Technical Notes and Correspondence

An Improved Stability Criterion for the Damped Mathieu Equation

Abstract—An improved stability criterion is derived for the damped Mathieu equation using periodic Lyapunov functions.

Considering the damped Mathieu equation

$$\ddot{x} + 0.5\delta\dot{x} + (\omega^2 + \epsilon \cos 2t)x = 0 \quad (1)$$

(with $\delta > 0$) Michael [1] has shown that for large ω and small ϵ , stability is assured if $|\epsilon| < 6$. In this correspondence, an improved stability criterion is obtained using periodic Lyapunov functions.

Putting $x = x_1$ and $\dot{x} = x_2$, (1) can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -0.5\delta x_2 - (\omega^2 + \epsilon \cos 2t)x_1. \end{aligned} \quad (2)$$

Consider the Lyapunov function

$$V_1 = (\omega^2 + 0.5k\delta + k\epsilon \sin 2t)x_1^2 + 2kx_1x_2 + x_2^2 \quad (3)$$

where k is to be chosen. Differentiating (3) with respect to time and using (2),

$$\dot{V}_1 = -2\omega^2 k x_1^2 - 2\epsilon(\cos 2t - k \sin 2t)x_1x_2 - (6 - 2k)x_2^2.$$

For $\dot{V}_1$ to be negative definite, we require

$$0 < 2k < \delta \quad (4)$$

and

$$\epsilon^2 < \frac{2\omega^2 k(\delta - 2k)}{(1 + k^2)} \quad (5)$$

Now choose the value of k as

$$k = \frac{1/\delta^2 + 4 - 2}{6} \quad (6)$$

which maximizes the right-hand side of (5) and also satisfies (4). It can be easily proved that, for the above value of k , the function V_1 is positive definite. Substitution of the value of k given in (6) into (5) yields the stability criterion as

$$|\epsilon| < [\sqrt{\delta^2 + 4} - 2]^{1/2}\omega. \quad (7)$$

Now consider another Lyapunov function given by

$$V_2 = (\omega^2 + \epsilon \cos 2t)x_1^2 + (0.5\delta x_1 + x_2)^2. \quad (8)$$

Differentiating (8) and using (2),

$$\dot{V}_2 = -[\delta\omega^2 + \delta\epsilon \cos 2t + 2\epsilon \sin 2t]x_1^2.$$

Here we see that $\dot{V}_2$ can only be negative semidefinite since $\dot{V}_2 \equiv 0$

Fig. 1. Stability region for $\delta = 0.5$.

when $x_1 \equiv 0$. It is still possible to prove asymptotic stability in the large (MIL) of the system (2) by using the theorem of Krasovskii and Barbashin [2] for periodic Lyapunov functions if we can prove that $\dot{V}_2 \equiv 0$ only when $x_1 \equiv x_2 \equiv 0$. For the system (2), it can be easily seen that when $x_1 \equiv 0$, x_2 is also identically zero. Hence it is sufficient if $\dot{V}_2$ is negative semidefinite which requires that

$$|\epsilon| < \frac{\delta \omega^2}{\sqrt{\delta^2 + 4}} \quad (9)$$

Combining (7) and (9), the stability criterion can be written as

$$|\epsilon| < \max \left[\left\{ \sqrt{\delta^2 + 4} - 2 \right\}^{1/2} \omega, \frac{\delta \omega^2}{\sqrt{\delta^2 + 4}} \right] \quad (10)$$

The stability region using (10) for $\delta = 0.5$ is shown in Fig. 1. It may be noted that the stability region obtained by Michael [1] is contained in the stability region obtained here.

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