yields a close approximation to the complex roots and one of the real roots. If one applies Lin's method to remove a real root from (9), convergence to the factor $(s+1)$ is obtained in three trials and the polynomial is reduced by an order. Reapplying Lin's method to obtain the complex roots (or the remaining real root), the complex roots are approximated by the third or fourth trial divisor.

The described extension enhances the value of the basic technique by indicating the regions of the roots from crude sketches of the root-loci. Increased accuracy may be obtained by any of three methods:

1) Both loci may be plotted carefully in the regions of intersection.
2) One of the loci may be plotted accurately in the regions of intersections and the points corresponding to K located on the locus.
3) The approximate intersection obtained from the crude sketch may be used as a first trial factor in another technique such as Lin's or Newton's method.

John D. Glomb
Central Res. Engrg. Div.
Continental Can Co.
Chicago, Ill.

Determination of Transfer Function Coefficients of a Linear Dynamic System from Frequency Response Characteristics*

Determination of transfer functions in standard form, from experimentally obtained frequency response characteristics, is a classical problem of extraordinary interest to designers of servo-systems. Levy[1] has recently presented a very versatile method which facilitates exact synthesis of frequency dependent systems. The method can be used to fix up an algebraic expression as the ratio of two polynomials, which could later be solved, with the aid of an Isograph,[2] to give the zeroes and poles of the system under study.

However, as pointed out by Levy himself, the utility of his method is limited by two major restrictions. The object of this correspondence is to suggest means of overcoming these restrictions in a very direct and elegant manner. The first restriction according to Levy is that systems with infinite gain at zero frequency cannot be handled by his method except by an indirect approach. These sytems are the familiar type I, type II, and type III Servo-systems. It is interesting to observe that this restriction is totally removed if one uses M-curves instead of G-curves. G and M refer to the open and closed-loop responses in the frequency domain, as per standard notation. M-curves can be easily obtained from the G-curves with the aid of a Nichols Chart. The use of M-curves, in preference to G-curves, results in several additional advantages. Firstly, in the case of unity feedback systems the denominator polynomial of the M-curve directly gives the characteristic equation of the system, which is useful for determining quantitatively the performance characteristic of the system in the time domain. Secondly, computation with M-curves is much easier because of the smooth nature of these curves. In any event, if one wishes to get the transfer function for G, it can be obtained as follows.

Let

$$M(j\omega) = \frac{f_1(j\omega)}{f_2(j\omega)}, \tag{1}$$

where f_1 and f_2 are polynomials in $j\omega$. As per standard notation,

$$M(j\omega) = \frac{G(j\omega)}{1 + G(j\omega)}. \tag{2}$$

From (1) and (2),

$$G(j\omega) = \frac{f_1(j\omega)}{f_2(j\omega) - f_1(j\omega)}. \tag{3}$$

The necessity for using M-curves in case of type I, type II or type III systems can be illustrated by a specific example.

Let

$$G(j\omega) = f(j\omega)\cdot\frac{1}{(j\omega)^n}, \quad \text{for } n = 1, 2, 3, \cdots, \tag{4}$$

where $f(j\omega)$ is of type

$$\frac{A_0 + A_1(j\omega) + A_2(j\omega)^2 + \cdots}{1 + B_1(j\omega) + B_2(j\omega)^2 + \cdots}. \tag{5}$$

Eq. (4) obviously gives infinite gain at zero frequency. Considering the corresponding M-function, one gets from (2) and (4),

$$M(j\omega) = \frac{f(j\omega)}{(j\omega)^n + f(j\omega)}. \tag{6}$$

Eq. (6) gives a finite gain even at zero frequency, thus overcoming the first restriction.

The second restriction of Levy's method pertains to the weighting of the function for the error criterion. The use of the type of weighted error function suggested by Levy implies that for a given value of ω, the magnitude of the error is directly proportional to the magnitude of the function. In order to minimize the errors in computation, Levy sugggests selecting a greater number of sampling points in the critical region of the curve. A more direct means of overcoming this restriction is by working in terms of the inverse plots. By working with the inverse function, the weighting is such as to make the error a minimum in the neighborhood of the local maxima of the given function.

Let

$$M(j\omega) = \frac{n(j\omega)}{d(j\omega)}, \tag{7}$$

be the required function, where $n(j\omega)$ is the numerator and $d(j\omega)$ the denominator of $M(j\omega)$. Let

$$H(j\omega) = R(\omega) + JI(\omega) \tag{8}$$

be the experimental curve, with $R(\omega)$ denoting its real part, and $I(\omega)$ its imaginary part.

Let the numerical difference between the two functions $1/M(j\omega)$ and $1/H(j\omega)$ represent the error in curve-fitting, that is

$$\delta(\omega) = \frac{1}{H(j\omega)} - \frac{d(\omega)}{n(\omega)}. \tag{9}$$

Multiplying both sides of (9) by $n(\omega)$, one gets

$$\delta(\omega)\cdot n(\omega) = \frac{n(\omega)}{H(j\omega)} - d(\omega). \tag{10}$$

The RHS of (10) is a function of real and imaginary terms, which may be separated to give

$$\delta(\omega)\cdot n(\omega) = a(\omega) + jb(\omega); \tag{11}$$

or at any specific value of frequency $\omega = \omega_K$,

$$[\delta(\omega)\cdot n(\omega)]^2_{\omega=\omega K} = a^2(\omega_K) + b^2(\omega_K). \tag{12}$$

Now, the weighted error E is defined as the function given in (9) summed over the sampling freqencies ω_K. Thus,

$$E = \sum_{K=0}^{m} [a^2(\omega_K) + b^2(\omega_K)]. \tag{13}$$

The unknown coefficients are determined on the basis of minimizing the function E. By the above derivation, it is clear that $[\delta(\omega)\cdot n(\omega)]^2$ is a minimum, say K^2, for that set of coefficients determined by the principle of least squares of the weighted eıror function. For this choice, $\delta(\omega)$ is a minimum $K/n(\omega)$, where $n(\omega)$, or the numerator of $M(j\omega)$ is a maximum. Thus, the error in fitting is a minimum in the neighborhood of local maxima of M, an aspect of obvious advantage for design purposes. It is easily seen that the suggested modifications widen the scope of Levy's method to a considerable extent without introducing any new errors.

Further, the constant term of the numerator polynomial is equal to the magnitude of the function at negligibly small values of ω. Utilizing this data reduces the order of the matrix equation by one.

Recently, Dudnikov[3] has suggested an interesting method of determining the coefficients of the polynomials from the initial portion of an experimentally obtained amplitude-phase characteristic.

The transformer analog principle[4] can be advantageously applied for developing a special purpose computer to mechanize the the evaluation of the manipulated parameters λ, S, T and U. It is proposed to use this equipment in conjunction with a Mallock's equation solver,[5] to determine straightaway the coefficients of the numerator and denominator polynomials. Work on the development and design of this special purpose analog computer is under way.

The authors wish to express their sincere thanks to Prof. H. N. Ramachandra Rao for his kind interest in this work.

P. Venkata Rao
(Mrs.) C. Lakshmi Bai
Dept. of Power Engrg.
Indian Inst. of Sci.
Bangalore, India

* Received by the PGAC, June 8, 1960.

[1] E. C. Levy, "Complex curve fitting," IRE Trans. on Automatic Control, vol. AC-4, pp. 37–43; May, 1959.

[2] P. Venkata Rao, "A novel type of isograph," IRE Trans. on Electronic Computers, vol. EC-7, pp. 97–103; June, 1958.

[3] E. E. Dudnikov, "Determination of transfer function coefficients of a linear system," *Automation and Remote Control* (English translation of *Avtomat. i Telemekh.*), vol. 20, pp. 552–559; May, 1959.

[4] P. Venkata Rao and G. Krishna, "The transformer analogue computer (TAC)," *Trans. AIEE*, vol. 34, pp. 732–739; January, 1958.

[5] M. W. Humphrey Davies and G. R. Slemon, "Transformer analogue network analyzer," *Proc. IEE*, vol. 100, pt. 2, pp. 469–486; 1953.