ON THE RELATIVE EXTREMA OF THE TURAN EXPRESSION FOR BESSEL FUNCTIONS

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SOME years ago Otto Szász[1] showed that the Bessel function $J_n(x)$ satisfies the inequality

$$\triangle_n(x) \equiv [J_n(x)]^2 - J_{n+1}(x) J_{n-1}(x) \geq \frac{1}{n+1} [J_n(x)]^2,$$
$$n \geq 0, \quad x \text{ real} \tag{1}$$

wherefrom he deduced the interesting inequality

$$\triangle_n(x) > 0 \quad \text{for} \quad n > 0, \quad x \text{ real.} \tag{2}$$

The Turán inequality (2) for the function $J_n(x)$ is an immediate consequence of the identity[2]

$$\triangle_n(x) = \frac{4}{x^2} \sum_{k=0}^{\infty} (n + 2k + 1) \left(J_{n+2k+1}(x)\right)^2 \tag{3}$$

which represents $\triangle_n(x)$ as a series of positive quantities whenever $n \geq -1$. V. R. Thiruvenkatachar and T. S. Nanjundiah[3] obtained a new positive representation for $\triangle_n(x)$ by proving the identity

$$\triangle_n(x) = \frac{1}{n+1} \left(J_n(x)\right)^2 + \frac{2}{n+2} \left(J_{n+2}(x)\right)^2$$
$$+ 2n \sum_{k=2}^{\infty} [(n + k - 1)(n + k + 1)]^{-1} \left(J_{n+k}(x)\right)^2 \tag{4}$$

and the above result (1) of Otto Szász follows from this immediately.

While the above representations for $\triangle_n(x)$ are quite interesting we may also proceed to deduce the inequality (2) by considering the behaviour of the function $\Delta_n(x)$ itself. This procedure leads to several questions concerning the sequences of relative maxima and minima of $\triangle_n(x)$ and the

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aim of the present note is the study of these sequences. We prove in particular (Theorem 4 below) that the r-th relative maximum of $\triangle_n(x)$ is larger than the r-th relative maximum of $\triangle_{n+1}(x)$, for any fixed value of r. This answers the question first raised by John Todd for the classical orthogonal polynomials and settled for the Hermite function $H_n(x)$ by Otto Szász[4] and for the Turán expression for the Hermite function by the author.[5]

THEOREM 1: When $n > 0$, the relative maxima of $\triangle_n(x)$ occur at the zeros of $J_{n-1}(x)$ and the relative minima occur at the zeros of $J_{n+1}(x)$.

On differentiating the expression

$$\triangle_n(x) = (J_n(x))^2 - J_{n+1}(x)\, J_{n-1}(x) \tag{5}$$

twice, we get

$$\frac{d}{dx}\triangle_n(x) = \frac{2}{x} J_{n+1}(x)\, J_{n-1}(x) \tag{6}$$

and

$$\frac{d^2}{dx^2}\triangle_n(x) = \frac{2}{x} J_{n+1}(x) \frac{d}{dx} J_{n-1}(x) + \frac{2}{x} J_{n-1}(x) \frac{d}{dx} J_{n+1}(x) - \frac{2}{x^2} J_{n+1}(x)\, J_{n-1}(x). \tag{7}$$

Let $0 < x_{1,s} < x_{2,s} < \dots$ denote the zeros of $J_s(x)$ in the ascending order. From (7) we see that

$$\left(\frac{d^2}{dx^2}\triangle_n(x)\right)_{x=x_{r,n-1}} = -4n\left\{\left(\frac{1}{x} J_n(x)\right)^2\right\}_{x=x_{r,n-1}} < 0$$

and

$$\left(\frac{d^2}{dx^2}\triangle_n(x)\right)_{x=x_{r,n+1}} = 4n\left\{\left(\frac{1}{x} J_n(x)\right)^2\right\}_{x=x_{r,n+1}} > 0.$$

Hence the theorem.

The statement in the above theorem is reversed if $n < 0$. In what follows we assume that $n > 0$.

If we denote the successive relative maxima of $\triangle_n(x)$ by $M_{1,n}$, $M_{2,n}$, ... and the successive relative minima by $m_{1,n}$, $m_{2,n}$, ... respectively, we have

$$M_{r,n} = \triangle_n(x_{r,n-1}) = (J_n(x_{r,n-1}))^2, \tag{8}$$

and

$$m_{r,n} = \Delta_n (x_{r,n+1}) = \big(J_n (x_{r,n+1})\big)^2. \tag{9}$$

We may also observe incidentally that

$$m_{r,n-1} = \big(J_{n-1} (x_{r,n})\big)^2 = \big(J_{n+1} (x_{r,n})\big)^2 = M_{r,n+1}.$$

Since the quantities $M_{r,n}$ and $m_{r,n}$ are positive for all values of n, it follows that

$$\Delta_n (x) > 0 \tag{10}$$

which is Turán's inequality for the Bessel function $J_n (x)$.

THEOREM 2: The sequences of relative maxima and relative minima of $\Delta_n (x)$ are decreasing beyond a certain value of r. More specifically

$$M_{r,n} > M_{r+1,n} \quad \text{if} \quad x_{r,n-1} > \xi = \sqrt{2n (n-2)}$$

and

$$m_{r,n} > m_{r+1,n} \quad \text{if} \quad x_{r,n+1} > \eta = \sqrt{2n (n+2)}.$$

If $0 < n \leq 2$, we define $\xi = 0$.

Consider the function

$$f(x) = \Delta_n (x) + \frac{2n}{x^2} \big(J_{n-1} (x)\big)^2.$$

Then, by differentiation we get

$$\begin{aligned} f'(x) &= \frac{2}{x} J_{n+1} (x) J_{n-1} (x) - \frac{4n}{x^3} \big(J_{n-1} (x)\big)^2 + \frac{4n}{x^2} J_{n-1} (x) \frac{d}{dx} J_{n-1} (x) \\ &= \frac{2}{x^3} \big(2n (n-2) - x^2\big) \big(J_{n-1} (x)\big)^2 \end{aligned}$$

so that $f(x)$ is increasing in $0 < x < \xi$ and decreasing in $x > \xi$. Also at the relative maxima of $\Delta_n (x)$, $f(x) = \Delta_n (x)$ and hence the assertion that $M_{r,n} > M_{r+1,n}$ if $x_{r+1,n-1} > x_{r,n-1} > \xi$. To prove the corresponding result for the relative minimum, consider the function

$$g(x) = \Delta_n (x) - \frac{2n}{x^2} \big(J_{n+1} (x)\big)^2.$$

Differentiating this we have

$$g'(x) = \frac{2}{x^3} \big(2n (n+2) - x^2\big) \big(J_{n+1} (x)\big)^2$$

so that $g(x)$ is decreasing in $x > \eta$. Since $g(x)$ coincides with $\Delta_n(x)$ at all the relative minima of $\Delta_n(x)$, it follows that $m_{r,n} > m_{r+1,n}$ if $x_{r+1,n+1} > x_{r,n+1} > \eta$.

This completes the proof of the theorem.

THEOREM 3: The r-th relative maximum of $\Delta_n(x)$ is greater than the r-th relative minimum, *i.e.*, $M_{r,n} > m_{r,n}$.

We have $\Delta_n(x_{r,n-1}) = M_{r,n}$ and $\Delta_n(x_{r,n+1}) = m_{r,n}$ and $(d/dx)\,\Delta_n(x) = (2/x)\,J_{n+1}(x)\,J_{n-1}(x)$. From the interlacing properties of the zeros of Bessel functions we know that[6, 7]

$$x_{r-1,n+1} < x_{r,n-1} < x_{r,n} < x_{r,n+1} < x_{r+1,n-1}.$$

Hence in the interval $(x_{r,n-1}, x_{r,n+1})$, $\operatorname{Sgn}(d/dx)\,\Delta_n(x) = (-1)^{2r-1} = -1$, so that $\Delta_n(x)$ is decreasing in the interval and $\Delta_n(x_{r,n-1}) > \Delta_n(x_{r,n+1})$ or $M_{r,n} > m_{r,n}$ which proves the theorem.

By employing the functions $f(x)$ and $g(x)$ already introduced, we can show further that

$$M_{r,n} > \left(1 + \frac{8n^3}{x^4_{r,n+1}}\right) m_{r,n} \quad \text{if} \quad x_{r,n-1} > \xi$$

and

$$M_{r,n} > \left(1 - \frac{8n^3}{x^4_{r,n-1}}\right)^{-1} m_{r,n} \quad \text{if} \quad x_{r,n-1} > \eta.$$

THEOREM 4: For a fixed value of r, the r-th relative maxima of $\Delta_n(x)$ form a sequence of decreasing functions of n, *i.e.*, $M_{r,n} > M_{r,n+1}$. We have already noticed that $\Delta_n(x_{r,n-1}) = M_{r,n}$. We may see easily that $M_{r,n+1} = \Delta_{n+1}(x_{r,n}) = \Delta_n(x_{r,n})$. Also from the relation $(d/dx)\,\Delta_n(x) = (2/x)\,J_{n+1}(x)\,J_{n-1}(x)$, it follows that in the interval $(x_{r,n-1}, x_{r,n})$, $\operatorname{Sgn}(d/dx)\,\Delta_n(x) = -1$, so that $\Delta_n(x)$ is decreasing in the interval and $\Delta_n(x_{r,n-1}) > \Delta_n(x_{r,n})$. Whence the theorem.

Analogously for the relative minima, we have

THEOREM 5: For a fixed value of r, the r-th relative minima of $\Delta_n(x)$ form a sequence of decreasing functions of n, *i.e.*, $m_{r,n} > m_{r,n+1}$.

We have noticed earlier that $m_{r,n-1} = M_{r,n+1}$. Combining this with the result of the previous theorem, we see that $m_{r,n-1} > m_{r,n}$ which proves the assertion.

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