M_1} in the sense that:

1) *If $|A_{M_0}| > A_{M_1}$ then $|A_M|$ will decrease until $|A_M| \le A_{M_1}$*
2) *If $|A_{M_0}| \le A_{M_1}$ then $|A_M| \le A_{M_1}$ where,*

$$A_{M_1} = \left(\frac{NV_M}{V_e}\right) A_{T\max}. \tag{42}$$

PROOF According to Theorem 1, if $\theta_0 \in N_\theta^-(\mathcal{A})$ then $\theta \in N_\theta^-(\mathcal{A})$ until interception. From Lemma 5, we have for $\theta \in N_\theta^-(\mathcal{A})$,

$$V_S \le -V_e < 0.$$

Since $V_S < 0$, we have,

$$-\frac{A_{T\max}}{V_e} \le -\frac{|A_T(t)|}{V_e} \le \frac{|A_T(t)|}{V_S} \le \frac{A_T(t)\cos(\theta-\beta)}{V_S} \le -\frac{|A_T(t)|}{V_S} \le \frac{|A_T(t)|}{V_e} \le \frac{A_{T\max}}{V_e}. \tag{43}$$

Now if $|A_{M_0}| > A_{M_1}$ then

$$\dot\theta_0 > \frac{A_{T\max}}{V_e} \qquad \text{or} \qquad \dot\theta_0 < -\frac{A_{T\max}}{V_e}$$

If $\dot\theta_0 > A_{T\max}/V_e$ then from (43) we have $\dot\theta_0 > A_{T0}\cos(\theta-\beta)/V_S$ and so, since $V_S < 0$, from (31) we obtain the result that $\dot\theta$ is a decreasing function of time. Similarly, if $\dot\theta_0 < -A_{T\max}/V_e$ then from (43), $\dot\theta_0 < A_{T0}\cos(\theta-\beta)/V_S$, and hence from (31) we see that $\dot\theta$ is an increasing function of time. This is sufficient to prove that $|A_M|$ will decrease until $|A_M| < A_{M_1}$. On the other hand, if $|A_{M_0}| < A_{M_1}$ then $|A_M| < A_{M_1}$ for all time.

We can obtain even tighter bounds on A_M if we assume that the entire pursuit is restricted to a tail chase which would imply that in addition to $\theta_0 \in N_\theta^-(\mathcal{A})$ we would also require that $\pi/2 \le \theta - \beta \le 3\pi/2$, i.e., $\cos(\theta-\beta) < 0$.

THEOREM 4 *For a missile using a PPN guidance law and pursuing a target maneuvering with a bounded piecewise continuous target maneuver profile $\mathcal{A}$, if $\theta_0 \in N_\theta^-(\mathcal{A})$, $\nu > \sqrt{2}$, $k\nu > 1$, and the entire chase is restricted to the rear of the target then the missile latax is bounded by the following conditions:*

a) *If $|A_{M_0}| > \nu A_{T\max}$, then $|A_M|$ decreases until $|A_M| \le \nu A_{T\max}$ is satisjied.*
b) *If $|A_{M_0}| \le \nu A_{T\max}$, then $|A_M|$ continues to satisfy this limit.*

PROOF Since $\theta_0 \in N_\theta^-(\mathcal{A})$, $\theta \in N_\theta^-(\mathcal{A})$ for all time t, and hence $V_r < 0$ for all time t, i.e.,

$$V_M\cos(k\theta - \phi_0) < V_T\cos(\theta-\beta) \qquad \text{for all } t. \tag{44}$$

From (32) and (44), we get,

$$V_S < NV_T\cos(\theta-\beta) \qquad \text{for all } t. \tag{45}$$

Now for a tail chase, we have $\cos(\theta-\beta) < 0$ and hence, $V_S < 0$ and,

$$\frac{\cos(\theta-\beta)}{V_S} < \frac{1}{NV_T}. \tag{46}$$

Therefore,

$$-\frac{A_{T\max}}{NV_T} < \frac{A_T(t)\cos(\theta-\beta)}{V_S} < \frac{A_{T\max}}{NV_T}. \tag{47}$$

If $|A_{M_0}| > \nu A_{T\max}$ then

$$\dot\theta_0 > \frac{A_{T\max}}{NV_T} \qquad \text{or} \qquad \dot\theta_0 < \frac{A_{T\max}}{NV_T}$$

If $\dot\theta_0 > A_{T\max}/(NV_T)$ then from (47) we have $\dot\theta_0 > A_{T_0}\cos(\theta-\beta)/V_S$. Since $V_S < 0$, from (31) we obtain the result that $\dot\theta$ is a decreasing function of time. On the other hand, if $\dot\theta_0 < A_{T_0}\cos(\theta-\beta)/V_S$ then $\dot\theta_0$ is an increasing function of time. This is sufficient to prove that $|A_M|$ will decrease until $|A_M| < \nu A_{T\max}$. Also, if $|A_M| < \nu A_{T\max}$, then it will remain below this bound for all subsequent time.

Theorems 3 and 4 here extend the results of [8, Theorems 3 and 4] to arbitrarily time-varying target maneuvers.

B. Bounds on Missile Latax in σ_r^-

THEOREM 5 ***For*** *a missile using a PPN guidance law and pursuing a target maneuvering with any piecewise continuous maneuver profile bounded by $A_{T\max}$, if*

a) $\theta_0 \in \sigma_r^-$
b) $N > 4$
c) $V_M > \sqrt{2}V_T$ (*i.e.,* $\nu > \sqrt{2}$)

and the initial missile latax is such that

1) *if $|A_{M_0}| > A_{M_2}$, then $|A_M|$ will decrease until $|A_M| \le A_{M_2}$*

2) *if* $|A_{M_0}| \leq A_{M_2}$, *then* $|A_M| \leq A_{M_2}$ *for all time t until interception where,*

$$A_{M_2} = \left(\frac{N}{N-4}\right) \nu A_{T\max}. \tag{48}$$

PROOF It is easy to see that $\theta_0 \in \sigma_r^-$ implies that $\theta \in \sigma_r^-$ for all time until interception, i.e., the missile always moves towards the target and $V_r < 0$ for all time. Therefore, we have,

$$\cos(k\theta - \phi_0) < -\frac{1}{\nu}. \tag{49}$$

Since $\cos(\theta - \beta) \leq 1$ always, from (31) we get,

$$V_S \leq -(N-4)V_T. \tag{50}$$

Thus, if $N > 4$ then $V_S < 0$, and so

$$\frac{A_{T\max}}{-(N-4)V_T} \leq \frac{A_{T\max}}{V_S} \leq \frac{A_T\cos(\theta - \beta)}{V_S}$$

$$\leq -\frac{A_{T\max}}{V_S} \leq \frac{A_{T\max}}{(N-4)V_T}. \tag{51}$$

By the same arguments as above $|A_M|$ decreases and remains below the defined threshold.

Theorem 5 is a generalization of [9, Theorem 1], since it proves that the limits proposed here hold not only for the constant maneuver profile, but also for all bounded piecewise continuous target maneuvers.

V. CONCLUDING REMARKS

In this paper, the qualitative analysis approach adopted by Guelman [8], for analyzing capture performance of the PPN law against constantly maneuvering target, is extended to targets having bounded piecewise continuous time-varying maneuver profiles. Sufficient conditions for capture are obtained for any given target maneuver profile. Sufficient conditions for capture against worst case target maneuver profiles are also obtained. The analysis shows that, in the absence of closed-form solution for the trajectory equations, the qualitative analysis approach is the ideal choice for obtaining satisfactory solutions to such problems. It provides superior results to the earlier proposed Liapunov technique in the sense that almost the whole of the guaranteed capture region can be obtained while the conditions in [11] can be used to obtain only a very small portion of the capture region.

The qualitative analysis approach given here also provides additional results on missile latax bounds for time-varying target maneuvers. These results are generalizations and extensions of earlier results for constant target maneuvers. The Liapunov function technique does not provide these results.

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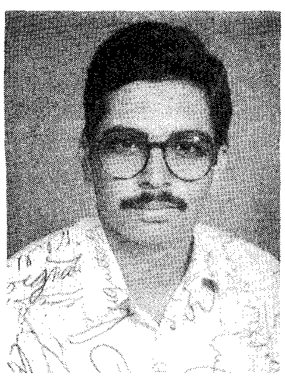

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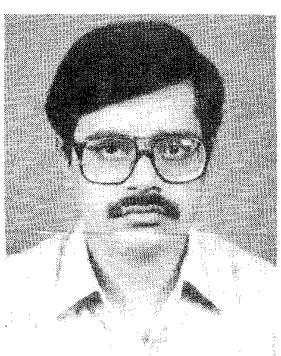

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