TWO PARAMETER UNIFORMLY ELLIPTIC STURM-LIOUVILLE PROBLEMS WITH EIGENPARAMETER DEPENDENT BOUNDARY CONDITIONS

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Abstract. We Consider the two parameter Sturm Liouville system

$$-y_1^{''} + q_1 y_1 = (\lambda r_{11} + \mu r_{12}) y_1 \text{ on } [0,1] \tag{1}$$

with the boundary conditions

$$\frac{y_1^{'}(0)}{y_1(0)} = \cot\alpha_1 \text{ and } \frac{y_1^{'}(1)}{y_1(1)} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1}$$

and

$$-y_2^{''} + q_2 y_2 = (\lambda r_{21} + \mu r_{22}) y_2 \text{ on } [0,1] \tag{2}$$

with the boundary conditions

$$\frac{y_2^{'}(0)}{y_2(0)} = \cot\alpha_2 \text{ and } \frac{y_2^{'}(1)}{y_2(1)} = \frac{a_2\mu + b_2}{c_2\mu + d_2},$$

subject to the uniform left definiteness and uniform ellipticity conditions; where q_i and r_{ij} are continuous real valued functions on $[0,1]$, the angle α_i is in $[0,\pi)$ and a_i, b_i, c_i, d_i are real numbers with $\delta_i = a_i d_i - b_i c_i > 0$ and $c_i \neq 0$ for $i,j = 1,2$. Results are given on asymptotics, oscillation of eigenfunctions and location of eigenvalues.

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1. Introduction

The Sturm-Liouville theory associated with the ordinary differential equation

$$-y^{''} + qy = \lambda r y \quad \text{on} \quad [0,1]$$

with $q,\ r$ continuous and $r > 0$ subject to the boundary conditions

$$y(0)\cos\alpha = y^{'}(0)\sin\alpha \quad \text{and} \quad y(1)\cos\beta = y^{'}(1)\sin\beta,$$

deals with existence, uniqueness, oscillation of eigenfunctions and completeness. Classical results about these are well known. The study of the above one parameter equation subject to the parameter dependent boundary conditions

$$\frac{y^{'}(0)}{y(0)} = \frac{a_0\lambda + b_0}{c_0\lambda + d_0} \text{ and } \frac{y^{'}(1)}{y(1)} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1}$$

have been investigated and results about the existence and oscillation theory are known [6]; there are also parameter dependence results and asymptotic expansions [6]. Klein's oscillation theorem for the equations (1) and (2) subject to the fixed boundary conditions

$$y_i(0)\cos\alpha_i = y_i^{'}(0)\sin\alpha_i \quad \text{and} \quad y_i(1)\cos\beta_i = y_i^{'}(1)\sin\beta_i,$$

and under the right definiteness condition $\det\begin{pmatrix} r_{11}(x) & r_{12}(x) \\ r_{21}(x) & r_{22}(x)\end{pmatrix} > 0$ for every $x \in [0,1]$ states that, for each nonnegative integer pair (m,n), there is a unique eigenvalue $(\lambda,\mu) \in \mathbb{R}^2$ and (up to scalar multiples) a unique pair of eigenfunctions (y_1, y_2) such that y_1 has m zeros and y_2 has n zeros in $(0, 1)$. A special case was proved by Klein, the general one (for continuous coefficients) by Ince [9].

The authors in [1] started the discussion of (1) and (2) subject to parameter dependent boundary conditions. Apart from the Sturm-Liouville theory, there are results on asymptotics and location of eigenvalues. The extension of (1) and (2) to several parameters with parameter independent or dependent boundary conditions were discussed by several authors, for example see [2, 11] and the references therein. Binding and Browne in [3] and [4] analyzed the abstract problem

$$(T_m - \sum_{n=1}^{k} \lambda_n V_{mn})x_m = 0 \quad \text{for} \quad (\lambda_1, \lambda_2, \cdots \lambda_k) \in \mathbb{R}^k \quad \text{and} \quad m = 1, 2, \cdots k,$$

under several definiteness conditions and provided an abstract Klein' s oscillation theorem. Here the operators T_m are selfadjoint and bounded below with compact resolvent and V_{mn} are bounded and selfadjoint.

In [1] the system (1) and (2) were studied under the Uniform Right Definiteness condition which is defined in Definition 1.1 below. There it was shown that each eigenvalue has a unique oscillation count (m, n) where m and n are the number of zeros of the corresponding eigenfunctions y_1 and y_2 respectively. In addition, there is an oscillation theorem [1, Theorem 4.4] which addresses the extent to which the converse is true.

We begin by stating the definiteness conditions, for which formulation of the problems in terms of Hilbert space operators is essential. In section 2 we prove the oscillation theorem in the Uniform Left Definite (ULD) case. As in [1], this result depends heavily on the asymptotic nature of the zeroth eigencurves of (1) and (2). In section 3, we remove the ULD assumption and retain only the Uniform Ellipticity (UE). The emphasis here is on finding the location of the eigenvalues coming out of the intersection of the first and second equation eigencurves. The bounded sets on which these eigenvalues are located arise from the study of the eigencurves of another system which is also explored.

The operator equivalent forms of (1) and (2) are as follows. Let AC be the subspace of $L^2[0, 1]$ consisting of absolutely continuous functions. Define linear functionals P_j and Q_j for $j = 1, 2$ on AC by

$$P_j(y) = b_j y(1) - d_j y^{'}(1), \quad Q_j(y) = a_j y(1) - c_j y^{'}(1).$$

Consider, the Hilbert space $L^2[0, 1] \oplus \mathbb{C}$ which has the inner product

$$< Y_1, Y_2 > = \int_0^1 y_1 \overline{y_2} \; + \alpha \overline{\beta}$$

where $Y_1 = \begin{pmatrix} y_1 \\ \alpha \end{pmatrix}$ and $Y_2 = \begin{pmatrix} y_2 \\ \beta \end{pmatrix}$ are in $L^2[0, 1] \oplus \mathbb{C}$. Now define the unbounded operators T_j for $j = 1, 2$ and the bounded operators V_{jk} for $j, k = 1, 2$ on $L^2[0, 1] \oplus \mathbb{C}$ by:

$$D(T_j) = \left\{ \begin{pmatrix} y \\ -Q_j(y) \end{pmatrix} \in L^2[0, 1] \oplus \mathbb{C} : y, y^{'} \in \text{AC}, \; -y^{''} + q_j y \in L^2[0, 1], \; y^{'}(0) = \cot \alpha_j y(0) \right\}$$

$$T_j \begin{pmatrix} y \\ -Q_j(y) \end{pmatrix} = \begin{pmatrix} -y^{''} + q_j y \\ P_j(y) \end{pmatrix} \text{ for } \begin{pmatrix} y \\ -Q_j(y) \end{pmatrix} \in D(T_j) \text{ and } V_{jk} \begin{pmatrix} y \\ \alpha \end{pmatrix} = \begin{pmatrix} r_{jk} y \\ \delta_{jk} \alpha \end{pmatrix}$$

where δ_{jk} is the Kronecker delta. Now (1) and (2) are equivalent to

$$(T_j - (\lambda V_{j1} + \mu V_{j2})) \begin{pmatrix} y_j \\ \alpha \end{pmatrix} = 0 \;\text{ for }\; \begin{pmatrix} y_j \\ \alpha \end{pmatrix} \in D(T_j), \; j = 1, 2.$$

For $Y = (Y_1, Y_2) \in (L^2[0, 1] \oplus \mathbb{C}) \times (L^2[0, 1] \oplus \mathbb{C})$, we set

$$t_j(Y) =< T_j(Y_j), Y_j >, v_{jk}(Y) =< V_{jk}(Y_j), Y_j >, \delta_0(Y) = \det[v_{jk}(Y)],$$

and $\delta_{0jk}(Y)$ = the cofactor of $v_{jk}(Y)$ in $\delta_0(Y)$. Let U be the unit sphere in $L^2[0,1] \oplus \mathbb{C}$.

Definition 1.1. *The basic definiteness assumptions used for the study of multiparameter Sturm-Liouville problems are defined as follows.*

(1) ***Uniform Right Definiteness(URD)*** *: For some $\gamma > 0$ and for each*

$$Y = (Y_1, Y_2) \in U \times U, \quad \delta_0(Y) \geq \gamma.$$

(2) ***Uniform Ellipticity(UE)*** *: For some $\gamma > 0$, for each $j, k = 1, 2$, and for each*

$$Y = (Y_1, Y_2) \in U \times U, \quad \delta_{0jk}(Y) \geq \gamma.$$

(3) ***Uniform Left Definiteness(ULD)*** *: UE holds and for some $\gamma > 0$ and for each $j = 1, 2$ and $Y = (Y_1, Y_2) \in U \times U$ with $Y_j \in D(T_j)$, $\quad t_j(Y) \geq \gamma$.*

2. Uniform Left Definiteness

In this section we discuss (1) and (2) subject to the uniform left definiteness condition. Since UE holds, $(-1)^{i+j} r_{ij}(x) > 0$ for $0 \leq x \leq 1$ and $i, j = 1, 2$ [2, Lemma 4.1]. The case when $\delta_0(u) > 0$ for all $u \in U$ is studied in [1] and the case when $\delta_0(u) < 0$ for all $u \in U$ is similar. Hence we shall consider the case when $\delta_0(u)$ changes sign for $u \in U$.

Eigencurves of the system (1) and (2) : Consider (2). If we fix λ and take μ as the parameter, equation (2) is then a one parameter Sturm-Liouville problem with one boundary condition depending on the parameter. There exist eigenvalues $\mu_{20}(\lambda) < \mu_{21}(\lambda) < \cdots$ with corresponding eigenfunctions $y_{20}, y_{21}, \ldots$. Also there exists a natural number N_2 depending on λ such that y_{2n} has n zeros for $n \leq N_2$ and $n-1$ zeros for $n > N_2$ in $(0,1)$, where $\mu_{2N_2} < -\frac{d_2}{c_2} \leq \mu_{2(N_2+1)}$. Moreover $\mu_{2n}(\lambda)$ are continuous strictly increasing functions of λ [1, Lemma 2.1, Theorem 3.1], [6, Theorem 3.1]. The graph of $\mu_{2n}(\lambda)$ for $\lambda \in \mathbb{R}$ are called the second equation eigencurves and we denote it by μ_{2n}. Similarly in (1), by fixing μ and taking λ as the parameter we get eigenvalues $\lambda_{10}(\mu) < \lambda_{11}(\mu) < \cdots$ with eigenfunctions $y_{10}, y_{11}, \ldots$. Also there exists a natural number N_1 depending on μ such that y_{1m} has m zeros for $m \leq N_1$ and $m-1$ zeros for $m > N_1$ in $(0,1)$ where $\lambda_{1N_1} < -\frac{d_1}{c_1} \leq \lambda_{1(N_1+1)}$ [6, Theorem 3.1]. Since $\lambda_{1m}(\mu)$ are continuous strictly increasing functions of μ, its inverse exists, say $\mu_{1m}(\lambda)$ and satisfies $\mu_{10}(\lambda) > \mu_{11}(\lambda) > \mu_{12}(\lambda) \cdots$. We call the graph of $\mu_{1m}(\lambda)$ as the first equation eigencurves and it is denoted by μ_{1m}.

The pair (λ, μ) is called an eigenvalue if there exist functions y_1 and y_2 such that (λ, μ, y_1, y_2) satisfies the system. The oscillation count of (λ, μ) is the pair (m, n), $m, n \geq 0$, where m and n are the number of zeros of y_1 and y_2 respectively in $(0,1)$.

It is well known that in the uniform left definite case, the first and second equation eigencurves intersect exactly twice. This follows from Theorem 3.3 of [3] and the subsequent discussion there. The intersection points are the eigenvalues of the system. So there are countably many eigenvalues for the system. With respect to the point $(-\frac{d_1}{c_1}, -\frac{d_2}{c_2})$ we consider the following quadrants:

$$Q_1 = \left\{(x,y) : x \geq \frac{-d_1}{c_1}, y \geq \frac{-d_2}{c_2}\right\} \quad \text{and} \quad Q_2 = \left\{(x,y) : x < \frac{-d_1}{c_1}, y \geq \frac{-d_2}{c_2}\right\},$$

$$Q_3 = \left\{(x,y) : x < \frac{-d_1}{c_1}, y < \frac{-d_2}{c_2}\right\} \quad \text{and} \quad Q_4 = \left\{(x,y) : x \geq \frac{-d_1}{c_1}, y < \frac{-d_2}{c_2}\right\}.$$

Let $(m,n) = k$. We denote the two intersection points of μ_{1m} and μ_{2n} as $(\lambda_1^k, \mu) = (\lambda_1^k, \mu_{1m}(\lambda_1^k))$ which is always in Q_3 and $(\lambda_2^k, \mu) = (\lambda_2^k, \mu_{1m}(\lambda_2^k))$ which is in Q_3 for the particular case $m = n = 0$, where $\lambda_1^k < \lambda_2^k$.

Lemma 2.1. *The graph of $\mu_{10}(\lambda)$ always lies on the left of the vertical line $\lambda = \frac{-d_1}{c_1}$ and $\lim_{\lambda \to \frac{-d_1}{c_1}} \mu_{10}(\lambda) = \infty$. On the other hand $\mu_{20}(\lambda) < \frac{-d_2}{c_2}$ for $\lambda \in \mathbb{R}$ and $\lim_{\lambda \to \infty} \mu_{20}(\lambda) = \frac{-d_2}{c_2}$.*

PROOF: For any given λ, the value of $\mu_{20}(\lambda)$ is obtained from the point of intersection of the leftmost branch B_0 of $f(\mu) = \cot\theta(1,\lambda,\mu)$ and the hyperbola $g(\mu) = \frac{a_2\mu+b_2}{c_2\mu+d_2}$ [1, Lemma 2.1], [6, Theorem 3.1].

The hyperbola $\nu = g(\mu)$ has horizontal asymptote $\nu = \frac{a_2}{c_2}$ and vertical asymptote $\mu = \frac{-d_2}{c_2}$. Since $\cot\theta(1,\lambda,\mu)$ decreases continuously on B_0, its intersection with the hyperbola must be on the left of $\mu = \frac{-d_2}{c_2}$. It follows that $\mu_{20}(\lambda) < \frac{-d_2}{c_2}$ and since μ_{20} is increasing, let $\lim_{\lambda\to\infty}\mu_{20}(\lambda) = l$. To show that $l = \frac{-d_2}{c_2}$, it is enough to show that $\lim_{\lambda\to\infty}\cot\theta(1,\lambda,\mu_{20}(\lambda)) = \infty$.

Choose $\eta > 0$ such that $\eta < \pi - \alpha_2$ and $\eta \le \frac{\pi}{2}$ where $\theta(0,\lambda,\mu_{20}(\lambda)) = \alpha_2 \in [0,\pi)$. Consider $S = \{x \in [0,1] : \eta \le \theta(x,\lambda,\mu_{20}(\lambda)) \le \pi - \eta\}$. By choosing η small enough we can assure that S is nonempty. Let x_0 be the infimum of S. Choose δ such that $\pi - \eta < \delta \le \pi$. For $x \in S$ and $\lambda > 0$, since $\sin\theta \ge \sin\eta$, we have

$$\begin{aligned}
\theta'(x,\lambda,\mu_{20}(\lambda)) &= \cos^2\theta + (\lambda r_{21} + \mu_{20}(\lambda) r_{22} - q_2)\sin^2\theta \\
&< 1 + (\lambda \sup_{x\in[0,1]} r_{21}(x) + l \sup_{x\in[0,1]} r_{22}(x))\sin^2\eta + \sup_{x\in[0,1]} |q_2(x)| \\
&< \frac{\eta-\delta}{1-x_0} \text{ for sufficiently large } \lambda.
\end{aligned}$$

Note that $\frac{\eta-\delta}{1-x_0}$ is the slope of the line segment h joining the points (x_0,δ) and $(1,\eta)$. Hence $(\theta-h)'(x) < 0$ for $x \in S$. This together with $(\theta-h)(x_0) < 0$ implies that

$$\theta(x,\lambda,\mu_{20}(\lambda)) < h(x) \text{ for } x \in [x_0,1]. \tag{3}$$

Let x_1 be the largest number such that $[x_0,x_1] \subset S$. Since θ is continuous in x, such a number exists. From (3), we get $x_1 \ne 1$. For any $x > x_1$, we have $x \notin S$, since θ is decreasing for all $x \in S$. Therefore $S = [x_0,x_1]$ and $\theta(x,\lambda,\mu_{20}(\lambda)) < \eta$ for $x > x_1$. In particular $\theta(1,\lambda,\mu_{20}(\lambda)) < \eta$. Since $\alpha_2 = \theta(0,\lambda,\mu_{20}(\lambda)) \ge 0$ and $\theta' > 0$ for $\theta \equiv 0(\bmod\ \pi)$, we know that $\theta(x,\lambda,\mu_{20}(\lambda))$ cannot be negative for $x \in [0,1]$, for otherwise θ' will have to be negative at the point where θ becomes zero. Hence $\theta(1,\lambda,\mu_{20}(\lambda)) \ge 0$. Since $\eta > 0$ is arbitrary, we are done.

Proceeding as above using the first differential equation, we get $\lambda_{10}(\mu) < \frac{-d_1}{c_1}$ and $\lim_{\mu\to\infty}\lambda_{10}(\mu) = \frac{-d_1}{c_1}$. Since $\mu_{10}(\lambda)$ is the inverse of $\lambda_{10}(\mu)$, the graph of $\mu_{10}(\lambda)$ lies on the left of $\lambda = \frac{-d_1}{c_1}$ and $\lim_{\lambda\to\frac{-d_1}{c_1}}\mu_{10}(\lambda) = \infty$. ☐

THEOREM 2.2. ***(Oscillation theorem)**: Let $M_1 = min\left\{m :\ (\lambda_2^{(m,0)},\mu) \in Q_4 \text{ and } (\lambda_2^{(m,1)},\mu) \in Q_1\right\}$ and $M_2 = min\left\{n :\ (\lambda_2^{(0,n)},\mu) \in Q_2 \text{ and } (\lambda_2^{(1,n)},\mu) \in Q_1\right\}$. With the exceptions below, each oscillation count corresponds to two eigenvalues.*

1. *For $m \ge M_1$ and $n \ge M_2$ each of the oscillation counts $(m,0)$ and $(0,n)$ corresponds to exactly three eigenvalues.*
2. *For $m < M_1$ and $n < M_2$, the oscillation count $k = (m,n)$ corresponds to at least two eigenvalues and at most five eigenvalues.*

PROOF: We have, $\mu_{1m}(\lambda)$ has the oscillation count m when $\lambda < \frac{-d_1}{c_1}$ and $m-1$ when $\lambda \ge \frac{-d_1}{c_1}$. $\mu_{2n}(\lambda)$ has the oscillation count n when $\mu_{2n}(\lambda) < \frac{-d_2}{c_2}$ and $n-1$ when $\mu_{2n}(\lambda) \ge \frac{-d_2}{c_2}$. Hence the oscillation count of the eigenvalue (λ_i^k,μ) is $(m-1,n-1)$(respectively $(m,n-1),(m,n)$, and $(m-1,n)$) if $(\lambda_i^k,\mu) \in Q_1$ (respectively $Q_2,\ Q_3,\ Q_4$).

Let Γ_i^k where $k = (m,n)$ denote the curvilinear cell defined by the vertices $(\lambda_i^{(m,n)},\mu)$, $(\lambda_i^{(m+1,n)},\mu)$, $(\lambda_i^{(m+1,n+1)},\mu)$, and $(\lambda_i^{(m,n+1)},\mu)$, for $i = 1,2$, and the corresponding eigencurve sections as edges. Note that Γ_1^k for any $k = (m,n)$ always lie in Q_3. Since the repeated oscillation counts must corresponds to the vertices of some cell, a given oscillation count $k = (m,n)$ corresponds to (i) the eigenvalue (λ_1^k, μ) from Γ_1^k, and (ii) at least one and at most four eigenvalues from Γ_2^k. Hence the minimum number of occurrences of an oscillation count should be two.

(1) For $m \geq M_1$, the oscillation count $(m, 0)$ occurs thrice, once each in Q_3, Q_4 and Q_1 corresponding to $(\lambda_1^{(m,0)}, \mu)$, $(\lambda_2^{(m+1,0)}, \mu)$ and $(\lambda_2^{(m+1,1)}, \mu)$. Similarly when $n \geq M_2$, the oscillation count $(0, n)$ corresponds to $(\lambda_1^{(0,n)}, \mu) \in Q_3$, $(\lambda_2^{(0,n+1)}, \mu) \in Q_2$ and $(\lambda_2^{(1,n+1)}, \mu) \in Q_1$.

(2) For $m \geq M_1$ and $n \geq M_2$, the cell $\Gamma_2^{(m,n)}$ is contained in Q_1; so when $m < M_1$ and $n < M_2$, the oscillation count (m, n) corresponds to at least two eigenvalues and at most five eigenvalues. □

REMARK 2.3. *Given an oscillation count, it may be possible that it corresponds to five eigenvalues. It may or may not happen depending on the problem. If it happens there is only one such case. There are finitely many cases where an oscillation count corresponds to four eigenvalues. However if there is an oscillation count which corresponds to five eigenvalues, then no oscillation count corresponds to four eigenvalues. There are always infinitely many cases where an oscillation count corresponds to three eigenvalues. Similarly there are always infinitely many cases where an oscillation count corresponds to two eigenvalues. There is no oscillation count which corresponds to one eigenvalue.*

THEOREM 2.4. *Let $m_1 < m_2 < ... < m_k$ be positive integers such that $\mu_{1m_1}, \mu_{1m_2} ..., \mu_{1m_k}$ intersects the line $\mu = \rho\lambda + c$ $(\rho \leq 0)$ at $\lambda_1, \lambda_2 ...., \lambda_k$ respectively. Then $\lambda_1 < \lambda_2 < ... < \lambda_k$ and for $\lambda \geq \lambda_k$ and $m \leq m_k$ we have $\mu_{1m}(\lambda) \geq \rho\lambda + c$.*

PROOF: For $m_i < m_j$, $1 \leq i < j \leq k$, since $\mu_{1m_i}(\lambda) > \mu_{1m_j}(\lambda)$, we have $\lambda_i < \lambda_j$. For, if $\lambda_i \geq \lambda_j$, then μ_{1m_i} and μ_{1m_j} intersect at some λ_{ij} and $\mu_{1m_i}(\lambda) \leq \mu_{1m_j}(\lambda)$ for $\lambda \geq \lambda_{ij}$, which is not possible.
Now for $\lambda \geq \lambda_k$ and $m \leq m_k$,

$$\mu_{1m}(\lambda) \geq \mu_{1m_k}(\lambda) \geq \mu_{1m_k}(\lambda_k) = \rho\lambda_k + c \geq \rho\lambda + c.$$

□

We state an analogue of the previous theorem for the eigencurves μ_{2n}. It can be proved in a similar way.

THEOREM 2.5. *Let $n_1 < n_2 < ... < n_k$ be positive integers such that $\mu_{2n_1}, \mu_{2n_2}, ..., \mu_{2n_k}$ intersects the line $\mu = \rho\lambda + c$ $(\rho \leq 0)$ at $\lambda_1, \lambda_2, ..., \lambda_k$ respectively. Then $\lambda_1 > \lambda_2 > ... > \lambda_k$ and for $\lambda \leq \lambda_k$ and $n \leq n_k$ we have $\mu_{2n}(\lambda) \leq \rho\lambda + c$.*

THEOREM 2.6. *The functions $\mu_{1m}(\lambda)$ and $\mu_{2n}(\lambda)$ and the eigenfunctions y_{1m} and y_{2n} are analytic in λ*

PROOF: Consider the operator equivalent form of (2),

$$(T_2 - (\lambda V_{21} + \mu V_{22})) \begin{pmatrix} y \\ \alpha \end{pmatrix} = 0 \quad \text{for} \quad \begin{pmatrix} y \\ \alpha \end{pmatrix} \in D(T_2).$$

$$\text{ie,} \quad \begin{pmatrix} \frac{-y'' + (q_2 - \lambda r_{21})y}{r_{22}} \\ P_2(y) \end{pmatrix} = \mu \begin{pmatrix} y \\ \alpha \end{pmatrix}$$

Define the operator $T_\lambda : D(T_2) \to L^2[0, 1] \oplus \mathbb{C}$ by

$$T_\lambda \begin{pmatrix} y \\ \alpha \end{pmatrix} = \begin{pmatrix} \frac{-y'' + (q_2 - \lambda r_{21})y}{r_{22}} \\ P_2(y) \end{pmatrix}.$$

The linear operator T_λ is a selfadjoint [2, Lemma 2.1] holomorphic family of type (A) [10, VII 2:1] defined for λ in any neighborhood of an interval I of the real axis. Let $\beta \in P(T_\lambda)$, the resolvent of T_λ. Then $(T_\lambda - \beta)^{-1} : L^2[0, 1] \oplus \mathbb{C} \to D(T_2)$ has the form

$$(T_\lambda - \beta)^{-1} \begin{pmatrix} v \\ \gamma \end{pmatrix} = \begin{pmatrix} Gv \\ -Q_2(y) \end{pmatrix} \text{ and is compact [8, II Theorem 6.9].}$$

Here $y = Gv$, where $G : L^2[0,1] \to L^2[0,1]$ is given by $(Gv)(x) = \int_0^1 g(x,t)v(t)dt$;

$$g(x,t) = \begin{cases} c^{-1}y_0(x)y_1(t) & 0 \le x \le t \le 1 \\ c^{-1}y_0(t)y_1(x) & 0 \le t \le x \le 1 \end{cases}$$

where c is the Wronskian of y_0 and y_1, and y_0 is a solution of $\frac{-y'' + (q_2 - \lambda r_{21})y}{r_{22}} - \beta y = 0$, $y'(0) = \cot\alpha_2 y(0)$ and y_1 is a solution of $\frac{-y'' + (q_2 - \lambda r_{21})y}{r_{22}} - \beta y = 0$, $P_2(y) + \beta Q_2(y) = \gamma$. It follows from Theorem 3.9 [10, VII 3:5] that the eigenvalues $\mu_{1m}(\lambda)$ and the eigenfunctions $\begin{pmatrix} y_{1m} \\ \alpha \end{pmatrix}$ of T_λ are analytic in λ. Consequently y_{1m} is also analytic.

In a similar way, considering the operator equivalent form of (1), we arrive at the conclusion that the eigenvalues $\lambda_{2n}(\mu)$ and the eigenfunctions y_{2n} are analytic. Here we take the operator $T_\mu : D(T_1) \to L^2[0,1] \oplus \mathbb{C}$ as

$$T_\mu \begin{pmatrix} y \\ \alpha \end{pmatrix} = \begin{pmatrix} \frac{-y'' + (q_1 - \mu r_{12})y}{r_{11}} \\ P_1(y) \end{pmatrix}.$$

Being the inverse of λ_{2n}, the function $\mu_{2n}(\lambda)$ is also analytic. □

The expression for the first derivatives of $\mu_{1m}(\lambda)$ and $\mu_{2n}(\lambda)$ with respect to λ are derived in [1, Theorem 3.1]. The second derivatives are given below.

THEOREM 2.7. *Assume that* $\frac{\partial^2 y''}{\partial\lambda^2} = \left(\frac{\partial^2 y}{\partial\lambda^2}\right)''$ *where* $' = \frac{\partial}{\partial x}$. *The second derivatives of* $\mu_{1m}(\lambda)$ *and* $\mu_{2n}(\lambda)$ *with respect to* λ *are given by*

$$\frac{d^2\mu_{1m}(\lambda)}{d\lambda^2} = (r_{12}(y_{1m}))^{-1}\left\{\frac{2\delta_1}{(c_1\lambda + d_1)^3}\left[c_1(y_{1m}(1)^2 - (c_1\lambda + d_1)y_{1m}(1)\frac{\partial y_{1m}(1)}{\partial\lambda}\right]\right.$$
$$\left. -2\frac{d\mu_{1m}(\lambda)}{d\lambda}\int_0^1 r_{12}y_{1m}\frac{\partial y_{1m}}{\partial\lambda} - 2\int_0^1 r_{11}y_{1m}\frac{\partial y_{1m}}{\partial\lambda}\right\}$$

$$\frac{d^2\mu_{2n}(\lambda)}{d\lambda^2} = \left[\frac{\delta_2(y_{2n}(1))^2}{(c_2\mu_{2n} + d_2)^2} + r_{22}(y_{2n})\right]^{-1}\left\{\frac{2\delta_2}{(c_2\mu_{2n} + d_2)^3}\left[c_2(y_{2n}(1))^2\left(\frac{d\mu_{2n}(\lambda)}{d\lambda}\right)^2\right.\right.$$
$$\left.\left. -(c_2\mu_{2n} + d_2)y_{2n}(1)\frac{d\mu_{2n}(\lambda)}{d\lambda}\frac{\partial y_{2n}(1)}{\partial\lambda}\right] - 2\frac{d\mu_{2n}(\lambda)}{d\lambda}\int_0^1 r_{22}y_{2n}\frac{\partial y_{2n}}{\partial\lambda} - 2\int_0^1 r_{21}y_{2n}\frac{\partial y_{2n}}{\partial\lambda}\right\}$$

where $r_{i2}(y_{im}) = \int_0^1 r_{i2}y_{im}^2$ *for* $i = 1, 2$.

PROOF: Differentiation of (1) twice with respect to λ yields,

$$(4) \qquad -\frac{\partial^2 y''}{\partial\lambda^2} + q_1\frac{\partial^2 y}{\partial\lambda^2} = (r_{11} + \frac{d\mu}{d\lambda}r_{12})\frac{\partial y}{\partial\lambda} + \frac{d^2\mu}{d\lambda^2}r_{12}y + (\lambda r_{11} + \mu r_{12})\frac{\partial^2 y}{\partial\lambda^2} + (r_{11} + \frac{d\mu}{d\lambda}r_{12})\frac{\partial y}{\partial\lambda}$$

Multiplying (1) by $\frac{\partial^2 y}{\partial\lambda^2}$, (4) by y and subtracting the former from the later, we get

$$-y\frac{\partial^2 y''}{\partial\lambda^2} + y''\frac{\partial^2 y}{\partial\lambda^2} = r_{12}y^2\frac{d^2\mu}{d\lambda^2} + 2(r_{11} + \frac{d\mu}{d\lambda}r_{12})y\frac{\partial y}{\partial\lambda}$$

Integrating over [0,1],

$$(5) \qquad \left[-y\frac{\partial}{\partial x}\left(\frac{\partial^2 y}{\partial\lambda^2}\right) + y'\frac{\partial^2 y}{\partial\lambda^2}\right]_0^1 = \frac{d^2\mu}{d\lambda^2}\int_0^1 r_{12}y^2 + 2\frac{d\mu}{d\lambda}\int_0^1 r_{12}y\frac{\partial y}{\partial\lambda} + 2\int_0^1 r_{11}y\frac{\partial y}{\partial\lambda}$$

We differentiate the boundary conditions in (1) twice with respect to λ and then use it to solve the left hand side of (5) to get the expression for $\frac{d^2\mu_{1m}(\lambda)}{d\lambda^2}$. Using (2) and following similar steps we get,

$$\left[-y\frac{\partial}{\partial x}\left(\frac{\partial^2 y}{\partial\lambda^2}\right)+y^{'}\frac{\partial^2 y}{\partial\lambda^2}\right]_0^1 = \frac{d^2\mu}{d\lambda^2}\int_0^1 r_{22}y^2+2\frac{d\mu}{d\lambda}\int_0^1 r_{22}y\frac{\partial y}{\partial\lambda}+2\int_0^1 r_{21}y\frac{\partial y}{\partial\lambda}$$

Now as in the computation above, we solve the left hand side using the boundary conditions of (2) to get the required result. □

3. Uniform Ellipticity

From now on we assume only uniform ellipticity for the system (1) and (2). UE condition implies that $(-1)^{i+j}r_{ij}(x) > 0$ for $0 \le x \le 1$ and $i, j = 1, 2$ [2, Lemma 4.1]. We permit $\delta_0(u)$ to take both positive and negative values for $u \in U$. We also assume that $r_{11}(x_1)r_{22}(x_2) - r_{12}(x_1)r_{21}(x_2)$ for $(x_1, x_2) \in [0,1]\times[0,1]$ is not identically zero and changes sign. The first and second equation eigencurves μ_{1m} and μ_{2n} can be derived exactly as in the uniform left definite case and we are following the same notations. In particular notice that Lemma 2.1 and Theorem 2.6 is valid in this case as well. The eigenvalues of the system will be obtained by knowing the intersection points of μ_{1m} and μ_{2n}.

Lemma 3.1. *The operators T_j for $j = 1, 2$ are selfadjoint and bounded below with compact resolvent.*

Proof: The selfadjointness of T_j follows from [2, Lemma 2.1]. For the compactness of the resolvent of T_j, see the proof of Theorem 2.6. Now let us have a look at the eigenvalues of T_j. $T_jY = \mu Y$ implies that

$$-y^{''}+q_j y=\mu y, \qquad \frac{y^{'}(0)}{y(0)}=\cot\alpha_j, \qquad \frac{y^{'}(1)}{y(1)}=\frac{a_j\mu+b_j}{c_j\mu+d_j}.$$

Theorem 3.1 of [6] shows that the system has countable number of eigenvalues $\mu_j^0 < \mu_j^1 < \mu_j^2 < \cdots$. So the spectrum of the operator T_j is bounded below. The discussion in [10, V 3:10 Page 278] concludes that T_j is bounded below. □

Theorem 3.2. *Given $n \ge 0$, there exists an integer $N(n) \ge 0$ such that μ_{2n} intersect with μ_{1m} in at least two points if and only if $m \ge N(n)$.*

Proof: For $(\lambda, \mu) \in \mathbb{R}^2$, since the operator $T_2 - \lambda V_{21} - \mu V_{22}$ is selfadjoint and bounded below with compact resolvent [4, Lemma 1], it has countable number of eigenvalues $\rho_2^0(\lambda,\mu) \le \rho_2^1(\lambda,\mu) \le \cdots$. For each $n \ge 0$ and $\lambda \in \mathbb{R}$, since $-< V_{22}(u), u > < 0$ for all $u \in U$, there exists a unique $\mu^{2n}(\lambda)$ such that $\rho_2^n(\lambda, \mu^{2n}(\lambda)) = 0$ and

$$\left(T_2-\lambda V_{21}-\mu^{2n}(\lambda)V_{22}\right)\begin{pmatrix} y^{2n} \\ \alpha \end{pmatrix} = 0.$$

Moreover $\mu^{2n}(\lambda)$ are continuous in λ and $\mu^{20}(\lambda) \le \mu^{21}(\lambda) \le \cdots$ [4, Theorems 2,3], [5, Theorem 2.1]. We claim that $\mu^{2n}(\lambda) = \mu_{2n}(\lambda)$. Since $\mu^{20}(\lambda) \le \mu^{21}(\lambda) \le \cdots$ and $\mu_{20}(\lambda) < \mu_{21}(\lambda) < \cdots$, it suffices to show that

$$\left\{\mu^{2n}(\lambda) \;:\; n \ge 0\right\} = \{\mu_{2n}(\lambda) \;:\; n \ge 0\}$$

For $\lambda \in \mathbb{R}$, the set $\{\mu_{2n}(\lambda) \;:\; n \ge 0\}$ form a complete set of eigenvalues for the equation

$$(T_2-\lambda V_{21}-\mu V_{22})\begin{pmatrix} y \\ \alpha \end{pmatrix} = 0.$$

Since $\left\{\mu^{2n}(\lambda) \;:\; n \ge 0\right\}$ are eigenvalues of this equation, we have

$$\left\{\mu^{2n}(\lambda) \;:\; n \ge 0\right\} \subseteq \{\mu_{2n}(\lambda) \;:\; n \ge 0\}$$

The eigenvalues $\mu_{2n}(\lambda)$ satisfy the equation $(T_2 - \lambda V_{21} - \mu_{2n}(\lambda)V_{22})\begin{pmatrix} y_{2n} \\ \alpha \end{pmatrix} = 0$. Hence $\rho_2^j(\lambda, \mu_{2n}(\lambda)) = 0$ for some $j \geq 0$. But $\rho_2^j(\lambda, \mu^{2j}(\lambda)) = 0$. Therefore $\mu_{2n}(\lambda) = \mu^{2j}(\lambda)$ by the uniqueness of $\mu^{2j}(\lambda)$. Thus the other inclusion holds.

Now consider $T_1 - \lambda V_{11} - \mu V_{12}$ for $(\lambda, \mu) \in \mathbb{R}^2$. Its eigenvalues can be ordered as $\rho_1^0(\lambda, \mu) \leq \rho_1^1(\lambda, \mu) \leq \cdots$. For each $m \geq 0$ and $\mu \in \mathbb{R}$, since $- < V_{11}(u), u > < 0$ for all $u \in U$, there exists a unique $\lambda^{1m}(\mu)$ such that $\rho_1^m(\lambda^{1m}(\mu), \mu) = 0$. Then by a similar procedure as above we can prove that $\lambda^{1m} = \lambda_{1m}$. Since $\lambda_{1m}(\mu)$ is a continuous strictly increasing function of μ [1, Theorem 3.1], its inverse $\mu_{1m}(\lambda)$ exists and

$$(T_1 - \lambda V_{11} - \mu_{1m}(\lambda)V_{12})\begin{pmatrix} y_{1m} \\ \alpha \end{pmatrix} = 0 \text{ with } \rho_1^m(\lambda, \mu_{1m}(\lambda)) = 0.$$

Now we are ready to apply corollary 4.2 of [3]. We know that

$$(T_2 - \lambda V_{21} - \mu_{2n}(\lambda)V_{22})\begin{pmatrix} y_{2n} \\ \alpha \end{pmatrix} = 0 \text{ for } \lambda \in \mathbb{R} \text{ and } \mu_{2n}(\lambda).$$

It remains to solve the equation $(T_1 - \lambda V_{11} - \mu_{2n}(\lambda)V_{12})\begin{pmatrix} y \\ \alpha \end{pmatrix} = 0$ for $\lambda \in \mathbb{R}$ and $\mu_{2n}(\lambda)$. In other words the problem is to find an $m \geq 0$ such that $\rho_1^m(\lambda, \mu_{2n}(\lambda)) = 0$ for two values of λ. By corollary 4.2 of [3], given $n \geq 0$, there exists an integer $N(n) \geq 0$ such that

$$(T_1 - \lambda V_{11} - \mu_{2n}(\lambda)V_{12})\begin{pmatrix} y \\ \alpha \end{pmatrix} = 0.$$

for two values of λ say λ_1 and λ_2 with $\rho_1^m(\lambda_i, \mu_{2n}(\lambda_i)) = 0$ for $i = 1, 2$ if and only if $m \geq N(n)$. In this case, since $\rho_1^m(\lambda_i, \mu_{2n}(\lambda_i)) = 0$, we have $\mu_{2n}(\lambda_i) = \mu_{1m}(\lambda_i)$ for $i = 1, 2$. □

A similar argument will give the following result.

Theorem 3.3. *For a given $m \geq 0$, there exists an integer $M(m) \geq 0$ such that μ_{1m} intersect with μ_{2n} in at least two points if and only if $n \geq M(m)$.*

Corollary 3.4. *The nonnegative integers $M(m)$ and $N(n)$ are nonincreasing in m and n respectively and $M(m_0) = N(n_0) = 0$ for some m_0 and n_0.*

Proof: Let $n < l$. Suppose $N(n) < N(l)$. From the preceding two theorems, we have μ_{2n} intersect with μ_{1m} for $m \geq N(l)$. Fix m; where $N(n) \leq m < N(l)$. Then since μ_{1m} intersect with μ_{2k} if and only if $k \geq M(m)$, in particular μ_{1m} intersects μ_{2l}, which is a contradiction.

Given $n = 0$, there exists $N(0)$ such that μ_{2n} intersects μ_{1m} if and only if $m \geq N(0)$. Now if $m \geq N(0)$, then μ_{1m} intersects μ_{20}. So $M(m) = 0$ for $m \geq N(0)$. The other assertions are proved in a similar way. □

Thus for $m \geq m_0$, the curve μ_{1m} intersect with all μ_{2n} where $n \geq 0$, and for $n \geq n_0$ the curve μ_{2n} intersect with all μ_{1m} where $m \geq 0$.

The study of the eigencurves of the following equation will enable us to find the location of the intersection points of μ_{1m} and μ_{2n}.

$$-y_1'' + (q_1 + Q - \lambda r_{11} - \mu_{2n}(\lambda)r_{12})\, y_1 = \Omega Q y_1 \text{ on } [0, 1] \tag{6}$$

$$\frac{y_1'(0)}{y_1(0)} = \cot \alpha_1, \quad y_1(1) = 0,$$

where Q is a positive constant to be suitably chosen, Ω is a real parameter and $(\lambda, \mu_{2n}(\lambda))$ for $n = 0, 1, \cdots$, are the eigenpairs of (2). For $\lambda \in \mathbb{R}$ and $\mu_{2n}(\lambda)$, where $n \geq 0$ is fixed, the eigenvalues can be ordered as $\Omega_{0,n}^D < \Omega_{1,n}^D < \cdots$ and $\Omega_{m,n}^D(\lambda)$, m = 0,1,2, $\cdots$ are analytic in λ [11, Lemma 3.1]. We now wish to investigate the nature of the eigencurves $\Omega_{m,n}^D$. Our analysis is similar to that of Sleeman [11].

First, let us form a differential equation. Multiply (2) by y_2 and integrate over $0 \le x_2 \le 1$, substitute the value of $\mu_{2n}(\lambda)$ so obtained into (6) to get the equation:

$$\frac{d^2y_1}{dx_1^2} + (\lambda a(x_1,\lambda) - H_1(x_1,\lambda) + H_2(x_1,\lambda) + Q\Omega - Q)y_1 = 0, \tag{7}$$

where

$$\begin{aligned}
a(x_1,\lambda) &= \frac{\int_0^1 (r_{11}r_{22} - r_{12}r_{21})y_{2n}^2 dx_2}{\int_0^1 r_{22}y_{2n}^2 dx_2},\\
H_1(x_1,\lambda) &= \frac{\int_0^1 q_1 r_{22}y_{2n}^2 dx_2 + \int_0^1 (-r_{12})q_2 y_{2n}^2 dx_2}{\int_0^1 r_{22}y_{2n}^2 dx_2},\\
H_2(x_1,\lambda) &= \frac{\int_0^1 (-r_{12})y_{2n}y_{2n}'' dx_2}{\int_0^1 r_{22}y_{2n}^2 dx_2}.
\end{aligned}$$

The following asymptotic result of the eigencurve μ_{2n} is useful in providing an estimate for $H_1(x_1,\lambda) - H_2(x_1,\lambda)$. Let $K = \inf\left\{\frac{-r_{21}(x)}{r_{22}(x)} : 0 \le x \le 1\right\}$. Then K is finite and $\lim_{\lambda\to\infty} \frac{\mu_{2n}(\lambda)}{\lambda} = K$ for $n > 0$ [1, Lemma 3.4], [2, Lemma 4.5]. Now for $\lambda \in \mathbb{R}$ and $\mu_{2n}(\lambda)$, where $n > 0$, we have,

$$\begin{aligned}
H_1(x_1,\lambda) - H_2(x_1,\lambda) &= q_1(x_1) + \frac{1}{\int_0^1 r_{22}y_{2n}^2}\left[\lambda\int_0^1 -r_{12}(x_1)r_{21}y_{2n}^2 dx_2 - \int_0^1 r_{12}(x_1)r_{22}y_{2n}^2 dx_2\right]\\
&= q_1(x_1) \ + \ O(\lambda) \quad \text{as} \quad \lambda \to \infty.
\end{aligned}$$

and for large positive λ and $\mu_{20}(\lambda)$, using Lemma 2.1,we have,

$$H_1(x_1,\lambda) - H_2(x_1,\lambda) \ \le \ q_1(x_1) + L_1\lambda + L_2 \times \frac{-d_2}{c_2},$$

where L_1 and L_2 are the upper bounds for the respective terms in $H_1 - H_2$.

In the (λ,Ω) plane we take $\Omega = 0$ as abscissa and $\lambda = 0$ as ordinate and introduce the angle ϕ as the angle which a ray through the origin makes with the positive λ -axis.

Define $G = -\sup_{(x_1,\lambda)\in[0,1]\times(-\infty,\infty)} \frac{a(x_1,\lambda)}{Q}$, and $g = -\inf_{(x_1,\lambda)\in[0,1]\times(-\infty,\infty)} \frac{a(x_1,\lambda)}{Q}$. Then $G < 0$ and $g > 0$, since $r_{11}r_{22} - r_{12}r_{21}$ changes sign in $[0,1]\times[0,1]$. Let $\phi_1 = \tan^{-1} G$, $\phi_1^* = \tan^{-1} g$, $\phi_2 = \pi + \phi_1^*$, $\phi_2^* = \pi + \phi_1$, where the principal branch of the inverse tangent is taken. Clearly $-\frac{\pi}{2} < \phi_1 < 0 < \phi_1^* < \frac{\pi}{2} < \phi_2^* < \pi < \phi_2 < \frac{3\pi}{2}$.

THEOREM 3.5. *If* $\phi_1^* \le \phi \le \frac{\pi}{2}$ *then in the* (λ,Ω) *-plane a straight line through the origin with slope* $\tan\phi$ *cuts each curve* $\Omega_{m,n}^D$ *in precisely one point* $(\lambda(\phi), \Omega_{m,n}^D(\phi))$*, for* $m = 0,1,2,\cdots$ *and* $\Omega_{0,n}^D(\phi) < \Omega_{1,n}^D(\phi) < \cdots$ *and* $\lim_{m\to\infty} \Omega_{m,n}^D(\phi) = \infty$.

PROOF: Consider the case $\phi_1^* \le \phi < \frac{\pi}{2}$. Since $\tan\phi = \frac{\Omega}{\lambda}$, we have from (7),

$$\frac{d^2y_1}{dx_1^2} + (\lambda Q F_\phi(x_1,\lambda) - H_1(x_1,\lambda) + H_2(x_1,\lambda) - Q)y_1 = 0.$$

where $F_\phi(x_1,\lambda) = \tan\phi + \frac{a(x_1,\lambda)}{Q}$. Since $\tan\phi_2^* \le -\frac{a(x_1,\lambda)}{Q} \le \tan\phi_1^*$, it follows that $F_\phi(x_1,\lambda) \ge 0$ and does not vanish identically for all $x_1 \in [0,1]$ and $\phi_1^* \le \phi < \frac{\pi}{2}$. Introduce the Prüfer transformations,

$$\begin{aligned}
y_1(x_1,\lambda,\lambda\tan\phi) &= r(x_1,\lambda,\phi)\sin\theta(x_1,\lambda,\phi)\\
y_1'(x_1,\lambda,\lambda\tan\phi) &= r(x_1,\lambda,\phi)\cos\theta(x_1,\lambda,\phi)
\end{aligned}$$

Then $\theta(x_1,\lambda,\phi)$ is the solution of the initial value problem,

$$\theta'(x_1,\lambda,\phi) = \cos^2\theta(x_1,\lambda,\phi) + [\lambda Q F_\phi(x_1,\lambda) - H_1(x_1,\lambda) + H_2(x_1,\lambda) - Q]\sin^2\theta(x_1,\lambda,\phi), \ \ \theta(0,\lambda,\phi) = \alpha_1.$$

We seek values of λ so that $\theta(1,\lambda,\phi) = m\pi + \pi$. If we take Q sufficiently large and positive and argue as in the proof of [7, VIII Theorem 2.1], we get $\theta(1,0,\phi) < \pi$.

Claim: $\theta(1,\lambda,\phi)$ is a strictly increasing function of λ.

Differentiate (6) with respect to λ, multiply the result by y_1 and substitute

$$\frac{d\mu_{2n}(\lambda)}{d\lambda} = -\int_0^1 r_{21}y_{2n}^2 \left[\frac{(a_2d_2 - b_2c_2)(y_{2n}(1))^2}{(c_2\mu_{2n}(\lambda)+d_2)^2} + \int_0^1 r_{22}y_{2n}^2\right]^{-1} \quad [1, \text{Theorem } 3.1]$$

and $\mu_{2n}(\lambda)$ from (6). Integration of this with respect to x_1 gives

$$y_1'(1,\lambda,\lambda\tan\phi)\frac{\partial y_1(1,\lambda,\lambda\tan\phi)}{\partial\lambda} - y_1(1,\lambda,\lambda\tan\phi)\frac{\partial y_1'(1,\lambda,\lambda\tan\phi)}{\partial\lambda}$$

$$= \left[\frac{(a_2d_2 - b_2c_2)(y_{2n}(1))^2}{(c_2\mu_{2n}(\lambda)+d_2)^2} + \int_0^1 r_{22}y_{2n}^2 dx_2\right]^{-1} \left\{\int_0^1 r_{22}y_{2n}^2 dx_2 \int_0^1 QF_\phi(x_1,\lambda)y_1^2(x_1,\lambda,\lambda\tan\phi)dx_1 \right.$$
$$\left. + \frac{(a_2d_2 - b_2c_2)(y_{2n}(1))^2}{(c_2\mu_{2n}(\lambda)+d_2)^2}\left[\int_0^1 r_{11}y_1^2 dx_1 + \int_0^1 Q\tan\phi y_1^2 dx_1\right]\right\}$$

The left hand side is $(r(1,\lambda,\phi))^2\frac{d\theta(1,\lambda,\phi)}{d\lambda}$ and the right hand side is positive. Hence $\theta(1,\lambda,\phi)$ is strictly increasing in λ. Furthermore $\theta(1,\lambda,\phi) \to \infty$ as $\lambda \to \infty$. This follows on taking Q sufficiently large, using the estimate for $H_1(x_1,\lambda) - H_2(x_1,\lambda)$ and arguing as in the proof of Theorem 2.1 [7, VIII]. Thus the equation $\theta(1,\lambda,\phi) = m\pi + \pi$ for $m = 0,1,2\cdots$ has a unique solution. Since $\theta(1,0,\phi) < \pi$, and θ is increasing in λ, these solutions form a strictly increasing sequence of positive numbers which tends to infinity as $m \to \infty$. Hence the theorem follows in this case.

Let $\phi = \frac{\pi}{2}$. Clearly $\Omega_{m,n}^D$ cuts the vertical axis in precisely one point. If $\Omega_{m,n}^D(\phi) \leq 0$ then $\Omega_{m,n}^D$ cuts some lines through the origin with slope $\tan\phi'$, $\phi_1^* \leq \phi' < \frac{\pi}{2}$, where $\Omega_{m,n}^D(\phi') \leq 0$; which is not possible. □

Theorem 3.6. *For all $\lambda \in (-\infty,\infty)$ the eigencurve $\Omega_{m,n}^D$, $m = 0,1,2,\cdots$ lies in the sector $\phi_1 < \phi < \phi_2$. Furthermore given any $\epsilon \in (0,\frac{\pi}{2})$, there is a positive number $N_{m,n}(\epsilon)$ such that for $\lambda \geq N_{m,n}(\epsilon)$, $\Omega_{m,n}^D(\lambda)$ lies in the sector $\phi_1 < \phi < \phi_1 + \epsilon$ and for $\lambda \leq -N_{m,n}(\epsilon)$, $\Omega_{m,n}^D(\lambda)$ lies in the sector $\phi_2 - \epsilon < \phi < \phi_2$.*

Proof: The result follows from [11, Theorem 4]. □

Theorem 3.7. *If $\Omega^* \leq 0$, then the line $\Omega = \Omega^*$ intersects each curve $\Omega_{m,n}^D$ in at least two points and at most finite number of points.*

Proof: For fixed $m \geq 0$, we have from Theorem 3.5, $\Omega_{m,n}^D(0) > 0$. By choosing $\epsilon > 0$ very small in Theorem 3.6 we arrive at the conclusion that $\lim_{\lambda\to\infty}\Omega_{m,n}^D(\lambda) = \lim_{\lambda\to-\infty}\Omega_{m,n}^D(\lambda) = -\infty$. Hence $\Omega_{m,n}^D$ intersects $\Omega = \Omega^*$ in at least one point with positive abscissa and in at least one point with negative abscissa. Since $\Omega_{m,n}^D$ is analytic there are at most finite number of points of intersection with each such points having nonzero abscissas. □

Let

$$\lambda_{m,n}^- = \min\left\{\lambda < 0 : \Omega_{m,n}^D \text{ intersects } \Omega = 0 \text{ at } \lambda\right\}$$
$$\lambda_{m,n}^+ = \max\left\{\lambda > 0 : \Omega_{m,n}^D \text{ intersects } \Omega = 0 \text{ at } \lambda\right\}$$

Remark 3.8. *Since $\Omega_{m,n}^D(\lambda)$ is analytic in λ, both the above sets contains only finite number of elements and hence $\lambda_{m,n}^-$ and $\lambda_{m,n}^+$ are finite. Also, note that if $\Omega_{m,n}^D(\lambda) \geq 0$ for some λ then λ must be in $[\lambda_{m,n}^-, \lambda_{m,n}^+]$. We denote this interval by $S_{m,n}$.*

Theorem 3.9. *Given m and n, all the intersection points of μ_{1m} and μ_{2n} are contained in the set $S_{m,n} \bigcup S_{m-1,n} \bigcup S_{m-2,n}$.*

Proof: Suppose μ_{1m} and μ_{2n} intersect at λ_1. Consider the equation

$$-y^{''} + (q_1 + Q - \lambda r_{11} - \mu_{2n}(\lambda) r_{12})\, y = \Omega Q y \tag{8}$$

$$\frac{y^{'}(0)}{y(0)} = \cot \alpha_1, \quad \frac{y^{'}(1)}{y(1)} = \frac{a_1 \lambda \Omega + b_1}{c_1 \lambda \Omega + d_1}.$$

For $\lambda \in \mathbb{R}$ and $\mu_{2n}(\lambda)$, the system has eigenvalues $\Omega_{0,n}(\lambda) < \Omega_{1,n}(\lambda) < \cdots$. Fix $\lambda = \lambda_1$. Then $\Omega_{0,n}(\lambda_1) < \Omega_{1,n}(\lambda_1) < \cdots$ and there exists a positive integer $M_1 = M_1(\lambda_1)$ where $\Omega_{M_1,n}(\lambda_1) < \frac{-d_1}{c_1 \lambda_1} \leq \Omega_{M_1+1,n}(\lambda_1)$, such that the eigenfunction y_l of $\Omega_{l,n}(\lambda_1)$ has l zeroes if $l \leq M_1$ and $l-1$ zeroes if $l > M_1$ [6, Theorem 3.1]. Since λ_1, $\mu_{1m}(\lambda_1) = \mu_{2n}(\lambda_1)$, and y_{1m} satisfy (1), we have $\Omega_{l_0,n}(\lambda_1) = 1$ for some $l_0 \geq 0$.

Case 1: $m \leq N_1$, where $N_1 = N_1(\mu_{1m}(\lambda_1))$. Then
1a: If $m < M_1$, then $l_0 = m$.
1b: If $m = M_1$, then $l_0 = m$ or $m+1$.
1c: If $m > M_1$, then $l_0 = m+1$.
We prove 1a. The proofs of 1b and 1c are similar. Let $m < M_1$. Suppose that $l_0 \neq m$. If $l_0 \leq M_1$, then y_{l_0} has l_0 zeroes. Since dimension of the eigenspace for $\Omega_{l_0,n}(\lambda_1)$ is one, $y_{l_0} = c y_{1m}$ where c is a constant. Thus the number of zeroes of y_{l_0} and y_{1m} are the same, which is not possible. Similarly for $l_0 > M_1$, we will get a contradiction.
(1a): $m < M_1$. So $\Omega_{m,n}(\lambda_1) = 1$. By construction $\Omega^D_{m-1,n}(\lambda_1) < \Omega_{m,n}(\lambda_1) = 1 < \Omega^D_{m,n}(\lambda_1)$, where $\Omega^D_{0,n}(\lambda_1) < \Omega^D_{1,n}(\lambda_1) < \cdots$ are the eigenvalues of the Dirichlet problem (6) [6, Theorem 3.1]. From Remark 3.8 we see that $\lambda_1 \in S_{m,n}$.
(1b): $m = M_1$. In this case $\Omega_{m,n}(\lambda_1) = 1$ or $\Omega_{m+1,n}(\lambda_1) = 1$. Then it follows from the inequality $\Omega^D_{m-1,n}(\lambda_1) < \Omega_{l_0,n}(\lambda_1) = 1 \leq \Omega^D_{m,n}(\lambda_1)$ where $l_0 = m$ or $m+1$ that $\lambda_1 \in S_{m,n}$.
(1c): $m > M_1$. Here $\Omega_{m+1,n} = 1$. Also we have $\Omega^D_{m-1,n}(\lambda_1) < \Omega_{m+1,n}(\lambda_1) = 1 < \Omega^D_{m,n}(\lambda_1)$. Hence $\lambda_1 \in S_{m,n}$.

Case 2: $m > N_1$. The following subcases can be proved as in case 1.
(2a): If $m \leq M_1$ then $l_0 = m-1$ so that $\lambda_1 \in S_{m-1,n}$.
(2b): $m = M_1 + 1$. Then $l_0 = m-1$ or m, and $\lambda_1 \in S_{m-1,n}$.
(2c): $m > M_1 + 1$. Then $l_0 = m-1$ or m. If $l_0 = m-1$ then $\lambda_1 \in S_{m-2,n}$. If $l_0 = m$ then $\lambda_1 \in S_{m-1,n}$.

Let λ_2 be another intersection point of μ_{1m} and μ_{2n}. By fixing λ_2 and $\mu_{2n}(\lambda_2)$ in (8), the eigenvalues of the equation can be arranged as $\Omega_{0,n}(\lambda_2) < \Omega_{1,n}(\lambda_2) < \cdots$ and there exists a positive integer $M_1(\lambda_2)$ such that the eigenfunction of $\Omega_{l,n}(\lambda_2)$ has l zeros if $l \leq M_1(\lambda_2)$ and $l-1$ zeros if $l > M_1(\lambda_2)$. Now as above, if $m \leq N_1$ where $N_1 = N_1(\mu_{1m}(\lambda_2))$, then $\lambda_2 \in S_{m,n}$ and if $m > N_1$, then λ_2 is in $S_{m-1,n}$ or $S_{m-2,n}$. Thus the theorem follows.

Corollary 3.10. *The eigencurves μ_{1m} and μ_{2n} intersect in at most finite number of points.*

Proof: Suppose there to be infinitely many points of intersection. Then by the previous theorem, these points lie in a bounded set. Since μ_{1m} and μ_{2n} are analytic, $\mu_{1m} \equiv \mu_{2n}$. Therefore μ_{1m} intersects μ_{1k} for $k \geq N(n)$, which is not possible. □

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