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Minimization of Quantization Noise amplification in 2-Channel IIR Bi-orthogonal Filter Bank

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Abstract

In this work we look at the class of uniform biorthogonal FIR, IIR and hybrid filter banks with $M = 2$ channels. when Quntization noise passes through synthesis filter bank it gets amplified, due to the multiplication of synthesis filter norms to the quantization noise in each channel. Better compression, or higher coding gain, will be achieved if the quantization noise amplification is reduced. Scaling down of the synthesis filters will not work **as** to maintain the property **of** perfect reconstruction we have to scale up the analysis filter bank causing no improvement. Minimization of such amplification has been proposed earlier **for** FIR filter banks. We are proposing an implementable and stable coloring filter **for** IIR filter bank which is the optimum coloring filter for the assumed filter structure. We deal with an example of a hybrid filter bank based on all pass delay chain structure.

1. Introduction

A subband coder, consisting of filter bank and quantization, offers *coding gain* while compressing **a** source. There are two ways the biorthogonal filter bank and its performance differs from that of the orthogonal case. the good news is the pass bands of the biorthogonal filter bank are not restricted to be flat unlike orthogonal case. Consequently, shaping of the passbands towards making the non-flat subband spectrum more flat adds to the gain. The bad news is the quantization noise amplification [1]. Coding gain **of a** biorthogonal subband coder with optimum bit allocation, using additive white noise model for the quantizers and uncorrelated noise assumption, is given in [1]. The additional terms in the denominator, *synthesis filter norms*, appear because the white quantization noise passing through the synthesis bank becomes colored. Consequently its variance is modified resulting in *quantization noise amplification* in the coder. Since the filter bank is perfect reconstruction, there exists an inverse relationship between the synthesis norm and the analysis norm (absorbed in $\sigma^2_{x_k}$). So synthesis norms can not be change independently. The question one may ask is if we can find out an implementable coloring filter at the analysisside, that colors the quantization noise, which brings down the effective synthesis filters norm. For the FIR polyphase matrices the answer is known in the affirmative [1]. In this work our aim is to find out the answer in the case **of** hybrid and IIR polyphase matrices.

2. Minimization of quantization noise amplification in filter banks

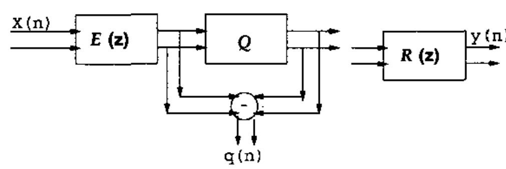

Figure 1: Subband coder showing quantization noise

Consider **a** PR filter bank with analysis polyphase matrix $\mathbb{E}(z)$ and synthesis polyphase matrix $\mathbb{R}(z)$ shown in figure(1). The input signal $x(n)$ passes through $\mathbb{E}(z)$ to get the subband signals $x_k(n)$. In absence of quantization, $x_k(n)$ then passes through $\mathbb{R}(z)$ to get the output signal $y(n)$. Since the system is PR, $y(n)$ is same **as** $x(n)$ except possible scaling and delay. Now consider quantizing the subbands. Assuming the additive noise model for the quantizers, after quantization we have $x_k(n) + q_k(n)$, the subband signals plus the quantization noise. Both of these pass through $\mathbb{R}(z)$ and result in the reconstructed output $x(n) + e(n)$. The reconstruction error $e(n)$ equals $y(n) - x(n)$. Due to linearity, $e(n)$ is the output of $\mathbb{R}(z)$ when the input is $q_k(n)$. Thus, we ***see*** the variances of $q_k(n)$ are scaled by the synthesis filter norms from $\mathbb{R}(z)$ in the reconstruction error variance.

Now introduce a coloring filter $\mathbb{A}(z)$ such that $q_k(n)$ (which is assumed white in the quantizer model) now passes

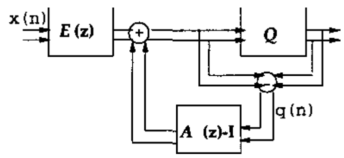

Figure 2: Analysis side with the coloring filter

through $\mathbb{A}(z)$ before getting added to $x_k(n)$. Figure 2 shows the block schematic. Then $e(n)$ becomes the output of $\mathbb{R}(z)\mathbb{A}(z)$ with the input $q_k(n)$. The total system is called the effective synthesis polyphase $\mathbb{R}_{eff}(z) = \mathbb{R}(z)\mathbb{A}(z)$. The signal component remains unaffected. The reconstruction error variance (and hence the coding gain) now depends on the effective synthesis filter norms obtained from $\dot{\mathbb{R}}_{eff}(z)$. An optimal choice of $\mathbb{A}(z)$ will minimize the quantization noise amplification, or maximize the coding gain.

For the FIR filter banks, the solution to the above minimization problem has been suggested in [1]. We briefly describe the solution below. Since the determinant of any polyphase matrix of any FIR PR filter bank is a scaled delay,

$$|\mathbb{R}(z)| = cz^{-L} \quad (1)$$

But $|\mathbb{R}(z)|$ can be decomposed into paraunitary and unimodular matrices [2]

$$\mathbb{R}(z) = \mathbb{P}(z)\mathbb{U}(z) \quad (2)$$

where $||\mathbb{U}(z)|| = ||\mathbb{R}(z)||$ as $||\mathbb{P}(z)|| = 1$. If $\mathbf{U}_0$ is the zeroth order coefficient of $\mathbb{U}(z)$, it can be decomposed as

$$\mathbf{U}_0 = \mathrm{PDL} \quad (3)$$

where P is an orthogonal matrix, D is a diagonal matrix and L is a lower triangular matrix. Assuming $\mathbf{S} = \mathrm{PD}$, the optimum coloring filter will he $\mathbb{U}^{-1}(z)\mathbf{S}$. Using the above coloring filter, the effective synthesis polyphase $\mathbb{R}_{eff}(z)$ is given by

$$\begin{aligned} \mathbb{R}_{eff}(z) &= \mathbb{R}(z)\mathbb{U}^{-1}(z)\mathbf{S} \\ &= \mathbb{P}(z)\mathbb{U}(z)\mathbb{U}^{-1}(z)\mathbf{S} \\ &= \mathbb{P}(z)\mathbf{S} \end{aligned} \quad (4)$$

Since a paraunitary matrix has unit norm, the effective filter norm will be $||\mathbf{S}||$ which is less than $||\mathbf{U}_0||$ which is again less than $||\mathbb{U}(z)||$. Thus the noise amplification is indeed reduced.

3. Proposed generalization of previous work

Only in case of FIR filter banks, the determinant of $\mathbb{R}(z)$ is a scaled delay. We consider here any 2 channel PR filter hank, which may be FIR, IIR, or hybrid. Then the determinant of $\mathbb{R}(z)$ has the most general form as $|\mathbb{R}(z)| = \frac{z^{-L}A_1(z)}{A_2(z)}$ for some $L \geq 0$ (assuming causality). If $\mathbb{R}(z) = \mathbf{R}_0 + z^{-1}\mathbf{R}_1 + z^{-2}\mathbf{R}_2 + \ldots$ then from the final value theorem

$$\begin{aligned} \lim_{z \to \infty} |\mathbb{R}(z)| &= \lim_{z\to\infty} |\mathbf{R}_0 + z^{-1}\mathbf{R}_1 + z^{-2}\mathbf{R}_2 + \ldots| \\ &= |\mathbf{R}_0| \end{aligned} \quad (5)$$

From the general form of $|\mathbb{R}(z)|$ at $z = \infty$,

$$\begin{aligned} \lim_{z\to\infty} |\mathbb{R}(z)| &= \lim_{z\to\infty} \frac{z^{-L}A_1(z)}{A_2(z)} \\ &= 0 \end{aligned} \quad (6)$$

since $L > 0$. From equations (5) and (6), we have $|\mathbf{R}_0| = 0$. Thus zeroth order matrix of $\mathbb{R}(z)$ has its determinant zero. As $\mathbf{R}_0$ is singular, there exists a non zero vector $\mathbf{v}$ such that

$$\mathbf{v}^T\mathbb{R}(z) = z^{-1}\mathbf{q}^T(z)$$

where $\mathbf{q}(z)$ is some causal vector whose degree is 1 less than the degree of $\mathbb{R}(z)$. This is because $\mathbf{v}^T\mathbf{R}_0 = 0$. We can always scale $\mathbf{v}$ to have unit norm. Since we consider 2 channel filter bank, let

$$\mathbf{v} = \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

Now we construct a non singular matrix having $\mathbf{v}^T$ as its second row. Since there is no restriction on the first row, we can make this non singular matrix unitary. Let the rotation matrix $\mathbf{R}_T$ be the constructed non singular matrix of discussion.

$$\mathbf{R}_T = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Prernultiplying $\mathbb{R}(z)$ with $\mathbf{R}_T$ we have,

$$\begin{aligned} \mathbf{R}_T\mathbb{R}(z) &= \mathbf{R}_T(\mathbf{R}_0 + z^{-1}\mathbf{R}_1 + z^{-2}\mathbf{R}_2 + \ldots) \\ &= \mathbf{R}_T\mathbf{R}_0 + z^{-1}\mathbf{R}_T\mathbf{R}_1 + z^{-2}\mathbf{R}_T\mathbf{R}_2 + \ldots \end{aligned} \quad (7)$$

Let $\mathbf{R}_0 = \begin{bmatrix} r_{0_{11}} & r_{0_{12}} \\ r_{0_{21}} & r_{0_{22}} \end{bmatrix}$. Since $\mathbf{R}_0$ is singular, that means $\frac{r_{0_{11}}}{r_{0_{21}}} = \frac{r_{0_{12}}}{r_{0_{22}}}$. Now

$$\mathbf{R}_T\mathbf{R}_0 = \begin{bmatrix} r_{0_{11}}\cos(\theta) - r_{0_{21}}\sin(\theta) & r_{0_{12}}\cos(\theta) - \sin(\theta)r_{0_{22}} \\ r_{0_{11}}\sin(\theta) + r_{0_{21}}\cos(\theta) & \sin(\theta)r_{0_{12}} + \cos(\theta)r_{0_{22}} \end{bmatrix}$$

We can select θ such that

$$\begin{aligned} r_{0_{11}}\sin(\theta) + r_{0_{21}}\cos(\theta) &= 0 \\ r_{0_{12}}\sin(\theta) + r_{0_{22}}\cos(\theta) &= 0 \\ \Rightarrow \theta &= \tan^{-1}\left(\frac{-r_{0_{22}}}{r_{0_{12}}}\right) \\ &= \tan^{-1}\left(\frac{-r_{0_{21}}}{r_{0_{11}}}\right) \end{aligned} \quad (8)$$

So $\mathbf{R}_T\mathbf{R}_0 = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$, and using this result in the earlier equations we have,

$$\mathbf{R}_T\mathbb{R}(z) = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix} + z^{-1}\mathbf{R}_T\mathbf{R}_1 + z^{-2}\mathbf{R}_T\mathbf{R}_2 + \ldots.$$

So it is obvious that the second row of $\mathbf{R}_T\mathbb{R}(z)$ will have z^{-1} in common. Finally we can write

$$\mathbf{R}_T\mathbb{R}(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \mathbb{R}^1(z)$$

where $\mathbb{R}^1(z)$ is a matrix with reduced degree of determinant (by one). This is because the product of the determinant of both sides of the above equation must be same, and the determinant of the left-hand-side has z^{-L} term. Since determinant of the first matrix on the right-hand-side is z^{-1}, determinant of $|\mathbb{R}^1(z)$ must have degree z^{-L+1}. Another point to be noted is that $\mathbf{R}_T$ and $\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}$ are paraunitary, so $||\mathbb{R}(z)|| = ||\mathbb{R}^1(z)||$.

Now, repeat this process i times till $|\mathbf{R}_0^i| \neq 0$, RE, being the zeroth order coefficient of $\mathbb{R}^i(z)$. $\mathbb{R}^i(z)$ will have the same norm as $\mathbb{R}(z)$ except that $|\mathbf{R}\ _0^i| \neq 0$. After i iterations $\mathbb{R}(z) = \Pi_{k=1}^{i}\mathbb{P}_k(z)\mathbb{R}^i(z)$, where $\mathbb{P}_k(z) = (\mathbf{R}_T^k)^{-1}\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}$.

Now $\mathbb{R}^i(z)$ may be used to design the optimal coloring filter. Neglecting the paraunitary factors $\mathbb{P}_k(z)$, since $\mathbb{R}_{eff}(z) = \mathbb{R}^i(z)\mathbb{A}(z)$, an immediate choice for $\mathbb{A}(z)$ is $(\mathbb{R}^i)^{-1}(z)$. But we need RE, to be upper triangular [1], so that the coloring filter is practically implementable. If $(\mathbb{R}^i)^{-1}(z)$ is implementable, effective synthesis filter norms can be made equal to 1 by using $(\mathbb{R}^i)^{-1}(z)$ as the coloring filter at the analysis side.

Now consider the case where $(\mathbb{R}^i)^{-1}(z)$ is not practically implementable. To make $(\mathbb{R}^i)^{-1}(z)$ practically implementable, we have to modify this filter such that zeroth order coefficient matrix becomes upper or lower triangular.

The zeroth order coefficient matrix of $(\mathbb{R}^i(z))^{-1}$ and $\mathbb{R}^i(z)$ are inverse of each other. Now to make $(\mathbb{R}^i(z))^{-1}$ implementable, its zeroth order coefficient matrix should be upper triangular/lower triangular. Express $\mathbb{R}^i(z)$ as

$$\begin{aligned} \mathbb{R}^i(z) &= \mathbf{R}_0^i + z^{-1}\mathbf{R}_1^i + z^{-2}\mathbf{R}_2^i + \ldots.. && (9) \\ &= \mathbf{PDU} + z^{-1}\mathbf{R}_1^i + z^{-2}\mathbf{R}_2^i + \ldots.. \\ &= \mathbf{SU} + z^{-1}\mathbf{R}_1^i + z^{-2}\mathbf{R}_2^i + \ldots.. && (10) \end{aligned}$$

where **PDU** is the decomposition of $\mathbf{R}_0^i$, zeroth order coefficient matrix of $\mathbb{R}^i(z)$, into paraunitary **P**, diagonal **D** and upper triangular **U**, and we have assumed **PD** equal to **S**.

Now writing the equation **for** $(\mathbb{R}^i(z))^{-1}$ from previous result, since our main concern is the zeroth order coefficient,

$$\begin{aligned} (\mathbb{R}^i(z))^{-1} &= (\mathbf{PDU})^{-1} + \ldots.. && (11) \\ &= \mathbf{U}^{-1}(\mathbf{PD})^{-1} + \ldots.. && (12) \\ &= \mathbf{LS}^{-1} + \ldots.. && (13) \end{aligned}$$

where **L** is a lower triangular matrix. Multiplying both sides by **S**, we have

$$\begin{aligned} (\mathbb{R}^i(z))^{-1}\mathbf{S} &= (\mathbf{LS}^{-1} + \ldots..)\mathbf{S} \\ &= \mathbf{L} + \ldots.. && (14) \end{aligned}$$

So $(\mathbb{R}^i(z))^{-1}\mathbf{S}$ is our required filter. It can be shown that it is the optimum coloring filter which is implementable as its zeroth order coefficient is lower triangular, with diagonal elements 1.

If we denote the optimum coloring filter by $\mathbb{A}_{opt}(z)$ then

$$\mathbb{A}_{opt}(z) = (\mathbb{R}^i(z))^{-1}\mathbf{S} \qquad (15)$$

Now the effective synthesis filter will be

$$\begin{aligned} \mathbb{R}(z)\mathbb{A}_{opt}(z) &= \mathbb{P}^i(z)\mathbb{R}^i(z)\mathbb{A}_{opt}(z) \\ &= \mathbb{P}^i(z)\mathbb{R}^i(z)(\mathbb{R}^i(z))^{-1}\mathbf{S} \\ &= \mathbb{P}^i(z)\mathbf{S} \\ &= \mathbb{P}^i(z)\mathbf{PD} && (16) \end{aligned}$$

So, it is obvious that only matrix **D** is causing the remaining amplification. The effective synthesis filter norms will be the square of the diagonal elements of **D.**

3.1 Optimum coloring filter will not increase the filter norms

We prove here that the square of the diagonal element of the diagonal matrix **D** will always be less than or equal to the initial synthesis filter norms. We know that RE, = **PDU.** Let

$$\begin{aligned} \mathbf{R}_0^i &= \begin{bmatrix} r_{01} & r_{02} \\ r_{03} & r_{04} \end{bmatrix} \\ \mathbf{D} &= \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \\ \mathbf{U} &= \begin{bmatrix} 1 & l_1 \\ 0 & 1 \end{bmatrix} && (17) \end{aligned}$$

Then the matrix equation will be

$$\begin{aligned} \begin{bmatrix} r_{01} & r_{02} \\ r_{03} & r_{04} \end{bmatrix} &= \mathbf{P}\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\begin{bmatrix} 1 & l_1 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} r_{01} & r_{02} \\ r_{03} & r_{04} \end{bmatrix} &= \mathbf{P}\begin{bmatrix} d_1 & d_1 l_1 \\ 0 & d_2 \end{bmatrix} && (18) \end{aligned}$$

P is paraunitary, so norm of the matrix $\mathbf{R}_0'$ and **DU** will be equal (a paraunitary matrix is only a rotation, it does not change the norm of any vector), i.e.,

$$r_{01}^2 + r_{03}^2 = d_1^2 \quad (19)$$
$$r_{02}^2 + r_{04}^2 = (d_1 l_1)^2 + d_2^2 \quad (20)$$

These are the initial filter norms. The point to be noted is that the effective filter norms after the use of optimum coloring filter is d_1^2 and d_2^2 respectively, and

$$d_1^2 \geq d_1^2 \quad (21)$$
$$(d_1 l_1)^2 + d_2^2 \geq d_2^2 \quad (22)$$

That means effective filter norms will be less than or equal to the norm contributed by the initial zeroth order coefficient matrix.

4 Minimizing the quantization noise for all pass delay chain filter bank

As an example, a hybrid filter hank is considered in this section. The filter bank based on all pass delay chain is PR, with IIR analysis and FIR synthesis filters [3].

The all pass function replacing the analysis delay chain is

$$A_p(z) = \frac{(z^{-1} - a)}{1 - az^{-1}} \quad (23)$$

for some a between -1 and 1. Also,

$$\mathbb{Y}(z) = \begin{bmatrix} \frac{-(z^{-1}+a)(1-az^{-1})}{(1-a^2)} & 0 \\ 0 & \frac{-(1-a^2z^{-2})}{(1-a^2)} \end{bmatrix}$$

$\mathbb{E}(z)$ and $\mathbb{R}(z)$ are some PR polyphase pairs, and here taken to be paraunitary by choice. If $\frac{a}{a^2-1} = p_1, \frac{a^2}{a^2-1} = p_2, \frac{1}{a^2-1} = p_3$ then $\mathbb{Y}(z)$ may be written as

$$\mathbb{Y}(z) = \begin{bmatrix} p_1 - z^{-1} - p_1 z^{-2} & 0 \\ 0 & p_3 - p_2 z^{-2} \end{bmatrix} \quad (24)$$

Let

$$\mathbb{R}(z) = \begin{bmatrix} R_{00}(z) & R_{01}(z) \\ R_{10}(z) & R_{11}(z) \end{bmatrix} \quad (25)$$

then the overall synthesis filters $F_1(z)$ and $F_2(z)$ are

$$\begin{aligned} [F_1(z) \;\; F_2(z)] &= [1 \;\; 1]\mathbb{Y}(z)\mathbb{R}(z^2) \\ F_1(z) &= (p_1 - z^{-1} - p_1 z^{-2})R_{00}(z^2) \\ &\quad + p_3 - p_2 z^{-2} R_{10}(z^2) \\ F_2(z) &= (p_1 - z^{-1} - p_1 z^{-2})R_{01}(z^2) \\ &\quad + p_3 - p_2 z^{-2} R_{11}(z^2) \end{aligned}$$

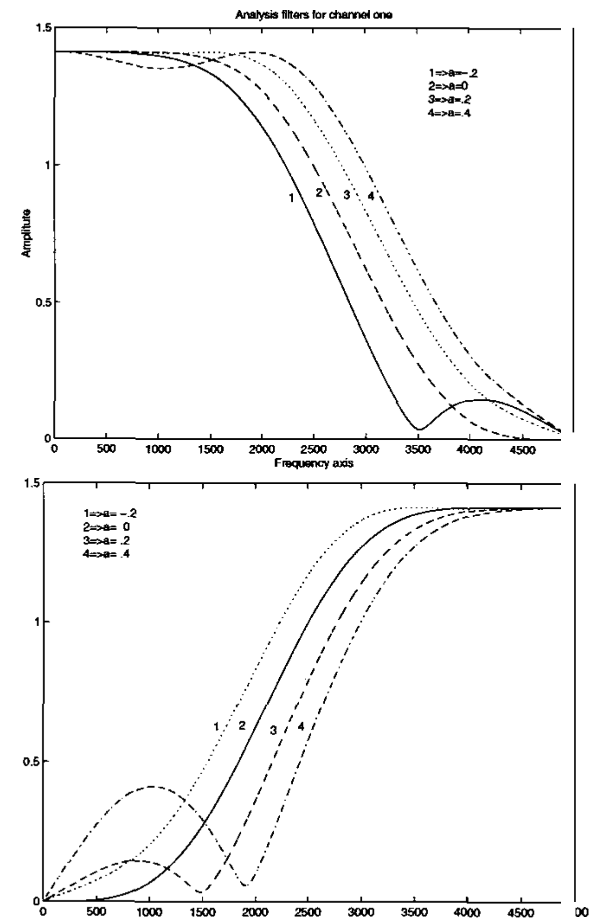

Figure 3 Analysis filter responses for different values of a

For a given all pass and a given polyphase matrix, after getting the overall synthesis filters we can write the overall synthesis polyphase matrix $\mathbb{R}_{all}(z)$. We can not comment whether in general the determinant of the zeroth order coefficient of this polyphase matrix will he zero or not. However, as discussed in the previous section, in either case we can find out the optimum coloring filter.

4.1 Results for DB4

All the results in this section are for an AR(1) source having correlation coefficient .90 and variance= **5.53987** and taking the basic paraunitary polyphase to be **DB4.**

Figure(3) shows the magnitude responses of the analysis filters for different values of a.

Figure(4) shows the magnitude responses of the corresponding coloring filters for those values of a.

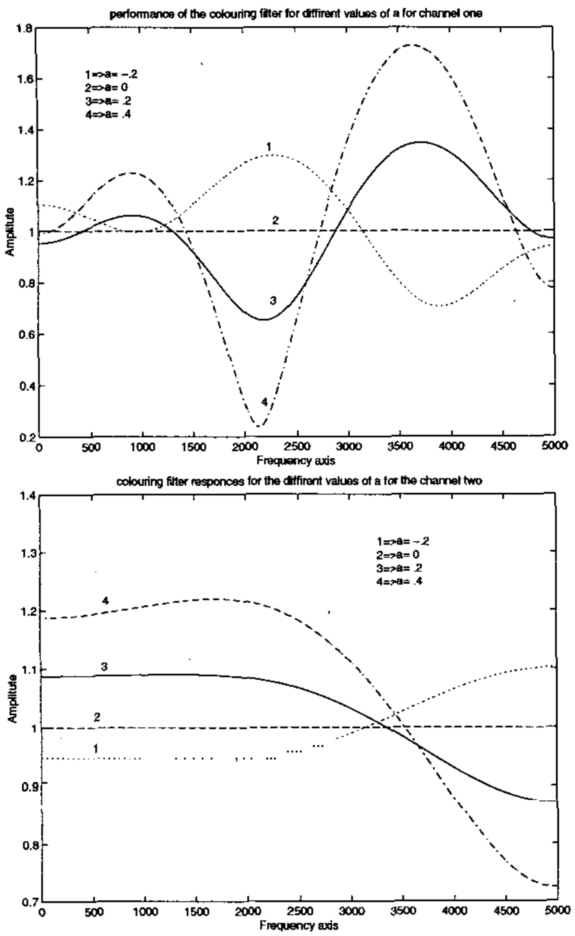

Figure **4:** Coloring filter responses for different values **of** *a*

a	TQ,R=5	MQ,R=5	TQ,R=1	MQ,R=1
-0.2	.00896	.0087	0.93820	0.91383
-0.1	.00791	.00785	0.82131	0.81616
0.0	.00725	.00725	0.74737	0.74737
0.1	.00697	.00692	0.71262	0.70793
0.2	.00717	.00694	0.72689	0.70615
0.3	.00814	.00749	0.81852	0.76041
0.4	.01051	.00887	1.04947	0.90297
0.5	.01573	.01162	1.56138	1.192888
0.6	.02763	.01694	2.734500	1.7691
0.7	.05892	.02866	5.83061	3.07035
0.8	.16733	.06363	16.60658	7.02726

Table 1: Theoretical error variances for $R = 5$ and $R = 1$

R	TQ	MTQ	SQ	MSQ
1	0.72689	0.70615	0.84035	0.83038
2	0.24296	0.23558	0.2711411	0.27047
3	0.07693	0.074535	0.08649	0.0862
4	0.02370	0.02295	0.02576	0.02569
5	0.00717	0.00694	0.00764	0.00762
6	0.00214	0.00207	0.00206	0.0020
7	0.00064	0.00062	0.00059	0.0005

Table **2:** Theoretical error variances for $a = \mathbf{0.2}$

Theoretical results for the allpass delay chain structure based filter hank are presented. Table(1) gives the theoretical quantization error variances(TQ) without any coloring filter and with the coloring filter (MQ), for different values of a for a hit rate R = **5** and R = 1bits/sample.

Table(2) give the theoretical quantization error variances without(TQ) and with coloring filter(MTQ), and simulated quantization error with(SQ) and without(SMQ) for different hit rates R for a specific value of a.

Here we can observe the deviation of the simulation result from the theoretical result because our assumption that the factor $(A_i(z) - I)Q_i(z)$ will not change the variance **of** the input signal is violated specially for large a like a = 0.7 (because in that case norm of the synthesis filter becomes very high) and for low bit rate R.

References

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