SUPERPOSABILITY OF STEADY AXI-SYMMETRICAL FLOWS IN A NON-NEWTONIAN FLUID

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Let $(\mathbf{q}_1, p_1, \Omega_1)$ and $(\mathbf{q}_2, p_2, \Omega_2)$ be the two flows of an incompressible fluid of uniform density ρ, kinematic viscosity ν and kinematic cross-viscosity ν_c, where $\mathbf{q}$, p, Ω denote the velocity vector, the presssure and the potential from which the external forces are assumed to be derived. The two flows are said to be superposable, if a pressure $(p_1 + p_2 + \pi)$ can be found such that $(\mathbf{q}_1 + \mathbf{q}_2, p_1 + p_2 + \pi, \Omega_1 + \Omega_2)$ is also a solution of the Stokes-Navier equation with the necessary modification in the initial and boundary conditions.

The equations of motion for a non-Newtonian fluid are given by,

$$\rho\left(\frac{\partial u_i}{\partial t} + u_{i,j}u_j\right) = t^i_{j,j} + f^i, \tag{0.1}$$

$$u_{j,j} = 0 \tag{0.2}$$

t_j^i being the stress tensor given by

$$t_j^i = -p\delta_j^i + \mathrm{F}_1 d_j^i + \mathrm{F}_2 d_a^i d_j^a, \tag{0.3}$$

where f^i is the body force,

$$d_j^i = u_{j,i} + u_{i,j} \tag{0.4}$$

is the rate of deformation tensor, F_1 and F_2 are functions of some material constants and the second and third invariants of the rate of deformation tensor d_j^i.

By taking $\mathrm{F}_1 = \mu$ and $\mathrm{F}_2 = \mu_c$ we recognize F_1 and F_2 to be the coefficient of viscosity and coefficient of cross-viscosity.

1. For the problem on hand it is convenient to use cylindrical co-ordinate system (r, θ, Z) in which z-axis is taken to be the axis of symmetry and r is the distance from the axis. The velocity components at any point

are given by u_r, u_θ and u_z, where the velocity vector $\boldsymbol{q}$ does not necessarily lie in the meridian plane, so that

$$\boldsymbol{q} = u_r \boldsymbol{i}_r + u_\theta \boldsymbol{i}_\theta + u_z \boldsymbol{i}_z,$$

where for axial-symmetry the velocity components are functions of r and z only besides the time t.

(*a*) *Equations of motion.*—The equations of steady motion in cylindrical co-ordinates, with terms of azimuthal variation neglected, are,

$$\rho\left(u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}\right) = \frac{\partial \mathrm{P}_{rr}}{\partial r} + \frac{\partial \mathrm{P}_{rz}}{\partial z} + \frac{\mathrm{P}_{rr} - \mathrm{P}_{\theta\theta}}{r}, \qquad (1.1)$$

$$\rho\left(u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right) = \frac{\partial \mathrm{P}_{r\theta}}{\partial r} + \frac{\partial \mathrm{P}_{\theta z}}{\partial z} + \frac{2\mathrm{P}_{r\theta}}{r}, \qquad (1.2)$$

$$\rho\left(u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z}\right) = \frac{\partial \mathrm{P}_{rz}}{\partial r} + \frac{\partial \mathrm{P}_{zz}}{\partial z} + \frac{\mathrm{P}_{rz}}{r}, \qquad (1.3)$$

along with the equation of continuity

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0, \qquad (1.4)$$

where the stress matrix P_{ij} and the rate of strain matrix e_{ij} in the case of non-Newtonian fluids are related as (Braun and Reiner[1])

$$\begin{pmatrix} \mathrm{P}_{rr} & \mathrm{P}_{r\theta} & \mathrm{P}_{rz} \\ \mathrm{P}_{\theta r} & \mathrm{P}_{\theta\theta} & \mathrm{P}_{\theta z} \\ \mathrm{P}_{zr} & \mathrm{P}_{z\theta} & \mathrm{P}_{zz} \end{pmatrix} = -p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} e_{rr} & e_{r\theta} & e_{rz} \\ e_{\theta r} & e_{\theta\theta} & e_{\theta z} \\ e_{zr} & e_{z\theta} & e_{zz} \end{pmatrix}$$

$$+ \mu_c \begin{pmatrix} e_{rr} & e_{r\theta} & e_{rz} \\ e_{\theta r} & e_{\theta\theta} & e_{\theta z} \\ e_{zr} & e_{z\theta} & e_{zz} \end{pmatrix} \begin{pmatrix} e_{rr} & e_{r\theta} & e_{rz} \\ e_{\theta r} & e_{\theta\theta} & e_{\theta z} \\ e_{zr} & e_{z\theta} & e_{zz} \end{pmatrix}, \qquad (1.5)$$

where $e_{ij} = e_{ji}$

$$\left.\begin{aligned} e_{rr} &= 2\frac{\partial u_r}{\partial r} & e_{r\theta} &= \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \\ e_{\theta\theta} &= 2\frac{u_r}{r} & e_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\ e_{zz} &= 2\frac{\partial u_z}{\partial z} & e_{\theta z} &= \frac{\partial u_\theta}{\partial z} \end{aligned}\right\} \qquad (1.6)$$

Using the relations (1.4), (1.5) and (1.6), the equations of motion (1.1), (1.2) and (1.3) for axially symmetric steady flow become,

$$u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) + 2\nu_c \left\{4\frac{\partial u_r}{\partial r}\cdot\frac{\partial^2 u_r}{\partial r^2}\right.$$

$$+\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\left(\frac{\partial^2 u_r}{\partial r \partial z} + \frac{\partial^2 u_z}{\partial r^2}\right) - \frac{u_r}{r}\frac{\partial^2 u_z}{\partial r \partial z} - \frac{u_r}{r}\frac{\partial^2 u_r}{\partial z^2}$$

$$\left. - \frac{1}{2r}\left(\frac{\partial u_r}{\partial z}\right)^2 + \frac{1}{2r}\left(\frac{\partial u_z}{\partial r}\right)^2 + \frac{2}{r}\left(\frac{\partial u_r}{\partial r}\right)^2 - \frac{2u_r^2}{r^3}\right\}$$

$$+ \nu_c \left\{2\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\frac{\partial}{\partial r}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) + \frac{\partial u_\theta}{\partial z}\frac{\partial}{\partial z}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\right.$$

$$\left. - \frac{1}{r}\left(\frac{\partial u_\theta}{\partial z}\right)^2 + \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\frac{\partial^2 u_\theta}{\partial z^2}\right\}, \tag{1.1 a}$$

$$u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right) + \nu_c \left\{\frac{\partial^2 u_\theta}{\partial r \partial z}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\right.$$

$$+ \frac{\partial u_\theta}{\partial z}\frac{\partial}{\partial r}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) - 2\frac{\partial^2 u_r}{\partial r \partial z}\frac{\partial u_\theta}{\partial z} + \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

$$\times \frac{\partial}{\partial z}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) + \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

$$- 2\frac{\partial u_z}{\partial z}\frac{\partial}{\partial r}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) - 2\frac{\partial^2 u_\theta}{\partial z^2}\frac{\partial u_r}{\partial r}$$

$$- 2\frac{\partial^2 u_z}{\partial r \partial z}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) - \frac{4}{r}\frac{\partial u_z}{\partial z}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\left. + \frac{2}{r}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\frac{\partial u_\theta}{\partial z}\right\}, \tag{1.2 a}$$

and

$$u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z + 2\nu_c \left\{4\frac{\partial u_z}{\partial z}\cdot\frac{\partial^2 u_z}{\partial z^2} - \frac{u_r}{r}\frac{\partial^2 u_r}{\partial r \partial z}\right.$$

$$+ \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\left(\frac{\partial^2 u_r}{\partial z^2} + \frac{\partial^2 u_z}{\partial r \partial z}\right) - \frac{u_r}{r}\frac{\partial u_z^2}{\partial r^2}$$

$$\left. - \frac{1}{r}\frac{\partial u_r}{\partial r}\cdot\frac{\partial u_r}{\partial z} - \frac{1}{r}\frac{\partial u_r}{\partial r}\cdot\frac{\partial u_z}{\partial r}\right\} + \nu_c \left\{\frac{\partial u_\theta}{\partial z}\left(2\frac{\partial^2 u_\theta}{\partial z^2} + \frac{\partial^2 u_\theta}{\partial r^2}\right)\right.$$

$$\left. + \frac{\partial^2 u_\theta}{\partial r \partial z}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)\right\}. \tag{1.3 a}$$

Expressing the velocity components in terms of the Stokes' stream function ψ, we have

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial z}, \quad u_\theta = \chi(r, z), \quad u_z = -\frac{1}{r}\frac{\partial\psi}{\partial r},$$

i.e.,

$$\boldsymbol{q} = \left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\boldsymbol{i}_r + \chi\boldsymbol{i}_\theta + \left(-\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\boldsymbol{i}_z. \tag{1.7}$$

(*b*) *The conditions of integrability.*—The conditions of integrability of the equations of motion in terms of the Stokes' stream function ψ turn out to be,

$$\begin{aligned} r\mathrm{J}\begin{pmatrix}\psi & r^{-2}\mathrm{E}^2\psi \\ z & r\end{pmatrix} - 2\chi\frac{\partial\chi}{\partial z} &= \nu\mathrm{E}^4\psi + 2\nu_c\left\{\frac{2}{r}\mathrm{J}\begin{pmatrix}\psi & r^{-2}\mathrm{E}^2\psi \\ z & r\end{pmatrix} - 2\mathrm{J}\begin{pmatrix}\frac{\partial\psi}{\partial r} & r^{-2}\mathrm{E}^2\psi \\ z & r\end{pmatrix}\right. \\ &\quad \left. - \frac{\partial}{\partial z}\left(\frac{1}{r}\mathrm{E}^2\psi\right)^2 - \frac{1}{r^2}\frac{\partial\psi}{\partial z}\mathrm{E}^4\psi + \frac{2}{r}\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{\mathrm{E}^2\psi}{r^2}\right)\right\} \\ &\quad + \nu_c\left\{r^2\mathrm{J}\begin{pmatrix}\mathrm{E}^2\chi & \frac{\chi}{r} \\ z & r\end{pmatrix} - 2r\mathrm{J}\begin{pmatrix}\frac{\partial\chi}{\partial r} & \frac{\chi}{r} \\ z & r\end{pmatrix} - \frac{3\chi}{r^2}\frac{\partial\chi}{\partial z}\right. \\ &\quad \left. - 5\frac{\partial\chi}{\partial z}\mathrm{E}^2\chi\right\} \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \frac{1}{r^2}\mathrm{J}\begin{pmatrix}\psi & r\chi \\ z & r\end{pmatrix} &= \nu\left(\nabla^2\chi - \frac{\chi}{r^2}\right) + \nu_c\left\{\mathrm{J}\begin{pmatrix}\nabla^2\psi & \frac{\chi}{r} \\ z & r\end{pmatrix} + \frac{2}{r}\mathrm{J}\begin{pmatrix}\frac{\partial\psi}{\partial z} & \frac{\partial\chi}{\partial z} \\ z & r\end{pmatrix}\right. \\ &\quad \left. + \frac{2}{r}\mathrm{J}\begin{pmatrix}\frac{\partial\psi}{\partial r} & \frac{\partial\chi}{\partial r} \\ z & r\end{pmatrix} + \frac{2}{r^2}\frac{\partial\psi}{\partial r}\frac{\partial^2\chi}{\partial r\partial z} + \frac{2}{r^2}\frac{\partial\psi}{\partial z}\frac{\partial^2\chi}{\partial z^2}\right\}, \end{aligned} \tag{1.9}$$

where

$$\mathrm{E}^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{r}\frac{\partial}{\partial r}$$

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

(*c*) *Conditions for superposability of two motions.*—Taking the two motions given by

$$\boldsymbol{q}_s = \left(\frac{1}{r}\frac{\partial \psi_s}{\partial z}\right)\boldsymbol{i}_r + \chi_s \boldsymbol{i}_\theta + \left(-\frac{1}{r}\frac{\partial \psi_s}{\partial r}\right)\boldsymbol{i}_z$$

where $s = 1, 2$ corresponds to two motions. Using the conditions of integrability (1.8) and (1.9), we obtain the following conditions for superposability of two motions:

$$\begin{aligned}
& r\mathrm{J}\begin{pmatrix}\psi_1 & r^{-2}\mathrm{E}^2\psi_2\\ z & r\end{pmatrix} + r\mathrm{J}\begin{pmatrix}\psi_2 & r^{-2}\mathrm{E}^2\psi_1\\ z & r\end{pmatrix} - 2\chi_1\frac{\partial\chi_2}{\partial z} - 2\chi_2\frac{\partial\chi_1}{\partial z}\\
&= 2\nu_c\left\{\frac{2}{r}\mathrm{J}\begin{pmatrix}\psi_1 & r^{-2}\mathrm{E}^2\psi_2\\ z & r\end{pmatrix} + \frac{2}{r}\mathrm{J}\begin{pmatrix}\psi_2 & r^{-2}\mathrm{E}^2\psi_1\\ z & r\end{pmatrix}\right.\\
&\quad - 2\mathrm{J}\begin{pmatrix}\frac{\partial\psi_1}{\partial r} & r^{-2}\mathrm{E}^2\psi_2\\ z & r\end{pmatrix} - 2\mathrm{J}\begin{pmatrix}\frac{\partial\psi_2}{\partial r} & r^{-2}\mathrm{E}^2\psi_1\\ z & r\end{pmatrix}\\
&\quad + \frac{2}{r}\frac{\partial\psi_1}{\partial z}\frac{\partial}{\partial r}\left(\frac{\mathrm{E}^2\psi_2}{r^2}\right) + \frac{2}{r}\frac{\partial\psi_2}{\partial z}\frac{\partial}{\partial r}\left(\frac{\mathrm{E}^2\psi_1}{r^2}\right) - \frac{1}{r^2}\frac{\partial\psi_1}{\partial z}\mathrm{E}^4\psi_2\\
&\quad \left.- \frac{1}{r^2}\frac{\partial\psi_2}{\partial z}\mathrm{E}^4\psi_1\right\} + \nu_c\left\{r^2\mathrm{J}\begin{pmatrix}\mathrm{E}^2\chi_1 & \frac{\chi_2}{r}\\ z & r\end{pmatrix} + r^2\mathrm{J}\begin{pmatrix}\mathrm{E}^2\chi_2 & \frac{\chi_1}{r}\\ z & r\end{pmatrix}\right.\\
&\quad - 2r\mathrm{J}\begin{pmatrix}\frac{\partial\chi_1}{\partial r} & \frac{\chi_2}{r}\\ z & r\end{pmatrix} - 2r\mathrm{J}\begin{pmatrix}\frac{\partial\chi_2}{\partial r} & \frac{\chi_1}{r}\\ z & r\end{pmatrix} - \frac{3\chi_1}{r^2}\frac{\partial\chi_2}{\partial z}\\
&\quad \left.- 3\frac{\chi_2}{r^2}\frac{\partial\chi_1}{\partial z} - 5\frac{\partial\chi_1}{\partial z}\mathrm{E}^2\chi_2 - 5\frac{\partial\chi_2}{\partial z}\mathrm{E}^2\chi_1\right\}
\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}
& \mathrm{J}\begin{pmatrix}\psi_1 & r\chi_2\\ z & r\end{pmatrix} + \mathrm{J}\begin{pmatrix}\psi_2 & r\chi_1\\ z & r\end{pmatrix}\\
&= \frac{\nu_c}{r}\left\{\mathrm{J}\begin{pmatrix}\nabla^2\psi_1 & \frac{\chi_2}{r}\\ z & r\end{pmatrix} + \mathrm{J}\begin{pmatrix}\nabla^2\psi_2 & \frac{\chi_1}{r}\\ z & r\end{pmatrix} + 2\mathrm{J}\begin{pmatrix}\frac{\partial\psi_1}{\partial z} & \frac{\partial\chi_2}{\partial z}\\ z & r\end{pmatrix}\right.
\end{aligned}$$

$$+ 2J\begin{pmatrix} \frac{\partial \psi_2}{\partial z} & \frac{\partial \chi_1}{\partial z} \\ z & r \end{pmatrix} + 2J\begin{pmatrix} \frac{\partial \psi_1}{\partial r} & \frac{\partial \chi_2}{\partial r} \\ z & r \end{pmatrix} + 2J\begin{pmatrix} \frac{\partial \psi_2}{\partial r} & \frac{\partial \chi_1}{\partial r} \\ z & r \end{pmatrix}$$

$$+ \frac{2}{r}\frac{\partial \psi_1}{\partial r}\frac{\partial^2 \chi_2}{\partial r \partial z} + \frac{2}{r}\frac{\partial \psi_2}{\partial r}\frac{\partial^2 \chi_1}{\partial r \partial z} + \frac{2}{r}\frac{\partial \psi_1}{\partial z}\frac{\partial^2 \chi_2}{\partial z^2} + \frac{2}{r}\frac{\partial \psi_2}{\partial z}\frac{\partial^2 \chi_1}{\partial z^2}\Big\} . \tag{1.11}$$

We can easily show that the conditions (1.10) and (1.11) are both necessary and sufficient for the two axially symmetric flows of a non-Newtonian fluid corresponding to the two stream functions ψ_1 and ψ_2 to be superposable.

2. CONDITIONS FOR SELF-SUPERPOSABILITY OF A FLOW:

Using the conditions (1.10) and (1.11), it follows that the flow with stream function ψ is self-superposable if and only if,

$$rJ\begin{pmatrix} \psi & r^{-2}E^2\psi \\ z & r \end{pmatrix} - 2\chi\frac{\partial \chi}{\partial z}$$

$$= 2\nu_c\left[\frac{2}{r}J\begin{pmatrix} \psi & r^{-2}E^2\psi \\ z & r \end{pmatrix} - 2J\begin{pmatrix} \frac{\partial \psi}{\partial r} & r^{-2}E^2\psi \\ z & r \end{pmatrix} + \frac{2}{r}\frac{\partial \psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{E^2\psi}{r^2}\right)\right.$$

$$\left. - \frac{1}{r^2}\frac{\partial \psi}{\partial z}E^4\psi\right] + \nu_c\left[r^2J\begin{pmatrix} E^2\chi & \frac{\chi}{r} \\ z & r \end{pmatrix} - 2rJ\begin{pmatrix} \frac{\partial \chi}{\partial r} & \frac{\chi}{r} \\ z & r \end{pmatrix}\right.$$

$$\left. - \frac{3\chi}{r^2}\frac{\partial \chi}{\partial z} - 5\frac{\partial \chi}{\partial z}E^2\chi\right] \tag{2.1}$$

and

$$J\begin{pmatrix} \psi & r\chi \\ z & r \end{pmatrix} = \frac{\nu_c}{r}\left[J\begin{pmatrix} \nabla^2\psi & \frac{\chi}{r} \\ z & r \end{pmatrix} + 2J\begin{pmatrix} \frac{\partial \psi}{\partial z} & \frac{\partial \chi}{\partial z} \\ z & r \end{pmatrix} + 2J\begin{pmatrix} \frac{\partial \psi}{\partial r} & \frac{\partial \chi}{\partial r} \\ z & r \end{pmatrix}\right.$$

$$\left. + \frac{2}{r}\frac{\partial \psi}{\partial r}\frac{\partial^2 \chi}{\partial r \partial z} + \frac{2}{r}\frac{\partial \psi}{\partial z}\frac{\partial^2 \chi}{\partial z^2}\right]. \tag{2 2}$$

3. PARTICULAR CASES

(*a*) *Newtonian fluids.*—If in the conditions of superposability and self-superposability by putting $\nu_c = 0$, we obtain the conditions of superposability and self-superposability for a Newtonian fluid as obtained by Lakshmana Rao.[3]

(*b*) *Velocity vector lies in the meridian plane.*—We can find the conditions of superposability and self-superposability in steady axi-symmetrical flows when the velocity vector $\boldsymbol{q}$ lies in the meridian plane,

i.e.,

$$\boldsymbol{q} = u_r \boldsymbol{i}_r + u_z \boldsymbol{i}_z \tag{3.1}$$

or in terms of the Stokes' stream function

$$\boldsymbol{q} = \left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right)\boldsymbol{i}_r + \left(-\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\boldsymbol{i}_z \tag{3.2}$$

and the vorticity is given by

$$\omega = \frac{1}{r}\,\mathrm{E}^2\psi. \tag{3.3}$$

In this case the equations of motion are given by (1.1 *a*), (1.2 *a*) and (1.3 *a*) with $X = 0$.

The condition of integrability (1.8) becomes

$$r\mathrm{J}\begin{pmatrix}\psi & r^{-2}\mathrm{E}^2\psi\\ z & r\end{pmatrix} = \nu\mathrm{E}^4\psi + 2\nu_c\left\{\frac{2}{r}\,\mathrm{J}\begin{pmatrix}\psi & r^{-2}\mathrm{E}^2\psi\\ z & r\end{pmatrix} - 2\mathrm{J}\begin{pmatrix}\frac{\partial\psi}{\partial r} & r^{-2}\mathrm{E}^2\psi\\ z & r\end{pmatrix}\right.$$
$$\left. - \frac{1}{r^3}\frac{\partial\psi}{\partial z}\mathrm{E}^4\psi - \frac{\partial}{\partial z}\left(\frac{1}{r}\,\mathrm{E}^2\psi\right)^2 + \frac{2}{r}\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{\mathrm{E}^2\psi}{r^2}\right)\right\} \tag{3.4}$$

while the condition (1.9) is automatically satisfied.

The condition of superposability (1.10) for two axi-symmetrical flows with stream functions ψ_1 and ψ_2 is given by,

$$\mathrm{J}\begin{pmatrix}\psi_1 & r^{-2}\mathrm{E}^2\psi_2\\ z & r\end{pmatrix} + \mathrm{J}\begin{pmatrix}\psi_2 & r^{-2}\mathrm{E}^2\psi_1\\ z & r\end{pmatrix}$$
$$= \frac{2\nu_c}{r}\left\{\frac{2}{r}\,\mathrm{J}\begin{pmatrix}\psi_1 & r^{-2}\mathrm{E}^2\psi_2\\ z & r\end{pmatrix} + \frac{2}{r}\,\mathrm{J}\begin{pmatrix}\psi_2 & r^{-2}\mathrm{E}^2\psi_1\\ z & r\end{pmatrix}\right.$$
$$- 2\mathrm{J}\begin{pmatrix}\frac{\partial\psi_1}{\partial r} & r^{-2}\mathrm{E}^2\psi_2\\ z & r\end{pmatrix} - 2\mathrm{J}\begin{pmatrix}\frac{\partial\psi_2}{\partial r} & r^{-2}\mathrm{E}^2\psi_1\\ z & r\end{pmatrix}$$

$$- \frac{1}{r^2} \frac{\partial \psi_1}{\partial z} E^4\psi_2 - \frac{1}{r^2} \frac{\partial \psi_2}{\partial z} E^4\psi_1 + \frac{2}{r} \frac{\partial \psi_1}{\partial z} \frac{\partial}{\partial r}\left(\frac{E^4\psi_2}{r^2}\right)$$

$$+ \frac{2}{r} \frac{\partial \psi_2}{\partial z} \frac{\partial}{\partial r}\left(\frac{E^4\psi_1}{r^2}\right)\Bigg\}. \tag{3.5}$$

The condition (1.11) is automatically satisfied. The condition for self-superposability of this flow is given by:

$$J\begin{pmatrix} \psi & r^{-2}E^2\psi \\ z & r \end{pmatrix}$$

$$= \frac{2\nu_c}{r} \left\{ \frac{2}{r} J\begin{pmatrix} \psi & r^{-2}E^2\psi \\ z & r \end{pmatrix} - 2J\begin{pmatrix} \frac{\partial \psi}{\partial r} & r^{-2}E^2\psi \\ z & r \end{pmatrix} \right.$$

$$\left. - \frac{1}{r^2}\left(\frac{\partial \psi}{\partial z} E^4\psi + \frac{2}{r} \frac{\partial \psi}{\partial z} \frac{\partial}{\partial r}\left(\frac{E^2\psi}{r^2}\right)\right) \right\}. \tag{3.6}$$

From the condition (3.5), we see that a rotational flow with suffix 2, is superposable on an irrotational flow with suffix 1, if

$$J\begin{pmatrix} \psi_1 & r^{-2}E^2\psi_2 \\ z & r \end{pmatrix}$$

$$= \frac{2\nu_c}{r} \left\{ \frac{2}{r} J\begin{pmatrix} \psi_1 & r^{-2}E^2\psi_2 \\ z & r \end{pmatrix} - 2J\begin{pmatrix} \frac{\partial \psi_1}{\partial r} & r^{-2}E^2\psi_2 \\ z & r \end{pmatrix} \right.$$

$$\left. - \frac{1}{r^2} \frac{\partial \psi_1}{\partial z} E^4\psi_2 + \frac{2}{r} \frac{\partial \psi_1}{\partial z} \frac{\partial}{\partial r}\left(\frac{E^2\psi_2}{r^2}\right) \right\}. \tag{3.7}$$

Using the conditions (3.5) and (3.6), we can prove the following theorems.

THEOREM 1.—*All irrotational motions are self-superposable and any two irrotational motions are superposable on each other.*

THEOREM 2.—*The two flows will be superposable if the vorticity of each flow is proportional to* r. *Also an axi-symmetrical flow will be self-superposable if its vorticity is proportional to* r.

THEOREM 3.—*If* ω_1/r *is constant along the stream lines of flow* 2, *while* ω_2/r *is constant along the stream lines of flow* 1, *then the two flows are superposable if in addition the velocity components and vorticities satisfy the condition*

$$2(u_1v_1 + u_2v_2) + r(v_1\omega_1 + v_2\omega_2) = 0.$$

(*where* ω_1 *and* ω_2 *are the vorticities of the two flows*).

THEOREM 4.—*An axi-symmetrical flow is self-superposable, if the vorticity* ω *is of the form* $\omega = \mathrm{f}(\psi)\,\mathrm{r}$ *and if the stream function* ψ *satisfies the condition*

$$E^4\psi = 2\mathrm{r}\,\frac{\partial}{\partial \mathrm{r}}\left(\frac{1}{\mathrm{r}^2}\;E^2\psi\right).$$

While the condition of integrability reduces to

$$\nu \mathrm{E}^4\psi = \frac{2\nu_c}{r^2}\frac{\partial}{\partial z}(\mathrm{E}^4\psi).$$

Also a rotational flow with stream function ψ_2 is superposable on an irrotational flow with stream function ψ_1, if in general

$$\omega_2 = f(\psi_1)\,r$$

and

$$\mathrm{E}^4\psi_2 = 2r\,\frac{\partial}{\partial r}\left(\frac{1}{r^2}\,\mathrm{E}^2\psi_2\right).$$

The corresponding theorems for a Newtonian fluid have been given by Bhatnagar and Verma.[3]

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