STRONG GRAVITY AND SO(3) GAUGE FIELD

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ABSTRACT

Strong gravity field coupled to SO(3) gauge field is investigated for a conformally flat space. This leads to decoupling of the two fields. A Yukawa like potential for the mass modified field equation is found.

1. INTRODUCTION

RECENT investigations on strong spin–two interaction (strong gravity)[1–5] suggest strong correlations with the physics of elementary particles, in particular hadrons. It is found that the field equations of strong gravity are similar to those given by Einstein for weak Newtonian gravity which has the virtue of being a gauge theory[3,5]. The modification of the Einstein type equation in the context of strong gravity arises from the presence of a mass term introduced directly or through the strong cosmological term Λ_f [$\sim (m_f c/\hbar)^2$, m_f being the mass of spin two f-mesons]. It should be noted that Λ_f is also related to the strong gravity coupling constant as[5]

$$\Lambda_f = \frac{8\pi\, G_f\, \rho_h}{c^2}, \tag{1}$$

where ρ_h is the hadronic density, $10^{17}\,\mathrm{g\,cm^{-3}}$.

While the solutions of the mass modified free field equations[6] have given interesting insight into the space-time structure and underlying symmetry and quantum numbers (external and internal)[5] in regions where such strong fields interact, it is worthwhile studying its interaction and coupling with other gauge fields. It is expected that solutions of strong gravity field equations coupled to SO(3) gauge fields will not only give an effective potential but also other solutions which may simulate particle-like behaviour. In fact, Krive and Sitenko[7] have made a beginning by investigating, the two-tensor (f–g) theory of Isham *et al.*[1] coupled to SO(3) gauge field. The asymptotic behaviour of the classical solutions given by them behaves like constant times r^2 in the region r tending to zero. However, for massive f-quanta involved in the strong gravity equations, Yukawa-like solutions are more relevant in the context of hadron physics.

In the present series of papers we intend to investigate the problem for various kinds of metrics for example, conformally flat metric, de Sitter metric, Gödel metric and a few others which we believe to be appropriate for hadrons.

2. LAGRANGIAN AND FIELD EQUATIONS

In what follows we consider the situation for a conformally flat space. The modified action for the f–g theory coupled to SO(3) gauge can be written as

$$I = \int [\mathcal{L}(g) + \mathcal{L}(f) + \mathcal{L}(fg) + \mathcal{L}(fW)]\, d^4x \tag{2}$$

where

$$\mathcal{L}(g) = \frac{1}{K_g^2}(-g)^{\frac{1}{2}}\, R_g$$

is the usual Einstein Lagrangian density for the weak gravity, R_g being the corresponding curvature scalar,

$$\mathcal{L}(f) = \frac{1}{K_f^2}(-f)^{\frac{1}{2}}\, R_f, \tag{3}$$

is the Lagrangian density for the strong (f) gravity with metric $f^{\mu\nu}$ and curvature scalar R_f; K_g and K_f are the constants signifying the coupling constants of weak (g) and strong (f) gravities. The f–g mixing term is chosen such that the combination of the two tensors describes a massive particle[1,6] thus

$$\mathcal{L}(fg) = -\frac{m_f^2}{4k_f^2}(-f)^{\frac{1}{2}}(f^{\mu\nu} - g^{\mu\nu})(f^{\rho\sigma} - g^{\rho\sigma}) \times (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma}), \tag{4}$$

with $f^{\mu\nu}$, $g^{\mu\nu}$ being the symmetric tensor metric fields and m_f the mass of the f-meson in units of reciprocal length.

Finally[7,8]

$$\mathcal{L}(fw) = \tfrac{1}{4}(-f)^{\frac{1}{2}} f^{\mu\nu} f^{\rho\sigma}\, F^a_{\mu\rho}\, F^a_{\nu\sigma}, \tag{5}$$

where

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + q\epsilon^{abc}\, W^b_\mu\, W^c_\nu, \tag{6}$$

where q = gauge charge.

In the above set of equations, the Greek letters denote 0, 1, 2, 3 and Roman letters 1, 2, 3. Further $F^a_{\mu\nu}$ represent a triplet of vector fields. Thus $\mathcal{L}(fW)$ as given above describes the coupling of the f-field with the SO(3) gauge field.

As done by earlier authors, we make use of the Wu–Yang ansatz[9] to write W^a_μ. Thus for a static spherically symmetric situation, we have

$$W^a_j = \epsilon^{jal}\, x^l\, \frac{Y(r)}{er^2}, \tag{7}$$

with the condition $W^a_0 = 0$

and $W^a_{j,0} = 0$, $\epsilon^{ja\gamma}$ being the totally antisymmetric tensor.

Now for the g-field we take the Minkowski metric $g^{\mu\nu} = \eta^{\mu\nu}$ with $\eta^{\mu\nu} = (1, -1, -1, -1)$. For the f-field we choose the conformally flat metric given by,

$$f^{\mu\nu} = e^{2\lambda}\, \eta^{\mu\nu} \tag{8}$$

λ being the conformal factor. The present simplified choice amounts to taking only one potential in the context of strong gravity rather than the ten of Einsteins theory of weak gravity.

Let us consider the problem in the lowest order of the quotient

$$\frac{(K_g)}{K_f} \sim \frac{G_N}{G_f}, \quad \text{since } G_f \sim 10^{38}\, G_N{}^5$$

We can neglect the effect of weak gravity in the prsent calculation.

Thus with these simplifications the Lagrangian density turns out to be

$$\mathcal{L} = \frac{1}{K_f^2}[-6r^2 e^{-2\lambda} \lambda'^2 - r^2 m_f^2 e^{-4\lambda} \times (-3e^{4\lambda} + 6e^{2\lambda} - 3) + \frac{K_f^2}{2q^2 r^2} \times \left[\frac{1}{2} Y^2 (2 + Y)^2 + r^2 Y'^2 \right]. \tag{9}$$

The energy functional for the system takes the form

$$E = \int \mathcal{L}\, d^3 x = \frac{4\pi}{K_f^2} \int \left[-6r^2 e^{-2\lambda} \lambda'^2 - r^2 m_f^2 e^{-4\lambda} - (3e^{4\lambda} + 6e^{2\lambda} - 3) + \frac{K_f^2}{2q^2 r^2} \left\{ \frac{1}{2} Y^2 (Y + 2)^2 + r^2 Y'^2 \right\} \right] dr. \tag{10}$$

The variation of E with respect to λ and Y gives the usual Euler-Lagrange equations. These are

$$r^2 \lambda'' + 2r\lambda' - r^2 \lambda'^2 - m_f^2 r^2 (e^{-2\lambda} - 1) = 0 \tag{11}$$

for λ and

$$r^2 Y'' = Y(Y + 1)\ (Y + 2) \tag{12}$$

for Y.

3. Solutions of Mass Modified Field Equations

It should be noted that owing to our choice of conformally flat space-time the differential equations for λ and Y get decoupled. That a similar situation happens in the asymptotically flat space-time limit has also been noted by others (Aragone and Chela-Flores[10], Raman[11]). It is nevertheless instructive to find the solutions of these equations in that they will give the behaviour of the strong gravity field in this limit.

Consider the differential equation for the gravitation potential, *i.e.*, involving λ [*cf.* eqn. (11)], using the transformation $\psi = re^{-\lambda}$, the equation takes the form

$$\psi'' - m_f^2\, \psi = -m_f^2 \frac{\psi^3}{r^2}. \tag{13}$$

For mass term $m_f = 0$, it has the simple form $\psi'' = 0$, whose solution can be taken as

$$\psi = r + a, \tag{14}$$

giving

$$e^{-\lambda} = 1 + a/r \ (15), \tag{15}$$

which looks like the Schwarzschild's spherically symmetric solution. In the asymptotic limit, $e^{-\lambda} \sim 1$, (flat space time, *i.e.*, $\lambda \to 0$) which shows that $\psi \to r$ in this limit.

Let us consider the situation with $m_f \neq 0$. On ignoring the non-linear term on the right-hand side *i.e.*, $-m_f^2 (\psi^3/r^2)$ of eqn. (13), we get the simple and soluble form

$$\psi'' - m_f^2\, \psi = 0, \tag{16}$$

with solution $\psi = c_1 e^{m_f r} + c_2 e^{-m_f r}$. (17)

For the boundary condition $\psi(\infty) \to 0$, we have to choose the second term, *i.e.*, $c_2 e^{-m_f r}$. Now the full non-linear form given in eqn. (13), is extremely difficult to solve. However we can get a solution which gives the right asymptotic flat space-time behaviour, *i.e.*, $\psi \to r$ or $e^{-\lambda} \to 1$. Thus we try to solve eqn. (13) in the following approximate form, *i.e.*,

$$\psi'' - m_f^2\, \psi = -m_f^2\, r. \tag{18}$$

Now the Green's function for the equation $\psi'' - m_f^2 \psi = 0$ with boundary conditions, $\psi(0)$ is finite, $\psi(\infty) = 0$ has the form

$$G(r, s) = \left.\begin{aligned} &\frac{-1}{2m_f} e^{-m_f |r-s|}, && 0 \leqslant r < s \\ &\frac{-1}{2m_f} e^{-m_f |s-r|}, && s < r < \infty \end{aligned}\right\} \tag{19}$$

Thus, we get with eqn. (19).

$$\psi(r) = -m_f^2 \int_0^\infty G(r, s)\, s\, ds = \frac{e^{-m_f r}}{2m_f} + r \tag{20}$$

Alternatively, from the general solution $\psi(r) = r + c_1 e^{m_f r} + c_2 e^{-m_f r}$ of eqn. (18), the physically acceptable solution for the present purpose is

$$\psi_f(r) \approx r + a_f \frac{e^{-m_f r}}{m_f} \tag{21}$$

or

$$e^{-\lambda_f} \approx 1 + \frac{a_f e^{-m_f r}}{m_f r}, \qquad (22)$$

where

$$a_f = (K_f M\, m_f) = \text{constant}.$$

It should be noted that the solution (21) does satisfy eqn. (18) and has the right asymptotic limit and gives $e^{-\lambda_f} \to 1$ for $r \to \infty$, flat space-time limit. The important point to note is that the second term of eqn. (22) has Yukawa-like behaviour. Thus the metric for the present situation can be written as

$$ds^2 = \left(1 + \frac{a_f e^{-m_f r}}{m_f r}\right)^2 (C^2\, dt^2 - dr^2 - r^2\, d\Omega^2). \qquad (23)$$

Thus, we have

$$e^{-2\lambda} \approx \left(1 + \frac{2a_f e^{-m_f r}}{m_f r} + \frac{a_f\, e^{-2m_f r}}{(M_f r)^2}\right) \qquad (24)$$

and the gravitation potential for the strong f-gravity (in analogy with Newtonian case) can be written as

$$\frac{1}{2}(f_{00} - 1) = \frac{1}{2}(e^{-2\lambda} - 1) \approx \frac{a_f e^{-m_f r}}{m_f r} \qquad (25)$$

which has exactly the Yukawa form mediated by a massive boson. This is to be expected in the present model for strong gravity having mass modified field equations.

This should be contrasted with the result of Krive and Sitenko who did not find this kind of behaviour. It may be argued that we have taken only one potential in the conformally flat space-time instead of the usual ten of Einstein's theory. However, in the context of strong (f) gravity one needs to know the behaviour of the potential and there is no necessity of exploring the role of all the ten tensor components which will make the calculation extremely complicated.

Let us now consider the solution of eqn. (12) which in the present situation (*i.e.*, decoupled from the strong gravity field) corresponds to the pure Yang-Mills field. The significance of the solutions $Y = 0$, $Y = -2$, and $Y = -1$, has already been discussed in reference 7. We should only like to mention that the linearsed form

$$r^2 Y'' = 2Y$$

has the general solution $Y = Ar^2 + B/r$.

A and B being constants. Thus

$$W(r) = \frac{1}{qr^2}\, Y(r) = \frac{A}{q} + \frac{B}{qr^3},$$

It should be noted that this general solution of the linearised equation has the correct asymptotic behaviour obtained by Krive and Sitenko, namely, $W(r) \to$ constant for $r \to 0$ and $W(r)\ ar^{-3}$ for $r \to \infty$.

Thus we find that the study of strong gravity field for the conformally flat space give the essential qualitative features of the field, *i.e.*, Yukawa-like potential of f-gravity and particle like solutions of the gauge field.

It is believed that a study of a more general metric (*e.g.*, de Sitter or Gödel) coupled to SO(3) fields will yield results which are of direct importance to hadrons.

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