Application of fuzzy probability in water quality management of a river system

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Water quality management of a river system is addressed in a fuzzy and probabilistic framework. Two types of uncertainty, namely randomness and vagueness, are treated simultaneously in the management problem. A fuzzy-set-based definition that is a more general case of the existing crisp-set-based definition of low water quality is introduced. The event of low water quality at a check-point in the river system is considered as a fuzzy event. The risk of low water quality is then defined as the probability of the fuzzy event of low water quality. Instead of constraining the risk of violation of water quality standard by a finite value of risk through a chance constraint, a fuzzy set of low risk that considers a range of risk levels with appropriate membership values is introduced. Different goals associated with the management problem are expressed as fuzzy sets. The resulting management problem is formulated as a fuzzy multiobjective optimization problem.

1. Introduction

Water quality management of a river system is generally characterized by various types of uncertainty. One of the uncertainties that has received much attention is that due to randomness associated with various components of the river system. Two major components of the river system that give rise to randomness are the river flow and the effluent flow. These in turn make the water quality indices of interest random. The most common approach to incorporate the randomness of water quality indices in a management model is the adoption of a suitable probabilistic mathematical programming technique (see for example Lohani and Thanh (1978, 1979), Bum and McBean (1985), Fujiwara *et al.* (1986, 1987, 1988) and Ellis (1987)). Another type of uncertainty that is as prominent as randomness and is little addressed in the area of water quality management is vagueness associated with the description of the goals and the quantification of the low water quality. Incorporation of the uncertainty due to vagueness in engineering systems is achieved using the principles of

fuzzy set theory (Zadeh 1965). Quantification of vagueness is made possible using the concepts of fuzzy sets and the associated membership functions. This is analogous to the quantification of randomness using the concepts of events and the associated probability density functions (PDFs). An uncertain system may be characterized by the dominance of randomness or vagueness or both. A close look at the water quality management problem reveals that, for a river system, randomness is not the only relevant uncertainty but vagueness also may be of considerable importance. As a first instance, setting up the water quality criteria for any particular use of the water body may be painted out for the presence of vagueness. A second instance is with the assignment of permissible risk levels for the violation of water quality standards. A management model that takes into account both randomness and vagueness may be expected to be a more realistic decision-making tool for water quality management of a river system. An attempt is made in this paper to incorporate the two types of uncertainty, namely the randomness and vagueness, in a water quality management model for a river system.

The management models that take the vagueness into account belong to the category of fuzzy optimization models. Fuzzy optimization models have been success-

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fully applied in the areas of structural optimization (Rao 1987 a, b, 1993, Rao *et al.* 1992) water quality management (Hathhorn and Tung 1989, Chang *et al.* 1997, Sasikumar and Mujumdar 1998), water resources allocation (Kindler 1992), hydroelectric generation scheduling (Liang and Hsu 1994), reservoir operation (Fontane *et al.* 1997, Owen *et al.* 1997), and solid waste management (Chang and Wang 1996). Fuzzy optimization models make use of the concepts of fuzzy set theory (Zadeh 1965), fuzzy decision making (Bellman and Zadeh 1970) and fuzzy mathematical programming (Zimmermann 1978, 1985). Fuzzy optimization may be viewed as a multiobjective optimization technique in which the objectives are defined as fuzzy sets with appropriate membership functions. These membership functions play a key role in the management model. A few applications of fuzzy optimization in water quality management of a river system are briefly reviewed in the following paragraph.

Hathhorn and Tung (1989) addressed the waste load allocation process for water quality management of a river system in a fuzzy optimization framework. The two objectives considered in their study are maximization of the waste discharge and minimization of the largest difference in an equity measure between the various dischargers. These two objectives are expressed as fuzzy sets with either linear or logistic membership functions. The multiobjective problem is then transformed into a fuzzy linear programming problem with constraints due to water quality requirements, definition of largest difference in equity and the limitations on fraction removal levels. Chang *et al.* (1997) combined grey and fuzzy programming methods in an interactive multiobjective framework for water pollution control in a river basin. In their study, the uncertainty due to the environmental and economic parameters is modelled using interval numbers, and that due to the imprecise objectives identified by decision makers is modelled using fuzzy membership functions. The resulting interactive fuzzy interval multiobjective mixed integer programming approach is used to evaluate optimal wastewater treatment strategies for pollution control in a river basin. Sasikumar and Mujumdar (1998) developed a fuzzy waste load allocation model for water quality management of a river system in a deterministic framework. The model incorporates the conflicting objectives of the pollution control agency and dischargers in the system. The vagueness associated with setting up the water quality criteria and the aspirations of the pollution control agency and dischargers are quantified using fuzzy goals with appropriate membership functions. The management problem is formulated as a fuzzy multiobjective optimization problem that can be solved using the linear programming technique. The work reported in this paper extends the model developed by

Sasikumar and Mujumdar (1998) by incorporating the uncertainty due to randomness together with the uncertainty due to vagueness.

Two important concepts for water quality management of a river system are considered in this paper. The first concept is related to viewing the low water quality as a fuzzy event. The second concept is related to the definition of fuzzy risk for the violation of water quality standards. The risk of a low water quality event is derived using the concept of probability of a fuzzy event (Zadeh 1968). A water quality management model for a river system is developed in this paper incorporating these concepts. When the water quality indices at a check-point become random variables, the associated goals of the pollution control agency are modified by incorporating the probabilistic nature of the water quality indices. The goals of the pollution control agency are defined in terms of the risk of low water quality which is a fuzzy event. The objective of achieving low risk at a check-point is then expressed as a fuzzy set with appropriate membership functions. In order to make the model general, the goals related to the deterministic water quality indices at certain check-points are also retained in the model formulation. For example, it may be possible to consider the water quality indices deterministic at some check-points due to a regulated flow from a reservoir with a fixed policy of operation. In general, the model identifies the goals of the pollution control agency and the aspirations of the dischargers as fuzzy sets. Appropriate membership functions are assigned to these fuzzy sets based on the nature of the goals and aspirations. The management problem is then formulated as a fuzzy multiobjective optimization problem. The solution of the model yields optimum fraction removal levels for the pollutants in the river system. Section 2 of the paper deals with the mathematical treatment of the concepts of low water quality and low risk. The details of the management model is presented in section 3. Section 4 illustrates the application of the model to a hypothetical river system. The river system considered for illustrating the application of the management model is relatively simple for which the well-known Streeter-Phelps equations (Thomann and Mueller 1987) are used as the water quality model. Such an approach of using a simple water quality model is mainly to focus more on the fuzzy probabilistic aspect of the water quality management problem. However, a problem in which sophisticated water quality models are to be used may be formulated in a fuzzy probabilistic framework similar to that proposed in this paper. Sections 5 and 6 give the results and discussion respectively, and the important conclusions drawn from the performance of the model.

2. Fuzzy description of low water quality

2.1. Description of the river system

A general river system is used for describing the details of the fuzzy waste load allocation model. The river system consists of a set of dischargers who are allowed to release the pollutants into the river after removing some fraction of the pollutants. These fraction removal levels of the pollutants are necessary in order to maintain the acceptable water quality condition in the river as prescribed by the pollution control agency. A common practice of the pollution control agency to ensure the acceptable water quality condition is to check the water quality at a finite number of locations in the river. These locations are called water quality check-points or simply check-points. The water quality at a check-point is described by means of some indicators called water quality indices. The dissolved oxygen (DO deficit) is an example of a water quality index. The water quality indices may be of two kinds:

- (1) indices whose desirable levels in the river are greater than the permissible levels (e.g. DO);
- (2) indices whose desirable levels are less than the permissible levels (e.g. DO deficit and toxic pollutant concentration).

Most of the water quality indices are of kind (2). However, for the sake of generality, both kinds of water quality index are considered in the present study. The level of a water quality index at a check-point is affected by the controllable as well as the uncontrollable sources of pollutants in the river system. The level of the water quality index at a check-point is obtained using a water quality model that determines the spatial distribution of the pollutants in the river system. In a water quality management model, the level of the water quality index is expressed in terms of the fraction removal levels for the pollutants released by the dischargers into the river system. These fraction removal levels form the set of decision variables in the optimization problem. The solution of the optimization problem gives the optimal fraction removal levels according to the objective function and constraints considered in the problem formulation.

Table I gives the description of a river system to which the model can be applied for water quality management. The relevant components of the system are identified as sets. Set Q_L represents the collection of water quality checkpoints in the river system. Set D_M is the collection of dischargers (e.g. industries). Set P_M is the collection of the pollutants in the system (e.g. point sources of biochemical oxygen demand (BOD), and a mixture of toxic substances). Set T_P is the collection of uncontrollable sources of pollutants in the system (e.g.

BOD addition due to run-off and scour in a stream). Set V_i is the collection of water quality indices whose desirable levels are greater than the permissible levels for acceptable water quality (e.g. DO concentration). Set S_j is the collection of water quality indices whose desirable levels are less than the permissible levels for acceptable water quality (e.g. DO deficit and toxic pollutant concentration). The elements in the sets V_i and S_j are denoted by i and j respectively. A pollutant is assumed to affect one or more than one water quality index in the sets V_i or S_j or both. Note that no water quality index is common to sets V_i and S_j . Concentration levels of the water quality indices i and j at the check-point l are denoted as c_{il} and c_{jl} respectively. The pollution control agency sets a desirable level c_{il}^D and a minimum permissible level c_{il}^L for the water quality index i at the check-point l ($c_{il}^D > c_{il}^L$). Similarly, c_{jl}^D and c_{jl}^H represent the desirable and maximum permissible levels respectively of the water quality index j at the check-point l ($c_{jl}^D < c_{jl}^H$).

A common index w is used for denoting the water quality indices i as well as j whenever no distinction is necessary between i and j . This common index w is used in order to avoid the repetition of formulae and other statements in which no distinction exists between i and j . The equivalence of w to i and j is indicated by the notation ' $w \Rightarrow i, j$ '. Whenever a discussion is relevant to any one of the indices i or j , that index is explicitly stated.

2.2. Low water quality as a fuzzy event

The concepts related to the water quality and the associated risk are explained with reference to the water quality indices i and j (table I) whose concentrations at a check-point l in the river system are denoted as c_{il} and c_{jl} . It may be noted that c_{il} denotes the concentration of the water quality index whose desirable level at the check-point l is greater than the permissible level (e.g. DO concentration). Similarly, c_{jl} corresponds to the water quality index whose desirable level is less than the permissible level (e.g. DO deficit and concentration of toxic substances). For c_{il} , the conventional water quality criterion at the check-point i is such that any concentration level of the water quality index less than a specified value, say c_{il}^L corresponds to a low water quality. In other words, a concentration level that is very close to but greater than c_{il}^L is not considered to be a low water quality. This stringent definition of low water quality may be expressed mathematically using a characteristic function of c_{il} as follows:

$$\mu(c_{il}) = \begin{cases} 1, & c_{il} < c_{il}^L, \\ 0, & c_{il} \geq c_{il}^L. \end{cases} \quad (1)$$

Table 1. Description of the river system.

Set	Description of the set	Element representation	Number of elements
Q_L	Water quality check-points	l	N_q
D_M	Dischargers	m	N_d
P_N	Pollutants	n	5
T_P	Uncontrollable sources of pollutants	p	N_t
V_i	Water quality indices: desirable level > permissible level	i	N_v
S_j	Water quality indices: desirable level < permissible level	j	N_s

Similarly, for c_{jl} , the conventional water quality criterion is such that any concentration level of the water quality index greater than a specified value, say, c_{jl}^H , corresponds to a low water quality. In other words, a concentration level that is very close to but less than c_{jl}^H is not considered to be a low water quality. This definition may be expressed mathematically using a characteristic function of c_{jl} as follows:

$$\mu(c_{jl}) = \begin{cases} 0, & c_{jl} \leq c_{jl}^H, \\ 1, & c_{jl} > c_{jl}^H. \end{cases} \quad (2)$$

The crisp set L_{wl} of Concentration levels that belong to low water quality at check-point l is defined as

$$L_{wl} = \{c_{wl} : \mu(c_{wl}) = 1\} \quad w \Rightarrow i, j. \quad (3)$$

Graphically the characteristic functions corresponding to L_{ij} and L_{jl} can be represented as Step functions as shown in figures 1 and 2 respectively. These definitions of low water quality lead to the following deterministic constraints in the management model:

$$c_{il} \geq c_{il}^L \quad \forall i, l, \quad (4)$$

$$c_{jl} \leq c_{jl}^H \quad \forall j, l. \quad (5)$$

When the water quality indices are random variables, chance constraints are introduced in place of the deterministic constraints. The random variables corresponding to c_{il} and c_{jl} are denoted as C_{il} and C_{jl}

Figure 1. Characteristic function for L_{il} . Figure 2. Characteristic function for L_{jl} .

respectively. The chance constraint corresponding to c_{il} ensures that the risk of low water quality, that is the probability that the concentration level is less than c_{il}^L , is constrained by a specified upper limit γ_{il} . Similarly, the chance constraint corresponding to c_{jl} ensures that the risk of low water quality, that is the probability that the concentration level is greater than c_{jl}^H , is constrained by a specified upper limit γ_{jl} . Mathematically these chance constraints are expressed as follows:

$$P(C_{il} \leq c_{il}^L) \leq \gamma_{il}, \quad (6)$$

$$P(C_{jl} \geq c_{jl}^H) \leq \gamma_{jl}. \quad (7)$$

Two observations are made with respect to these chance constraints:

- (1) Any concentration level that is greater than but very close to c_{il}^L , and that is less than but very close to c_{jl}^H is not considered as a low water quality. This leads to stringent definitions of low water quality.
- (2) Fining the upper bounds γ_{il} and γ_{jl} for the risks of low water quality is quite arbitrary. Further, any water quality situation having risks higher than γ_{il} and γ_{jl} , even by an infinitesimal amount, is excluded.

To overcome these limitations and to account for vagueness in the description of low water quality, a fuzzy definition is introduced in place of the crisp defini-

on of low water quality. The set of concentration levels corresponding to the low water quality at check-point i is defined as a fuzzy set $W_{wi}(w \Rightarrow i, f)$ instead of the crisp set L_{wi} given by (3). Each concentration level in the fuzzy set W_{wi} is assigned a membership value that lies in the closed interval $[0, 1]$. Mathematically the fuzzy set W_{wi} is expressed as follows:

$$W_{wi} = \{c_{wi} : 0 \leq \mu_{W_{wi}}(c_{wi}) \leq 1\} \quad w \Rightarrow i, f. \quad (8)$$

In the crisp definition of low water quality, each concentration level in the crisp set L_{wi} is assigned a membership value of either one or zero depending on whether that concentration level corresponds to low water quality or not. On the other hand, the fuzzy set W_{wi} allows partial membership also for the concentration levels. The membership value indicates the degree of compatibility of the concentration level with the notion of low water quality. The membership function that assigns membership values to the elements of the fuzzy set of low water quality thus modifies the conventional definition of low water quality and makes it more realistic. Figures 3 and 4 show examples of membership functions of the fuzzy sets W_{wi} and W_{ji} of low water quality. It may be seen from figure 3 that the concentration levels greater than c_{ji}^L also correspond to low water quality but to a lesser degree than those smaller than c_{ji}^L . In figure 3, c_{ji}^D corresponds to the desirable level for the

water quality index i . It may be noted that c_{ji}^L is less than c_{ji}^D . Similarly, it may be seen from figure 4 that the concentration levels less than c_{ji}^H also correspond to low water quality but to a lesser degree than those greater than c_{ji}^H . Mathematically the membership functions corresponding to W_{ji} and W_{ji} may be expressed as follows:

$$\mu_{W_{ji}}(c_{ji}) = \begin{cases} \exp\left(\frac{a_{1ji}(c_{ji})^{n_{1ji}}}{c_{ji} - c_{ji}^D}\right), & 0 \leq c_{ji} < c_{ji}^D, \\ 0, & c_{ji} \geq c_{ji}^D, \end{cases} \quad (9)$$

$$\mu_{W_{ji}}(c_{ji}) = \begin{cases} 1 - \exp\left(\frac{a_{1ji}(c_{ji})^{n_{1ji}}}{c_{ji} - c_{ji}^H}\right), & 0 \leq c_{ji} < c_{ji}^H, \\ 1, & c_{ji} \geq c_{ji}^H, \end{cases} \quad (10)$$

where a_{1ji} , a_{1ji} , n_{1ji} and n_{1ji} are parameters of the membership functions. These parameters may be assigned non-zero positive values depending on the desired shape of the membership functions. For the membership function $\mu_{W_{ji}}(c_{ji})$ shown in figure 3, the values of the parameters a_{1ji} and n_{1ji} may be so assigned that the membership value corresponding to the permissible level c_{ji}^L is approximately equal to one (i.e. $\mu_{W_{ji}}(c_{ji}^L) \approx 1$). Similarly, for the membership function $\mu_{W_{ji}}(c_{ji})$ shown in figure 4, values for a_{1ji} and n_{1ji} may be selected such that the membership value corresponding to the desirable level c_{ji}^D is approximately equal to zero (i.e. $\mu_{W_{ji}}(c_{ji}^D) \approx 0$). In general, assignment of numerical values to these parameters is governed by the permissible and desirable levels of the water quality index and the desired shape of the membership function. The membership functions discussed above are exponential type. Nevertheless, other types of membership function, such as linear or power type, may also be assigned for the fuzzy set of low water quality. For example, linear membership functions in the closed intervals $[c_{ji}^L, c_{ji}^D]$ and $[c_{ji}^D, c_{ji}^H]$ may be used for the fuzzy sets W_{ji} and W_{ji} respectively. The aspirations and requirements of the pollution control agency may be taken as the deciding factors in setting the guidelines for Selection of appropriate membership functions for the fuzzy sets of low water quality.

2.3. Fuzzy risk of low water quality

The definition of low water quality as a fuzzy event necessitates appropriate modification of the risk of low water quality given by the left-hand side of the inequalities (6) and (7). The modification of the risk of low water quality is introduced using the concept of probability of a fuzzy event (Zadeh 1968). The risk of low water quality is defined as the probability of occurrence

Figure 3. Membership function for W_{ji} .

Figure 4. Membership function for W_{ji} .

of the fuzzy event of low water quality. Mathematically this can be stated as follows:

$$\text{fuzzy risk} = \mathbf{P}(\text{fuzzy event of low water quality}) \quad (11)$$

$$= \tilde{P}(\text{low water quality}), \quad (12)$$

where $\tilde{P}$ denotes the probability of a fuzzy event. Zadeh (1968) has defined the probability of a fuzzy event A_f as follows:

$$\tilde{P}(A_f) = \int_{\mathbf{R}^n} \mu_{A_f}(y) dP, \quad (13)$$

where $\mathbf{R}^n$ is Euclidean n space, $\mu_{A_f}(y)$ is the membership function of the fuzzy event A_f and $\mathbf{P}$ is a probability measure over $\mathbf{R}^n$. A point in $\mathbf{R}^n$ is denoted by y . $\tilde{P}(A_f)$ may be rewritten using $dP = f(y) dy$ as follows:

$$\tilde{P}(A_f) = \int_{\mathbf{R}^n} \mu_{A_f}(y) f(y) dy \quad (14)$$

where $f(y)$ is the PDF of the random variable Y . For the fuzzy event of low water quality, $\mathbf{R}^n$ is a one-dimensional space ($n = 1$) of concentration levels defined by $[0, \infty]$. The upper limit ∞ of Concentration level is used here only for mathematical convenience. Physically, this upper limit will have a finite value for the concentration level. The fuzzy risk $r_{wl}(w \Rightarrow i, j)$ of low water quality can then be defined as follows:

$$r_{wl} = \int_0^{\infty} \mu_{W_{wl}}(c_{wl}) f(c_{wl}) dc_{wl} \quad w \Rightarrow i, j, \quad (15)$$

where $\mu_{W_{wl}}(c_{wl})$ is the membership function of the fuzzy set W_{wl} of low water quality and $f(c_{wl})$ is the PDF of the concentration level of water quality index w at the check-point I in the river system. Based on the PDF $f(c_{wl})$ of the water quality index w and the membership function $\mu_{W_{wl}}(c_{wl})$ of the fuzzy set W_{wl} of low water quality, direct or numerical integration may be performed to evaluate the fuzzy risk r_{wl} . The risk defined by (15) will be used in the subsequent sections of the paper to model the uncertainty. It may be noted that, when (1) and (2) are substituted successively in (15) with w replaced by i and j respectively, and the fuzzy set W_{wl} replaced by the crisp set L_{wl} , we obtain the Conventional definitions for risk of low water quality given by the left-hand sides of the inequalities (6) and (7). Thus, the crisp definition of low water quality may be considered as a particular case of the more general fuzzy-set-based definition. The definition of probability of a fuzzy event given by Zadeh (1968) acts as an effective tool in linking the uncertainty due to vagueness in the definition of low water quality and the uncertainty due to randomness of the water quality index. The concept of fuzzy risk is discussed in the following Section.

2.4. Low fuzzy risk as a fuzzy set

Instead of assigning arbitrary upper bounds for the risk of low water quality as given in the chance constraints (6) and (7), the concept of low fuzzy risk is introduced. The low fuzzy risk of low water quality at the check-point I is represented as a fuzzy set R_{wl} . Different risk levels that belong to the fuzzy set R_{wl} are assigned membership values between zero and one depending on the conformity of the risk levels with the notion of low risk. The membership function, denoted by $\mu_{R_{wl}}(r_{wl})$ of the fuzzy set of low risk is essentially a mapping from the set of risk levels $[0, 1]$ to the set of membership values $[0, 1]$. Different membership functions that are monotonically non-increasing functions of risk level can be assigned for the fuzzy set R_{wl} . Figure 5 shows a suitable membership function for the fuzzy set of low risk. Mathematically this membership function can be expressed as follows:

$$\mu_{R_{wl}}(r_{wl}) = \begin{cases} \exp\left(\frac{a_{2wl}(r_{wl})^{n_{2wl}}}{r_{wl} - r_{wl}^M}\right), & 0 \leq r_{wl} < r_{wl}^M, \\ 0, & r_{wl}^M \leq r_{wl} \leq 1 \quad w \Rightarrow i, j, \end{cases} \quad (16)$$

where a_{2wl} and n_{2wl} are parameters of the membership function and r_{wl}^M is the maximum permissible risk level. This particular exponential-type membership function is used in the model formulation, as it is very general in providing a wide choice of membership functions based on the parameters a_{2wl} and n_{2wl} . However, other forms of membership functions such as linear or power types may also be used for the fuzzy set of low risk. The parameters a_{2wl} and n_{2wl} may be assigned non-zero positive values such that the membership values of the acceptable and maximum permissible risk levels are nearly equal to one and zero respectively. The argument r_{wl} in the membership function given by (16) can be substituted from (15). Equation (15) evaluates the fuzzy risk of the event of low water quality, and the membership

Figure 5. Membership function for the fuzzy set R_{wl}

function defined by (16) assigns a membership value to this risk in the fuzzy set of low risk. Thus the two levels of fuzziness, one of low water quality and the other of low risk, have been quantified using fuzzy sets and the associated membership functions. The concepts discussed in this section are used in developing the fuzzy optimization model for water quality management of a river system.

The goals of the pollution control agency and the dischargers are transformed to fuzzy goals in the next section using fuzzy sets. The fuzzy goals are then used in formulating the fuzzy decision for the water quality management problem.

3. Fuzzy optimization model

3.1. Water quality management goals

For the sake of generality of the model, two types of check-point, depending on whether the water quality indices are deterministic or probabilistic at the check-point, are considered in the model formulation. For example, a river system may have regulated deterministic flow (e.g. flow from a reservoir with a fixed policy of operation) in some reaches and hence the water quality indices at the check-points in those reaches may be considered as deterministic. Let N_S denote the number of check-points where the water quality indices are probabilistic. The water quality indices are deterministic for the remaining $N_q - N_S$ check-points, where N_q denotes the total number of check-points in the river system (table I). That is, the water quality indices are probabilistic at the check-points 1 to N_S and are deterministic at the check-points $N_S + 1$ to N_q . The concentrations of the deterministic water quality indices are denoted as c_{il} and c_{dl} . When the water quality indices are probabilistic at a check-point l , the randomness has to be incorporated in the definition of related water quality goals of the pollution control agency.

3.1.1. Goals of the pollution control agency. These are as follows.

Goal E_{wl} Make the concentration level c_{dl} of the water quality index w ($w \Rightarrow i, j$) at the check-point l as close as possible to the desirable level so that the water quality at the check-point l is enhanced for all w and $l = N_S + 1$ to N_q .

Goal G_{wl} Make the risk r_{wl} of low water quality with respect to the water quality index w ($w \Rightarrow i, j$) at the check-point l as low as possible for all w and $l = 1$ to N_S .

3.1.2. Goal of the dischargers. This is as follows

Goal F_{wmn} Make the fraction removal level x_{wmn} as close as possible to the aspiration level x_{wmn}^L for all w ($w \Rightarrow i, j$) and n .

The quantities of interest are the concentration level c_{dl} of the water quality index, the risk r_{wl} of low water quality and the fraction removal levels x_{wmn} of the pollutants that affect the concentration levels of the water quality indices. The quantity x_{wmn} is the fraction removal level of the pollutant n from the discharger m to control the water quality index w ($w \Rightarrow i, j$). The aspiration level of the discharger m with respect to x_{wmn} is represented as x_{wmn}^L . The corresponding maximum fraction removal level acceptable to the discharger m is represented as x_{wmn}^M .

Using the fuzzy decision proposed by Bellman and Zadeh (1970), and the subsequently developed fuzzy multiobjective optimization based on the fuzzy decision (Zimmermann 1978, 1985, Feng 1983, Kindler 1992, Rao 1993, Sakawa 1995), the fuzzy decision Z for the water quality management problem may be given as follows:

$$Z = \left[\bigcap_{i,j} G_{il} \right] \cap \left[\bigcap_{j,l} G_{jl} \right] \cap \left[\bigcap_{i,l} E_{il} \right] \cap \left[\bigcap_{j,l} E_{jl} \right] \cap \left[\bigcap_{i,m,n} F_{imn} \right] \cap \left[\bigcap_{j,m,n} F_{jmn} \right]. \quad (17)$$

The membership function corresponding to the decision Z is given by

$$\mu_Z(X) = \text{minimum}_{i,j,l,m,n} [\mu_{G_{il}}(r_{il}), \mu_{G_{jl}}(r_{jl}), \mu_{E_{il}}(c_{dl}), \mu_{E_{jl}}(c_{dl}), \mu_{F_{imn}}(x_{imn}), \mu_{F_{jmn}}(x_{jmn})], \quad (18)$$

where X is the space of alternatives composed of r_{wl} , c_{dl} and x_{wmn} ($w \Rightarrow i, j$). The corresponding optimal decision X^* is given by

$$\mu_Z(X^*) + \lambda^* = \max_{X \in Z} [\mu_Z(X)]. \quad (19)$$

The details of the membership functions of the various fuzzy goals are discussed in the following section.

3.2. Membership functions for the fuzzy goals

3.2.1. Goal G_{wl} . The membership function (16) for the low risk given in section 2.4 can be considered as the membership function for the fuzzy goal G_{wl} , $l = 1$ to N_S . The pollution control agency may set acceptable and maximum permissible risk levels for the violation of water quality standard. The parameters a_{2wl} and n_{2wl} may be assigned appropriate non-zero positive values so that the membership values of the acceptable and maximum permissible risk levels are nearly equal

to one and zero respectively. The membership function for the fuzzy goal G_{wl} (with R_{wl} in (16) replaced by G_{wl}) is given as follows:

$$\mu_{G_{wl}}(r_{wl}) = \begin{cases} \exp\left(\frac{a_{2wl}(r_{wl})^{n_{2wl}}}{r_{wl} - r_{wl}^M}\right), & 0 \leq r_{wl} < r_{wl}^M, \\ 0, & r_{wl}^M \leq r_{wl} \leq 1 \quad w \Rightarrow i, j. \end{cases} \quad (20)$$

The membership function for the fuzzy goal G_{wl} may be interpreted as the variation in the satisfaction level of the pollution control agency with respect to the risk. It may be noted that a stringent specification of risk level is absent and instead, a wide range, even from zero to one, of risk levels can be incorporated using appropriate membership functions. This provides a wider perspective for uncertainty modelling. In addition to this, the risk itself is evaluated after incorporating the vagueness present in the definition of low water quality.

3.2.2. Goal E_{il} The pollution control agency sets up a desirable level c_{dil}^D and a minimum permissible level c_{dil}^L for the water quality index i at the check-point l ($l = N_S + 1$ to N_q). A parametric membership function that is nonlinear in the domain $[0, c_{dil}^D]$ may be considered for the fuzzy goal E_{il} . An example of such a membership function is shown in figure 6. Mathematically this membership function can be expressed as follows:

$$\mu_{E_{il}}(c_{dil}) = \begin{cases} 1 - \exp\left(\frac{a_{3il}(c_{dil})^{n_{3il}}}{c_{dil} - c_{dil}^D}\right), & 0 \leq c_{dil} < c_{dil}^D, \\ 0, & c_{dil} \geq c_{dil}^D, \end{cases} \quad (21)$$

where a_{3il} and n_{3il} are the parameters of the membership function. These parameters are assigned non-zero positive values such that the membership values corresponding to the desirable and minimum permissible levels of c_{dil} are nearly equal to one and zero respectively. The membership function for the fuzzy goal E_{dil}

figure 6. Membership function for the goal E_{il}

may be interpreted as the variation in satisfaction level of the pollution control agency with regard to the concentration c_{dil} of the water quality index i at the check-point l . The satisfaction of the pollution control agency decreases as the concentration level c_{dil} of the water quality index decreases from c_{dil}^D .

3.2.3. Goal E_{jl} The pollution control agency sets up a desirable level c_{djl}^D and a maximum permissible level c_{djl}^H for the water quality index j at the check-point l ($l = N_S + 1$ to N_q). A parametric membership function that is nonlinear in the domain $[0, c_{djl}^H]$ may be considered for the fuzzy goal E_{jl} . An example of such a membership function is shown in figure 7. Mathematically this membership function can be expressed as follows:

$$\mu_{E_{jl}}(c_{djl}) = \begin{cases} \exp\left(\frac{a_{3jl}(c_{djl})^{n_{3jl}}}{c_{djl} - c_{djl}^H}\right), & 0 \leq c_{djl} < c_{djl}^H, \\ 0, & c_{djl} \geq c_{djl}^H, \end{cases} \quad (22)$$

where a_{3jl} and n_{3jl} are the parameters of the membership function. These parameters are assigned non-zero positive values such that the membership values corresponding to the desirable and permissible levels of c_{djl} are nearly equal to one and zero respectively. The membership function for the fuzzy goal E_{djl} may be interpreted as the variation in satisfaction level of the pollution control agency with regard to the concentration c_{djl} of the water quality index j at the check-point l . The satisfaction of the pollution control agency increases as the concentration level c_{djl} of the water quality index decreases from c_{djl}^H .

3.2.4. Goal F_{wmn} The membership function for the fuzzy goal $F_{wmn}(w, i, j)$ is shown in figure 8. The exponent β_{wmn} appearing in the membership function is a positive real number. The fraction removal level x_{wmn}^K corresponding to the aspiration level of the discharger m with regard to x_{wmn} is assigned a membership value

Figure 7. Membership function for the goal E_{jl} .

Figure 8. Membership function for the goal F_{wmn} .

of 1. The maximum acceptable level x_M^M is assigned a membership value of zero. The membership function for the fuzzy goal F_{wmn} is expressed as

$$\mu_{F_{wmn}}(x_{wmn}) = \begin{cases} 1, & x_{wmn} \leq x_L^T \\ \left(\frac{x_M^M - x_{wmn}}{x_M^M - x_L^T} \right)^{\beta_{wmn}}, & x_L^T < x_{wmn} < x_M^M \\ 0, & x_{wmn} \geq x_M^M \end{cases} \quad (23)$$

This membership function may be interpreted as the variation of satisfaction level of the discharger m in treating the pollutant n to control the water quality index w in the river system. Using the membership functions of the fuzzy goals G_w^i , E_w^i and F_{wmn}^i ($w \Rightarrow i, j$), the water quality management model is expressed as follows:

$$\text{maximize } \lambda \quad (24)$$

subject to

$$\mu_{G_w^i}(r_j^i) \geq \lambda \quad \forall i, \quad l = 1, \dots, N_S \quad (25)$$

$$\mu_{G_w^i}(r_j^i) \geq \lambda \quad \forall j, \quad l = 1, \dots, N_S \quad (26)$$

$$\mu_{E_w^i}(c_{dl}^i) \geq \lambda \quad \forall i, \quad l = N_S + 1, \dots, N_q \quad (27)$$

$$\mu_{E_w^i}(c_{dl}^i) \geq \lambda \quad \forall j, \quad l = N_S + 1, \dots, N_q \quad (28)$$

$$\mu_{F_{wmn}^i}(x_{wmn}^i) \geq \lambda \quad \forall i, m, n \quad (29)$$

$$\mu_{F_{wmn}^i}(x_{wmn}^i) \geq \lambda \quad \forall j, m, n \quad (30)$$

$$g_h(X) \leq 0 \quad \forall h \quad (31)$$

$$0 \leq \lambda \leq 1 \quad (32)$$

where the set of constraints (31) denotes the set of crisp constraints associated with the management problem. The crisp constraints generally depend on the restric-

tions on the domains of the membership functions of the various fuzzy goals, minimum fraction removal levels imposed by the pollution control agency and other specific features of the problem. Incorporating the fuzzy goals, the fuzzy optimization model is given as follows:

$$\text{maximize } \lambda \quad (33)$$

subject to

$$\exp \left(\frac{d_{2jl}(r_j^i)^{\beta_{2jl}}}{r_j^i - r_j^M} \right) \geq \lambda \quad \forall i, \quad l = 1, \dots, N_S \quad (34)$$

$$\exp \left(\frac{d_{2jl}(r_j^i)^{\beta_{2jl}}}{r_j^i - r_j^M} \right) \geq \lambda \quad \forall j, \quad l = 1, \dots, N_S \quad (35)$$

$$1 - \exp \left(\frac{d_{3jl}(c_{dl}^i)^{\beta_{3jl}}}{c_{dl}^i - c_{dl}^D} \right) \geq \lambda \quad \forall i, \quad l = N_S + 1, \dots, N_q \quad (36)$$

$$\exp \left(\frac{d_{3jl}(c_{dl}^i)^{\beta_{3jl}}}{c_{dl}^i - c_{dl}^D} \right) \geq \lambda \quad \forall j, \quad l = N_S + 1, \dots, N_q \quad (37)$$

$$\left(\frac{x_{wmn}^M - x_{wmn}^L}{x_{wmn}^M - x_{wmn}^L} \right)^{\beta_{wmn}} \geq \lambda \quad \forall i, m, n \quad (38)$$

$$\left(\frac{x_{wmn}^M - x_{wmn}^L}{x_{wmn}^M - x_{wmn}^L} \right)^{\beta_{wmn}} \geq \lambda \quad \forall j, m, n \quad (39)$$

$$0 \leq r_j^i \leq r_j^M \quad \forall i, \quad l = 1, \dots, N_S \quad (40)$$

$$0 \leq r_j^i \leq r_j^M \quad \forall j, \quad l = 1, \dots, N_S \quad (41)$$

$$0 \leq c_{dl}^i \leq c_{dl}^D \quad \forall i, \quad l = N_S + 1, \dots, N_q \quad (42)$$

$$0 \leq c_{dl}^i \leq c_{dl}^D \quad \forall j, \quad l = N_S + 1, \dots, N_q \quad (43)$$

$$x_{wmn}^L \leq x_{wmn}^i \leq x_{wmn}^M \quad \forall i, m, n \quad (44)$$

$$x_{wmn}^L \leq x_{wmn}^i \leq x_{wmn}^M \quad \forall j, m, n \quad (45)$$

$$x_{wmn}^L \leq x_{wmn}^i \leq x_{wmn}^M \quad \forall i, m, n \quad (46)$$

$$x_{wmn}^L \leq x_{wmn}^i \leq x_{wmn}^M \quad \forall j, m, n \quad (47)$$

$$0 \leq \lambda \leq 1 \quad (48)$$

The constraints (34)–(39) define the parameter λ as the minimum satisfaction level in the system. The parameter λ is also a decision variable, in addition to the fraction removal levels, in the optimization problem. The crisp constraints (40)–(48) determine the space of alternatives. The constraints (40)–(43) are based on the water quality requirements set by the pollution control agency. The aspiration level and maximum acceptable level of fraction removal levels set by the dischargers are expressed in constraints (44) and (45). However, the pollution control agency imposes minimum fraction removal levels that are expressed as the lower bounds

$x_{imn}^{\min}$ and $x_{jmn}^{\min}$ in constraints (46) and (47). The upper bounds $x_{imn}^{\max}$ and $x_{jmn}^{\max}$ in these constraints represent the technologically possible maximum fraction removal levels. Observing that the maximum acceptable level x_{wmn}^M of pollutant treatment cannot exceed the technologically possible upper limit $x_{wmn}^{\max}$, the constraints (44) and (46) can be simplified to a single constraint given by (49) and, similarly, the constraints (45) and (47) can be simplified to a single constraint (50), as follows:

$$\max(x_{imn}^L, x_{imn}^{\min}) \leq x_{imn} \leq x_{imn}^M \quad \forall i, m, n, \quad (49)$$

$$\max(x_{jmn}^L, x_{jmn}^{\min}) \leq x_{jmn} \leq x_{jmn}^M \quad \forall j, m, n, \quad (50)$$

The concentration levels c_{wi} and c_{dwi} ($w \Rightarrow i, j$) of the water quality index at the check-point I can be related to the fraction removal levels x_{wmn} with the help of transfer functions. These transfer functions can be derived using appropriate mathematical models that determine the spatial distributions of the water quality index due to the pollutants in the river system. The risk r_{wi} of low water quality at the check-point I can be evaluated using (15). The PDF $f(c_{wi})$ is derived from the PDFs of the various random elements that affect the concentration level c_{wi} . The PDF thus derived will be in terms of the vector of fraction removal levels x_{wmn} . The derivation of PDF depends on the problem at hand. The example given in section 4 illustrates this with respect to the DO deficit as the water quality index that is random due to the random nature of river Row. Substitution for r_{wi} , c_{wi} and c_{dwi} in the constraints of the above optimization model transforms the constraints in terms of the fraction removal levels. In general, the resulting model will be a nonlinear optimization model and suitable algorithms need to be adopted to solve the model. The solution of the model gives X^* and λ^* where X^* is the vector of optimum fraction removal levels and λ^* is the max-min satisfaction level in the system. The application of the fuzzy optimization model to a river system is discussed in the following section.

4. Model application

Application of the model is illustrated using a hypothetical river system shown in figure 9. Description of the river system with respect to the notation used in the model is given in table 2. To keep the emphasis on simultaneous treatment of randomness and vagueness, the illustration is kept simple with respect to other detailed aspects of water quality modelling for a river system. For example, more detailed water quality models (see for example Gromiec *et al.* (1983), Thomann and Mueller (1987) and James and Elliot (1993)) may be used in place of the one-dimensional BOD-DO model used in the illustration. The effects due to the randomness of other variables such as the

Figure 9. A hypothetical river system

temperature and reaction rate coefficients that affect the probability distribution of the water quality index may also be treated in a similar way as discussed in the illustration for random river flow. The objective of the illustrative example presented in this section is to demonstrate the applicability of the fuzzy probabilistic model to a simplified but realistic hypothetical river system.

River reaches 1-4 form the main stem of the river network. River reaches 5 and 6 form a tributary to the main stem. The river Row Q in the main stem of the river network is considered as a random variable. The PDF and the cumulative distribution function (CDF) of the flow Q are assumed to be known. All other Rows such as the tributary Row and effluent Rows are assumed to be deterministic. The magnitude of the deterministic Row in the tributary which is composed of reaches 5 and 6 is $3.9821 \times 10^6 \text{ m}^3 \text{ day}^{-1}$. The pollutant in the system is the point sources of BOD waste load. The water quality index of interest is the DO deficit at a finite number of check-points in the river network due to those point-sources of BOD. Water quality is checked at the 16 check-points shown in figure 9. The data pertaining to the river flows and effluent Rows are given in table 3. The mean time of Row in the sixth column of table 3 corresponds to the travel time of Row from the beginning to the end of the reach. The middle check-point in each reach corresponds to half the mean time of Row from the beginning to the end of the reach. The deoxygenation and reaeration rate coefficients, and the saturation DO concentration remain constant for all the check-points in the same river reach.

The DO deficit at a check-point can be expressed in terms of the fraction removal levels of BOD associated with various dischargers located upstream of the check-point under consideration through the transfer function. The transfer function that expresses the DO deficit at a check-point in terms of the concentrations of the point sources of ROD and the fraction removal levels are obtained using the one-dimensional steady-state BOD-DO equations (Gromiec *et al.* 1983, Thomann and

Table 2. Details of the hypothetical river system.

Set	Description of the ref	Element representation	Number of elements
Q_L	Water quality check-points	I	$N_0 = 16$
D_M	Dischargers	m	$N_d = 6$
P_N	Pollutants: point sources of BOD	n	$N_p = 1$
T_P	Uncontrollable sources of pollutants: nil	p	$N_i = 0$
V_i	Water quality indices: nil; desirable level > permissible level	i	$N_r = 0$
S_j	Water quality indices: DO deficit; desirable level < permissible level	j	$N_s = 1$

Table 3. Effluent flow and river flow data.

Effluent flow data				River flow data				
Discharger	Effluent flow rate ($10^6 \text{ m}^3 \text{ day}^{-1}$)	BOD concentration (mg l^{-1})	DO concentration (mg l^{-1})	River reach u_e	mean time of flow (days)	Deoxygenation rate constant (day^{-1})	Reaeration rate constant (day^{-1})	Saturation DO concentration (mg l^{-1})
D_1	2.114	1250	1.23	u_1	0.316	0.111	0.841	9.70
D_2	6.321	1415	2.40	u_2	0.732	0.320	0.754	9.80
D_3	7.554	2080	1.70	u_3	1.218	0.357	0.670	9.85
D_4	5.180	935	2.16	u_4	1.823	0.390	0.210	9.4
D_5	6.415	765	1.93	u_5	0.642	0.328	0.741	9.80
D_6	4.126	1880	1.80	u_6	1.312	0.378	0.532	9.90

Mueller 1987, James and Elliot 1993). Different methods that employ the one-dimensional Streeter-Phelps equations to predict the spatial distribution of DO deficit in a river network with multiple reaches are available in literature (see for example Arbabi and Elzinga (1975) and Fugiwara *et al.* (1986)). A suitable adoption of the model proposed by Fugiwara *et al.* (1986) is used in the present study.

Since only a single water quality index (DO deficit) and only one type of pollutant (point source of BOD from the dischargers) are considered in this illustrative example, the notation for the different variables are simplified by retaining only the subscripts I (check-point) and m (discharger) with the various quantities of interest. For example, c_l , r_l and x_m denote, the DO deficit at the check-point l , the risk of low water quality at the check-point l and the fraction removal level of BOD for the discharger m respectively. Similarly, the goals of the pollution control agency are denoted by G_l (for probabilistic case) and E_l (for deterministic case), and the goals of the dischargers are denoted by F_m .

Owing to the random nature of the river flow Q in the main stem of the river network, the DO deficit at a check-point l ($l=1-11$) in the main stem (reaches 1-4)

may be considered as a random variable C_l . The DO deficit at any check-point l ($l=12-16$), other than that in the main stem, in the river network is assumed to be deterministic. The deterministic DO deficits at these check-points can be expressed as linear functions of the fraction removal levels using the method given by Fugiwara *et al.* (1986). The constant term and the coefficients of the fraction removal levels in the DO deficit expressions for the check-points $l=12-16$ are given in table 4. The DO deficits at these check-points are functions of only the fraction removal levels x_5 and x_6 corresponding to the dischargers D_5 and D_6 respectively.

Table 4. Data for DO deficit expression: c_l ($l=12-16$)

Checkpoint l	Constant term	~ 1 × following coefficients					
		x_1	x_2	x_3	x_4	x_5	x_6
12	1.1741	0	0	0	0	1.0758	0
13	1.8943	0	0	0	0	1.8168	0
14	2.0589	0	0	0	0	1.7976	0
15	6.9033	0	0	0	0	3.0579	3.6612
16	9.1225	0	0	0	0	3.5533	5.4391

For example, the DO deficit at the check-point 15 is expressed as follows using the data given in table 4:

$$c_{15} = 6.9033 - 3.0579x_5 - 3.6612x_6. \quad (51)$$

Treatment of the random variable C_l of the DO deficit at the check-points 1–11 is considered next. Following Fujiwara *et al.* (1986), the DO deficit C_l at the check-point l can be expressed in general form as

$$C_l = \frac{A_l Q + B_l}{Q + V_l}, \quad l = 1, \dots, 11, \quad (52)$$

where A_l is a constant depending on the BOD and DO deficit with which the river flows enter the main stem (at the beginning of reach 1) and tributary (at the beginning of reach 5) of the river network, B_l is a function of the vector $\mathbf{X}$ of fraction removal levels, and V_l is a constant depending on the deterministic component of the total flow (deterministic component of river flow plus effluent flow) at the check-point l . The details of A_l , B_l and V_l at the check-points 1–11 are obtained using the method suggested by Fujiwara *et al.* (1986) for evaluating the DO deficit at a check-point in a river system. The factor B_l is a linear function of the fraction removal levels. Table 5 gives the values for A_l , B_l and V_l . In the table, the third column gives the constant term, and the fourth to ninth columns give the coefficients of the fraction removal levels in the expression for B_l . It may be noted that, in (52), Q should be expressed in cubic metres per day so as to conform to the data given in table 5.

Owing to the dilution of waste-water flow by the river flow, the DO deficit at a check-point decreases as the river flow increases when all other factors that affect the DO deficit remain invariant (Thomann and Mueller

1987). As a consequence of this, the probability that the DO deficit is less than or equal to a certain DO deficit value c_l at the check-point l may be equated to the probability that the river flow is greater than or equal to an associated river flow value q , that is

$$P(C_l \leq c_l) = P(Q \geq q), \quad l = 1, \dots, 11. \quad (83)$$

Using (52) and (53), the CDF $F_{C_l}(c_l)$ of DO deficit at the check-point l can be derived as follows:

$$\begin{aligned} F_{C_l}(c_l) &= P(C_l \leq c_l) \\ &= P\left(\frac{A_l Q + B_l}{Q + V_l} \leq c_l\right) \\ &= P\left(Q \geq \frac{B_l - V_l c_l}{c_l - A_l}\right) \\ &= 1 - P\left(Q \leq \frac{B_l - V_l c_l}{c_l - A_l}\right) \\ &= 1 - F_Q\left(\frac{B_l - V_l c_l}{c_l - A_l}\right) \\ &= 1 - F_Q(q), \end{aligned} \quad (54)$$

where $F_Q(q)$ is the CDF of river flow. The PDF of DO deficit at the check-point l is then obtained by differentiating $F_{C_l}(c_l)$ with respect to c_l as follows:

$$f(c_l) = \frac{d}{dc_l} [F_{C_l}(c_l)] = -\frac{d}{dc_l} \left[F_Q\left(\frac{B_l - V_l c_l}{c_l - A_l}\right) \right]. \quad (55)$$

Derivation of the PDF $f(c_l)$ using (55) requires the CDF $F_Q(q)$ of the river flow to be a differentiable function, but the CDF $F_Q(q)$ may not be available in a closed-form expression in some cases and hence the differentiation will not be possible (e.g. the CDF corresponding to

Table 5. Data for DO deficits at checkpoints 1–11.

Checkpoint l	A_l	Constant term ($\times 10^7$)	B_l $\times 10^7 \times$ following coefficients						V_l ($\times 10^6$)
			x_1	x_2	x_3	x_4	x_5	x_6	
I	0	0.1429	0.1271	0	0	0	0	0	0.0213
	0	0.2456	0.2318	0	0	0	0	0	0.0213
3	0.1	3.9824	0.2318	0	0	0	1.4531	2.2243	4.1741
4	0.0756	4.8127	0.4066	0.8613	0	0	1.3314	2.1581	4.1741
5	0.0576	5.2659	0.5150	1.4203	0	0	1.2212	2.0673	4.1741
6	0.1076	5.3484	0.5150	1.4203	0	0	1.2212	2.0673	4.2496
7	0.0715	8.4192	0.6453	2.0718	2.5030	0	1.1172	1.9990	4.2496
8	0.0476	9.5108	0.6727	2.2847	3.6782	0	0.9884	1.8315	4.2496
4	0.0976	9.5721	0.6727	2.2847	3.6782	0	0.9884	1.8315	4.3014
III	0.0806	1.5651	0.8887	3.1269	5.7191	1.3111	1.1520	2.1992	4.3014
11	0.0665	16.6962	0.9674	3.4514	6.7123	2.0017	1.1866	2.2975	4.3014

[the normally distributed river Row). In such cases, appropriate numerical techniques may be used to determine the PDF of c_l . Starting with PDF of Q . The differentiation of the known or derived CDF $F_Q(q)$ is carried out with respect to c_l after substituting $(B_l - V_l c_l)/(c_l - A_l)$ for q in the expression for $F_Q(q)$. Since the management problem is more relevant for low Row conditions, a bounded, exponential distribution for the river Row Q has been used in this example. Such a distribution is commonly used for low Rows in hydrological studies (Yevjevich 1972, Beran and Rodier 1985). The closed-form expression for the CDF $F_Q(q)$ available in the case of the bounded exponential distribution makes it possible to obtain the PDF $f(c_l)$ in a closed form which can subsequently be used in (15) to evaluate the risk r_l . Mathematically the bounded exponential PDF of the Row is expressed as

$$f(q) = \theta \exp[-\theta(q - q_L)], \quad \theta > 0, \quad q \geq q_L. \quad (56)$$

The values of the parameters q_L and θ of the PDF are $5.8263 \times 10^6 \text{ m}^3 \text{ day}^{-1}$ and $0.4 \times 10^{-6} \text{ day m}^{-3}$ respectively. The CDF of the river Row is given as

$$F_Q(q) = 1 - \exp[-\theta(q - q_L)], \quad \theta > 0, \quad q \geq q_L. \quad (57)$$

Substituting for q using $(B_l - V_l c_l)/(c_l - A_l)$ in $F_Q(q)$ in (54), the CDF of DO deficit is obtained as

$$F_{c_l}(c_l) = \exp \left[-\theta \left(\frac{B_l - V_l c_l}{c_l - A_l} - q_L \right) \right]. \quad (58)$$

Differentiating the CDF given by (58), the PDF of DO deficit is obtained as

$$f(c_l) = \frac{\theta(B_l - A_l V_l)}{(c_l - A_l)^2} \exp \left[-\theta \left(\frac{B_l - V_l c_l}{c_l - A_l} - q_L \right) \right]. \quad (59)$$

This PDF of the DO deficit is used for evaluating the risk defined by (15). The membership function $\mu_{w_l}(c_l)$ for the fuzzy set of low water quality required in (15) is expressed as

$$\mu_{w_l}(c_l) = 1 - \exp \left(\frac{a_l(c_l)^{n_l}}{c_l - S_l} \right), \quad l = 1, \dots, 11, \quad (60)$$

where a_l and n_l are non-zero positive constants and S_l is the saturation DO concentration level at the check-point l . The limits of integration for evaluating the risk are obtained by considering the minimum and maximum values of DO deficit, denoted by $c_l^{\min}$ and $c_l^{\max}$ respectively, as follows. The minimum and maximum values of DO deficit depend on the upper and lower limits respectively of the PDP of the river Row. The minimum DO deficit $c_l^{\min}$ occurs when the river Row is maximum (i.e. as $Q \rightarrow \infty$) and the maximum DO deficit $c_l^{\max}$ occurs when the river Row is minimum (i.e. $Q = q_L$). Thus,

$$c_l^{\min} = \lim_{Q \rightarrow \infty} \left(\frac{A_l Q + B_l}{Q + V_l} \right) = A_l, \quad (61)$$

$$c_l^{\max} = \left(\frac{A_l q_L + B_l}{q_L + V_l} \right) \quad (62)$$

The risk of low water quality is then evaluated using (15) as follows:

$$r_l = \int_{c_l^{\min}}^{c_l^{\max}} \mu_{w_l}(c_l) f(c_l) dc_l. \quad (63)$$

A crisp constraint for the maximum and minimum concentration levels is written as

$$c_l^{\min} \leq c_l \leq c_l^{\max} \leq S_l. \quad (64)$$

Nonlinear membership functions are used for the water quality goals E_l at the check-points 12–16 where the DO deficit is deterministic, and for the risk goals G_l related to the low water quality at the check-points 1–11 where the DO deficit is random. Linear membership functions are assigned for the pollutant treatment goals F_m . The parameters of the various membership functions are fixed on the basis of the desired shape of the membership functions. Use of these parameters help to design the membership functions so as to reflect the qualitative trends associated with the fuzzy goals of the pollution control agency and dischargers, and the fuzzy sets of low water quality. The membership function given by (22) is used for all the fuzzy goals E_l assigning the values 0.01 and 4 to the parameters n_{3l} and n_{3l} respectively for all l ($l = 12-16$). The values of these parameters are so chosen that the membership values corresponding to the DO deficits of 2 mg l^{-1} (acceptable level) and 6 mg l^{-1} (maximum permissible level) are nearly equal to one and zero respectively. The parameter c_l^H in the membership function for the goal E_l takes the value of the saturation DO concentration S_l at the check-point l . As an example, the membership function for the goal E_{12} is shown in figure 10. The membership

Figure 10. Membership function for the goal E_{12} .

function given by (16) is used for all the fuzzy goals G_l . The parameters a_{2l} , n_{2l} and r_l^M of this membership function are assigned the values of 500, 4 and 1 respectively for all l ($l = 1-11$). The values of these parameters are so chosen that the membership values corresponding to the risk levels of 0.05 (acceptable) and 0.25 (maximum permissible) are nearly equal to one and zero respectively. This membership function is shown in figure 11. A linear membership function is used for the fuzzy goal F_m and is given as

$$\mu_{F_m}(x_m) = \begin{cases} 1, & 0 \leq x_m \leq x_m^L, \\ \frac{x_m^M - x_m}{x_m^M - x_m^L}, & x_m^L \leq x_m \leq x_m^M, \\ 0, & x_m \geq x_m^M \end{cases} \quad (65)$$

The parameters x_m^L and x_m^M of the membership functions F_m are given in table 6. Also given in table 6 are the values of the minimum fraction removal level $x_m^{\min}$ set by the pollution control agency for the discharger m . The membership function for the fuzzy goal F_1 is shown in figure 12. The membership function $\mu_{W_l}(c_l)$ for the fuzzy set of low water quality is given by (60). The values assigned to the parameters a_{1l} and n_{1l} are 0.01 and 4 respectively for all l such that the membership values corresponding to the DO deficits of 2 mg l^{-1} (desirable level) a 6 mg l^{-1} (permissible level) are nearly equal to zero and one respectively. As an example, the member-

Figure 11. Membership function for the goal G_l

Table 6. Details of membership functions for goals F_m .

Discharger D_m	x_m^L	x_m^M	$x_m^{\min}$
D_1	0.25	0.75	0.30
D_2	0.35	0.80	0.30
D_3	0.25	0.90	0.30
D_4	0.30	0.85	0.30
D_5	0.35	0.75	0.30
D_6	0.30	0.90	0.30

Figure 12. Membership function for the goal F_1 .

ship function $\mu_{W_{11}}(c_{11})$ for the fuzzy set of low water quality at the check-point 11 is shown in figure 13. The fuzzy optimization model for water quality management of the river system shown in figure 9 is now presented as follows:

$$\text{maximize } A \quad (66)$$

subject to

$$\exp\left(\frac{500 r_l^4}{r_l - 1}\right) \geq \lambda, \quad l = 1, \dots, 11, \quad (67)$$

$$\exp\left(\frac{0.01 c_l^4}{c_l - S_l}\right) \geq A, \quad l = 12, \dots, 16, \quad (68)$$

$$\frac{x_m^M - x_m}{x_m^M - x_m^L} \geq \lambda, \quad m = 1, \dots, 6, \quad (69)$$

$$0 \leq r_l \leq 1, \quad l = 1, \dots, 11, \quad (70)$$

$$c_l^{\min} \leq c_l^{\max} \leq S_l, \quad l = 1, \dots, 11, \quad (71)$$

$$0 \leq c_l \leq S_l, \quad l = 12, \dots, 16, \quad (72)$$

$$\max(x_m^L, x_m^{\min}) \leq x_m \leq x_m^M, \quad m = 1, \dots, 6, \quad (73)$$

$$0 \leq \lambda \leq 1. \quad (74)$$

Figure 13. Membership function for the fuzzy set F_1 .

The nonlinear constraints (67) and (68) render the fuzzy optimization model to be a nonlinear programming problem. The terms r_l and c_l in the above constraints are both functions of the fraction removal levels x_m . The resulting nonlinear programming problem has been solved using the Optimization Toolbox (The Maths Works, Inc. 1994) available in MATLAB (The Maths Works, Inc. 1995). The `constr` function in the Optimization Toolbox is used for solving the model. The constraint (67) requires special treatment as it involves the risk level r_l that is to be evaluated through an integration given in (63). These constraints are invoked using M-File Function (The Maths Works, Inc. 1995). The risk is expressed in the M-File, using Simpson's general formula for numerical integration. As the `constr` function may not give the global optimum (The Maths Works, Inc. 1994), different local optima are obtained starting with different initial vector of decision variables. The extreme value among such local optima is chosen as the global optimum and the corresponding vector of decision variables is taken as the optimal solution.

5. Results and discussion

The results of the analysis are summarized in table 7. The optimal value of the objective function is $\lambda^* = 0.2541$, and it corresponds to the max-min satisfaction level in the system. Also shown in the table 7 is

the DO deficit information at the various water quality check-points. The DO deficit values corresponding to the optimum fraction removal levels are given for the water quality check-points 12–16 where the DO deficit is deterministic. For the checkpoints 1–11, the risks of low water quality corresponding to the optimum fraction removal levels are computed using (63). The limits of the integration in (63), namely c_l^{min} and c_l^{max} , are computed using (61) and (62), which in turn depend on the data given in table 5 and the optimum fraction removal levels given in table 7. The PDF $f(c_l)$ and the membership function $\mu_{W_l}(c_l)$ in the expression for r_l are expressed as functions of c_l by means of (59) and (60) respectively, and then the risk r_l is evaluated from (63) using numerical integration. The risk levels (corresponding to the optimum fraction removal levels) obtained by performing the numerical integration are given in the fourth column of table 7, for the checkpoints 1–11. Figure 14 shows the variation in the risk of low water quality at the check-points 1–11. It may be seen that, as we proceed downstream, the risk of low water quality increases and attains a maximum value of 21.54% at the check-point 11. The CDF of DO deficit at a check-point l is obtained from (58) after substituting the value of B_l corresponding to the optimum fraction removal levels. As an example, figure 15 shows the CDF of DO deficit at the check-point 11. Similar information is obtained for other check-points also.

Table 7. Summary of results

Discharger D_m	Optimum fraction removal level x_m^*	River reach u_e	Water quality	
			Risk r_l of low water quality at check-point l (%)	DO deficit c_l at check-point (mg l^{-1})
D_1	0.6229	u_1	$r_1 \approx 0$	ND
D_2	0.6857	u_2	$r_2 \approx 0$	ND
D_3	0.7348	u_3	$r_3 = 0.12$ $r_4 = 0.24$ $r_5 = 0.34$	ND
D_4	0.7102	u_4	$r_6 = 1.70$ $r_7 = 1.99$ $r_8 = 2.69$ $r_9 = 3.38$	ND
D_5	0.6484	u_5	$r_{10} = 15.41$ $r_{11} = 21.54$	$c_{12} = 0.4766$ $c_{13} = 0.7163$ $c_{14} = 0.8933$ $c_{15} = 2.1823$ $c_{16} = 2.7528$
D_6	0.7475	u_6	ND	

ND, value not defined.

Figure 14. Variation in the risk of low water quality.

Figure 15. CDF of DO deficit at the check-point 11.

The fuzzy risk proposed in this study may be used as an effective measure to describe the fuzzy probabilistic aspects of the water quality indices governed by both randomness and vagueness. It may be expected that the fuzzy risk will be greater than the risk obtained by considering only the randomness of the water quality indices. This is because the fuzzy risk considers the full range of concentration levels of the water quality indices as low water quality with varying degrees of low water quality. On the other hand, the risk obtained from only the probability consideration takes into account only those concentration levels which are either smaller (for water quality index i whose desirable level is greater than the permissible level) or greater (for water quality index j whose desirable level is less than the permissible level) than a particular concentration level and not the full range of concentration levels as low water quality. Moreover, the pollution control agency has the flexibility to consider a wide range of risk levels by choosing the parameters $a_{w/l}$, $n_{w/l}$ and $r_{w/l}^M$ appropriately in the membership functions for the goals $G_{w/l}$ related to the risk of low water quality at the check-point l . For example, it may be possible for the pollution control agency to put a stringent condition on the risk level by

setting a lower value for $r_{w/l}^M$. A consequence of this will be an increased fraction removal level for the dischargers. A wide spectrum of decisions can be arrived at by choosing appropriate membership functions for the different fuzzy goals and the fuzzy sets of low water quality. By a mutual interaction between the pollution control agency and the dischargers, an agreement upon the various fuzzy goals and the related membership functions can be arrived at and this will help to take a best compromise decision acceptable to both the pollution control agency and the dischargers. The results obtained from the fuzzy optimization model incorporating the risk of low water quality described in a more realistic framework may be used in the planning and design phases of water quality management of a river system.

6. Conclusion

A fuzzy waste load allocation model is developed in the present study to incorporate the uncertainties due to randomness and vagueness associated with the water quality management of a river system. Considering the two forms of uncertainty (randomness and vagueness) simultaneously, water quality management of a river system is addressed in a fuzzy probabilistic framework. The goals of the pollution control agency related to low water quality are redefined in view of the random nature of the water quality indices. For this, a fuzzy set-based definition that is a more general case of the existing crisp-set-based definition of low water quality is considered. The event of low water quality at a check-point in the river system is considered as a fuzzy event. The fuzzy risk of low water quality is then defined as the probability of the fuzzy event of low water quality. Instead of constraining the risk of violation of water quality standard by a finite value of risk through a chance constraint, a fuzzy set of low risk that considers a range of risk levels with appropriate membership values is introduced. Different goals associated with the water quality aspects of the management problem are expressed as fuzzy sets. The exponential-type membership functions used for these goals provide flexibility in expressing the aspirations appropriately with regard to the water quality. The resulting management problem is formulated as a fuzzy multiobjective optimization problem. The model is applied to a hypothetical river system to illustrate the fuzzy probabilistic modelling in the water quality management of a river system. Assigning appropriate values for the parameters of the membership functions helps the decision makers to account for the aspirations regarding the attainment of respective goals. The fuzzy optimization model may be expected to act as an aid to decision making for water quality management of a river system.

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