FLOW OF AN ELECTRICALLY CONDUCTING NON-NEWTONIAN FLUID BETWEEN TWO ROTATING COAXIAL CONES IN THE PRESENCE OF EXTERNAL MAGNETIC FIELD DUE TO AN AXIAL CURRENT

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Received October 24, 1963

ABSTRACT

Bhatnagar and Rathna (*Quar. Journ. Mech. Appl. Maths.*, 1963, **16**, 329) investigated the flows of Newtonian, Reiner-Rivlin and Rivlin-Ericksen fluids between two rotating coaxial cones. In case of the last two types of fluids, they predicted the breaking of secondary flow field in any meridian plane. We find that such breaking is avoided by the application of a sufficiently strong azimuthal magnetic field arising from a line current along the axis of the cones.

NOTATIONS

$\mathbf{B}$ = magnetic induction vector

$\mathbf{E}$ = electric intensity

$\mathbf{H}$ = magnetic field

$\mathbf{D}$ = displacement vector

$\mathbf{J}$ = current density vector

ρ_e = excess charge density

σ = electrical conductivity

ϵ = dielectric constant

j = axial current

$\mathbf{V}$ = velocity vector

$\mathbf{T}$ = stress tensor

e = rate of strain tensor

D = acceleration gradient tensor

I = idem tensor

P = pressure

ρ = density

η = coefficient of viscosity

η_c = coefficient of cross-viscosity

η_v = coefficient of viscoelasticity

Ω = characteristic angular velocity

1. INTRODUCTION

THE flows of Newtonian, Reiner-Rivlin and Rivlin-Ericksen fluids between two rotating coaxial cones having the same vertex have been recently investigated by Bhatnagar and Rathna. [1] They have predicted the breaking of secondary flow in case of the last two categories of fluids. In the present note we include the effect of external magnetic field produced by current along the axis of the cones to examine its effect on the secondary flows.

The flow is characterized by the five non-dimensional parameters.

$$R = \frac{\rho L^2 \Omega}{\eta}, \text{ the Reynolds number}$$

$$S = \frac{\eta_c \Omega}{\eta}, \text{ the cross-viscosity parameter}$$

$$K = \frac{\eta_v \Omega}{\eta}, \text{ the viscoelasticity parameter}$$

$$M = \sigma L^2 \Omega,$$

$$N = \frac{j^2}{\eta L^2 \Omega},$$

where

$$\sqrt{MN} = \sqrt{\frac{\sigma j^2}{\eta}} = H_a, \text{ the Hartmann number,}$$

and L is a standard length.

We find that a large axial current suppresses the breaking of the secondary flow in case of non-Newtonian fluid. The axial current necessary for this purpose is more when the angular gap between the cones is small.

2. Basic Equations

The equations governing the steady flow of an electrically conducting liquid are:

Maxwell's equations

$$\nabla \times \mathbf{H} = \mathbf{J} \text{ (neglecting displacement current)}, \tag{2.1}$$

$$\nabla \times \mathbf{E} = 0, \tag{2.2}$$

$$\nabla \cdot \mathbf{D} = \rho_e, \nabla \cdot \mathbf{B} = 0, \tag{2.3}$$

$$\mathbf{D} = \Sigma \mathbf{E}, \mathbf{B} = \mathbf{H} \text{ (taking magnetic permeability to be unity)}, \tag{2.4}$$

Current equation

$$\mathbf{J} = \sigma [\mathbf{E} + \mathbf{V} \times \mathbf{B}) \text{ (neglecting convection current)}, \tag{2.5}$$

Continuity equation

$$\nabla \cdot \mathbf{V} = 0, \tag{2.6}$$

Momentum equation

$$\rho (\mathbf{V} \nabla) \mathbf{V} = \nabla \cdot \mathrm{T} + (\nabla \times \mathbf{H}) \times \mathbf{H}, \tag{2.7}$$

and the rheological equation of state:

$$\mathrm{T} = -p\mathrm{I} + \eta e + \eta_c e \cdot e + \eta_v \mathrm{D}, \tag{2.8}$$

where $\{\mathrm{T}_{ij}\}$, $\{e_{ij}\}$ and $\{\mathrm{D}_{ij}\}$ denote the stress tensor, the rate of strain tensor and the acceleration gradient trensor.

3. The Primary Motion

We have taken the primary motion to be that of a Newtonian fluid with the neglect of inertial terms. The stream lines and the lines of magnetic induction are circles in planes perpendicular to the axis of the cones, with their centres lying on it.

We shall work in spherical polar co-ordinates r, θ, ϕ with the origin at the common vertex of the cones, θ being measured from the axis of the cones and ϕ from a convenient meridian plane.

The variables are rendered dimensionless by means of a length L, a velocity L Ω with $\Omega = |\Omega_1| + |\Omega_2|$, Ω_1 and Ω_2 being the angular velocities of the inner and outer cones respectively and a magnetic field j/L, where j is the applied axial current. In our scheme, hydrostatic pressure will be given by $p\eta\Omega$.

Denoting the quantities of the primary motion by the suffix zero, we have

$$H_0 = \left(0,\ 0,\ \frac{2}{r \sin \theta}\right),$$

$$V_0 = (0,\ 0,\ r \sin \theta \omega_0),$$

$$\omega_0 = A_1\left((\log \tan \frac{\theta}{2} - \cot \theta \operatorname{cosec} \theta\right) + A_2,$$

$$A_1 = \frac{\Omega_1 - \Omega_2}{K},$$

$$A_2 = \frac{1}{\Omega K}\left[\Omega_1\left(\cot \theta_2 \operatorname{cosec} \theta_2 - \log \tan \frac{\theta_2}{2}\right) + \Omega_2\left(\log \tan \frac{\theta_1}{2} - \cot \theta_1 \operatorname{cosec} \theta_1\right)\right],$$

$$K = \log \tan \frac{\theta_1}{2} - \log \tan \frac{\theta_2}{2} + \cot \theta_2 \operatorname{cosec} \theta_2 - \cot \theta_1 \operatorname{cosec} \theta_1,$$

where θ_1 and θ_2 are the semi-vertical angles of the inner and outer cones respectively.

4. SECONDARY FLOW

We now consider the effect of inclusion of the inertial terms in Oseen's approximation and the first-order effect of cross-viscosity and viscoelasticity retaining only the first powers of S and K. Denoting the perturbation velocity by (u, v, w) and the induced magnetic field by (h_r, h_θ, h_ϕ), the linearized equations determining the perturbed magnetic field and velocity are

$$E^4 x = 0, \tag{4.1}$$

$$h_r = -\frac{1}{r^2 \sin \theta}\frac{\partial x}{\partial \theta},$$

$$h_\theta = \frac{1}{r \sin \theta}\frac{\partial x}{\partial r},$$

$$E^4 = \left[\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\left(\frac{\partial}{\partial \cos \theta}\right)^2\right]^2,$$

$$\triangle h_\phi - \frac{h_\phi}{r^2 \sin^2\theta} = \frac{2M}{r}\left[\frac{\partial}{\partial r}\left(\frac{u}{\sin \theta}\right) + \frac{\partial}{\partial \theta}\left(\frac{v}{r \sin \theta}\right)\right], \tag{4.2}$$

$$\frac{-Rw_0^2}{r} = -\frac{\partial p}{\partial r} + \Delta u - \frac{2u}{r^2} - \frac{2\cot\theta v}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}$$

$$\frac{-2N}{r^2}\cdot\frac{1}{\sin^2\theta}\frac{\partial}{\partial r}(r\sin\theta h_\phi) - \frac{2S}{r}\left(\frac{d\omega_0}{d\theta}\right)^2\sin^2\theta$$

$$-\frac{2K}{r}\left[\omega_0^2 + \left(\frac{d\omega_0}{d\theta}\right)^2\right],^* \tag{4.3}$$

$$\frac{-Rw_0^2\cot\theta}{r}$$

$$= -\frac{\partial p}{r\partial\theta} + \Delta v - \frac{v}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial u}{\partial\theta}$$

$$-\frac{2N}{r^3}\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}(r\sin\theta h_\phi) - \frac{4S}{r}\sin\theta\cos\theta\left(\frac{d\omega_0}{d\theta}\right)^2$$

$$-\frac{6K}{r}\sin\theta\cos\theta\left(\frac{d\omega_0}{d\theta}\right)^2, \tag{4.4}$$

$$\frac{R}{r}\left[v\frac{\partial w_0}{\partial\theta} + 2uw_0 + vw_0\cot\theta\right] = \Delta w - \frac{w}{r^2\sin^2\theta}, \tag{4.5}$$

$$\frac{\partial u}{\partial r} + \frac{2u}{r} + \frac{\partial v}{r\partial\theta} + \frac{v\cot\theta}{r} = 0, \tag{4.6}$$

where

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

Equations (4.1)–(4.6) have to be solved under the boundary conditions: when $\theta = \theta_1, \theta_2$

$$h_r = h_\theta = h_\phi = 0 \text{ (continuity of the magnetic field)}, \tag{4.7}$$

$$u = v = w = 0 \text{ (no slip condition)}. \tag{4.8}$$

With the help of equation (4.1) and boundary conditions (4.7) we can easily show that

$$h_r = h_\theta = 0.$$

The induced magnetic field in the azimuthal direction is given by (4.2),

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* The details of these equations may be seen in [1].

As in [1], we assume solutions of the form

$$u = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta},$$

$$v = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},$$

$$\psi = Rr^5 F(\theta) + Sr^3 \bar{F}(\theta) + Kr^3 \tilde{F}(\theta),$$

$$h_\phi = Rr^3 V(\theta) + Sr\bar{\gamma}(\theta) + Kr\tilde{\gamma}(\theta).$$

It is evident that the successive terms in the above experessions for ψ and h_ϕ denote the contributions of inertial terms, cross-viscosity and visco-elasticity respectively. Equations (4.3)–(4.5) then give the following set of simultaneous equations for the determination of the functions F, $\bar{F}$, $\tilde{F}$, γ, $\bar{\gamma}$ and $\tilde{\gamma}$:

$$(D^2 - \cot\theta\, D + 6)(D^2 - \cot\theta\, D + + 20) F(\theta)$$
$$+ 4N(D - 3\cot\theta)\gamma(\theta) = 4 A_1\omega_0, \tag{4.9}$$

$$\sin^2\theta\left(D^2 + \cot\theta\, D + 12 - \frac{1}{\sin^2\theta}\right)\gamma(\theta) = 4M(D - 5\cot\theta) F(\theta), \tag{4.9 a}$$

$$(D^2 - \cot\theta\, D + 6)(D^2 - \cot\theta)\bar{F}(\theta) + 4N(D - \cot\theta)\bar{\gamma}(\theta)$$
$$= 32 A_1^2 \cot\theta\, \mathrm{cosec}^3\theta, \tag{4.10}$$

$$\sin^2\theta\left(D^2 + \cot\theta\, D + 2 - \frac{1}{\sin^2\theta}\right)\bar{\gamma}(\theta) = 4M(D - 3\cot\theta)\bar{F}(\theta), \tag{4.10 a}$$

$$(D^2 - \cot\theta\, D)(D^2 - \cot\theta\, D + 6)\tilde{F}(\theta) + 4N(D - \cot\theta)\tilde{\gamma}(\theta)$$
$$= 2\sin^2\theta\,[4\cos\theta\, D\omega_0^2 - \sin\theta\,\omega_0 D\omega_0 - \cos\theta\,\omega_0^2], \tag{4.11}$$

and

$$\sin^2\theta\left[D^2 + \cot\theta\, D + 2 - \frac{1}{\sin^2\theta}\right]\tilde{\gamma}(\theta) = 4M(D - 3\cot\theta)\tilde{F}(\theta), \tag{4.11 a}$$

where

$$\mathrm{D} \equiv \frac{d}{d\theta} \cdot$$

5. Solution for Small Angular Gap Between the Cones

When the angular gap between the cones is small, say α^c, we obtain solutions of the pairs of equations (4.9)–(4.11) in the form

$$\mathrm{F} = \sum_0^\infty a_n\phi^n,\ \mathrm{F} = \sum_0^\infty \bar{a}_n\phi^n,\ \bar{\mathrm{F}} = \sum_0^\infty \tilde{a}_n\phi^n,$$

$$\gamma = \sum_0^\infty b_n\phi^n,\ \bar{\gamma} = \sum_0^\infty \bar{b}_n\phi^n,\ \breve{\gamma} = \sum_0^\infty \breve{b}_n\phi^n,$$

where

$$\phi = \theta_2 - \theta.$$

In view of the boundary conditions (4.7) and (4.8) we have

$$a_0 = a_1 = \bar{a}_0 = \bar{a}_1 = \tilde{a}_0 = \tilde{a}_1 = b_0 = \bar{b}_0 = \breve{b}_0 = 0.$$

Equating the coefficients of various powers of ϕ in equations (4.9)–(4.11), we get

$$(b_2, \bar{b}_2, \breve{b}_2) = \tfrac{1}{2}\cot\theta_2\,(b_1, \bar{b}_1, \breve{b}_1), \tag{5.1}$$

$$6\sin^2\theta_2\, b_3 = -\,8\mathrm{M}a_2 + (3\cos^2\theta_2 + 1 - 12\sin^2\theta_2)\, b_1, \tag{5.2}$$

$$12\sin^2\theta_2\, b_4$$
$$= -\,12\mathrm{M}a_3 - 20\mathrm{M}a_2\cot\theta_2 - b_1\Big[1 - 12\sin 2\theta_2 - 5\cos^2\theta_2$$
$$+ \frac{5}{2}\cot\theta_2\,(3\cos^2\theta_2 + 1 - 12\sin^2\theta_2)\Big], \tag{5.3}$$

$$6\sin^2\theta_2\,(\bar{b}_3, \breve{b}_3) = -\,8\mathrm{M}a_2 + (3\cos^2\theta_2 + 1 - 2\sin^2\theta_2)\,(\bar{b}_1, \breve{b}_1), \tag{5.4}$$

$$12\sin^2\theta_2\,(\bar{b}_4, \breve{b}_4)$$
$$= -\,12\mathrm{M}a_3 - 12\mathrm{M}a_2\cot\theta_2) - \Big[1 - 12\sin^2\theta_2 - 5\cos^2\theta_2$$
$$+ \frac{5}{2}\cot\theta_2\,(3\cos^2\theta_2 + 1 - 2\sin^2\theta_2)\,(\bar{b}_1, \breve{b}_1)\Big], \tag{5.5}$$

for Newtonian liquid:

$$24a_4 + 12 \cot \theta_2 a_3 + (28 + 3 \cot^2\theta_2)\, 2a_2 = 4A_1\omega_2 + 4Nb_1, \qquad (5.6)$$

$$120a_5 + 48 \cot \theta_2 a_4 + 30\,(6 + \cot^2\theta_2)\, a_3 + 2a_2 \cot \theta_2\,(35 + 9 \cot^2\theta_2)$$
$$= -\,8A_1^2 \operatorname{cosec}^3\theta_2 + 4N\,(2b_2 + 3 \cot \theta_2 b_1), \qquad (5.7)$$

with the boundary conditions

$$\sum_2^\infty a_n\alpha^n = \sum_2^\infty n a_n \alpha^{n-1} = \sum_1^\infty b_n \alpha^n = 0; \qquad (5.8)$$

for Reiner-Rivlin fluid:

$$24a_4 + 12 \cot \theta_2 a_3 + 2\,(8 + 3 \cot^2\theta_2)\, a_2$$
$$= 4Nb_1 + 32A_1^2 \cot \theta_2 \operatorname{cosec}^3\theta_2, \qquad (5.9)$$

$$120a_5 + 48 \cot \theta_2 a_4 + 30a_3\,(2 + \cot^2\theta_2) + 6a_2 \cot \theta_2\,(5 + 3 \cot^2\theta_2)$$
$$= 4N\,(2b_2 + \cot \theta_2 b_1) + 32A_1^2 \operatorname{cosec}^3\theta_2\,(4 \operatorname{cosec}^2\theta_2 - 3), \qquad (5.10)$$

the boundary conditions being the same as in (5.8). The corresponding equations for Rivlin-Ericksen fluid are obtained by including

$$-\,4A_1\omega_2 - 2 \sin^2\theta_2 \cos\theta_2 \omega_2^2$$

and

$$8A_1^2 \operatorname{cosec}^3\theta_2 + 2\,(2 \sin \theta_2 - 3 \sin^3\theta_2)\, \omega_2^2 + 8A_1\omega_2 \cot \theta_2$$

respectively in R.H.S.'s of (5.9) and (5.10).

It was found difficult to get general information about the flows from these equations and consequently we have studied numerically the following cases:

$$\theta_2 = \frac{\pi}{2}, \quad \alpha = \frac{\pi}{45}.$$

Case (i)	*Case (ii)*
$\frac{\Omega_1}{\Omega_2} = 20,$	$\Omega_2 = 0,$
$H_a = 1,\ 100,\ 1000,\ 2000$	$H_a = 1,\ 100,\ 1000,\ 2000.$

the method of determining the coefficients being the same as in [1]. The values of the constants a's are recorded in Table I.

The upper values correspond to $\Omega_1/\Omega_2 = 20$ and the lower cones to $\Omega_2 = 0$,

TABLE I

	$H_a = 1$	$H_a = 100$	$H_a = 1000$	$H_a = 2000$
$\bar{a}_2$	− 0·000217029	− 0·00022202	− 0·01269267	0·01573677
	− 0·00023403	− 0·00019757	− 0·0148364	0·018944
$\bar{a}_2$	0·00785146	0·00793821	0·00631353	0·00593
	0·00930075	0·0094035	0·00747894	0·007
$\tilde{a}_2$	0·0100069	− 0·009929	0·007912	0·00735
	0·011626	− 0·01157112	0·0078441	0·461
a_3	0·00643617	0·00654667	0·3475	− 0·555
	0·0050927	0·00426336	0·405482	− 0·11347
$\bar{a}_3$	− 0·168513	− 0·171786	− 0·1246	− 0·1344
	− 0·199618	− 0·202555	− 0·147554	− 0·14163
$\tilde{a}_3$	− 0·216569	− 0·214352	− 0·1567	− 0·16807
	− 0·249523	− 0·247112	− 0·15476	3·5212
a_4	− 0·050796	− 0·051993	− 2·1422	4·2457
	0·00054595	− 0·00052412	− 2·484	− 0·39726
$\bar{a}_4$	− 0·0045355	0·0351338	− 0·3177	− 0·47059
	− 0·006202	0·0146767	− 08·3764	− 0·4881
$\tilde{a}_4$	0·0447197	0·0291716	−0381	− 0·595306
	− 0·0077524	− 0·0249	−0·3947	− 0·461032
a_5	0·0448826	0·0487536	−3·309	− 2·0942
	− 0·3478678	− 0·2865747	4·0112	− 2·80634
$\bar{a}_5$	11·57494	11·41298	11·553	11·554
	13·80023	13·71302	13·6855	13·687
$\tilde{a}_5$	14·38455	14·3814	14·3546	14·347
	17·13943	16·406	14·3536	17·099

6. STREAM FUNCTION

The stream lines to our approximation are given by

$$\psi \equiv \mathrm{R}r^5\,\mathrm{F}(\theta) + \mathrm{S}r^3\,\bar{\bar{\mathrm{F}}}(\theta) + \mathrm{K}r^3\,\tilde{\mathrm{F}}(\theta) = \text{constant}.$$

In the cone-plate arrangement considered above, $\mathrm{F}(\theta)$ is negative for $\mathrm{H}_a = 1$ and 100 but is positive for $\mathrm{H}_a \geqslant 1000$, whereas $\bar{\bar{\mathrm{F}}}(\theta)$ and $\tilde{\mathrm{F}}(\theta)$ are positive for all values of H_a. Figures 3 and 4 show the stream lines for

$\mathrm{H}_a = 100$ and $\mathrm{H}_a = 1000$ respectively.

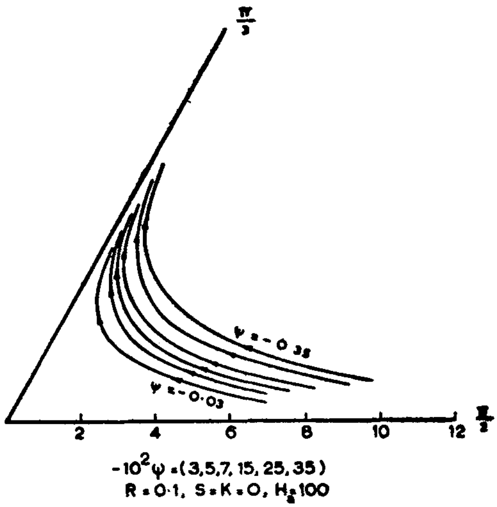

FIG. 1. Stream Lines in the Cone-Plate Arrangement.

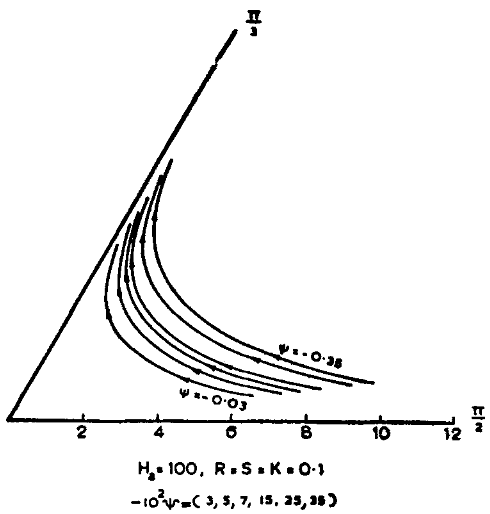

FIG. 2. Stream Lines in the Cone-Plate Arrangement.

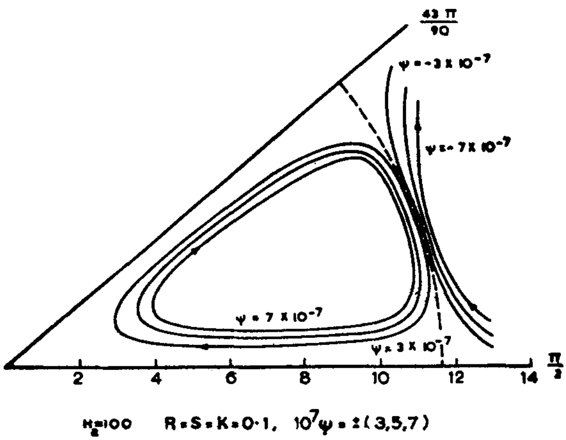

FIG. 3. Stream Lines in the Cone-Plate Arrangement.

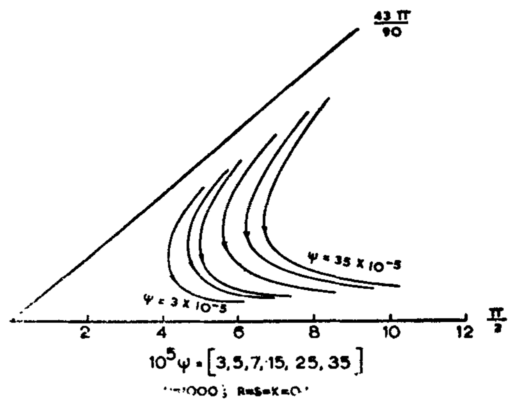

FIG. 4. Stream Lines in the Cone-Plate Arrangement.

It is interesting to note that in the case of flow separation (Fig. 3) in the presence of magnetic field, the dividing stream line is circular as was the case in [1].

7. Large Axial Current

In this section we shall assume the axial current and hence the Hartmann number H_a is large for any angular gap between the cones.

Writing

$$F(\theta) = \sin^5\theta\phi(\theta),$$

$$\bar{F}(\theta) = \sin^3\theta\bar{\phi}(\theta),$$

$$\bar{\bar{F}}(\theta) = \sin^3\theta\bar{\bar{\phi}}(\theta),$$

and eliminating $D\phi(\theta)$, $D\bar{\phi}(\theta)$, $D\breve{\phi}(\theta)$ between the pairs of equations (4.9)–(4.11), we have

$$\frac{A_1}{N}(D - \cot\theta)\,\omega_0$$

$$= \frac{1}{16H_a^2}[\sin^5\theta\, D^4 + 22\sin^4\theta\cos\theta\, D^3 + (147\sin^3\theta$$

$$- 167\sin^5\theta)\,D^2 + (333\sin^2\theta - 506\sin^4\theta)\cos\theta\, D$$

$$+ 192\sin\theta - 672\sin^3\theta + 504\sin^5\theta]$$

$$\times\left[\frac{1}{\sin^3\theta}\left(D^2 + \cot\theta\, D + 12 - \frac{1}{\sin^2\theta}\right)\gamma(\theta)\right]$$

$$+ (D - \cot\theta)(D - 3\cot\theta)\,\gamma(\theta), \tag{7.1}$$

$$\frac{8A_1^2}{N}(D + \cot\theta)\cot\theta\,\mathrm{cosec}^3\theta$$

$$= \frac{1}{16H_a^2}[\sin^3\theta\, D^4 + 14\sin^2\theta\cos\theta\, D^3$$

$$- \sin\theta\,(20 - 71\cos^2\theta)\,D^2 - \cos\theta\,(48 - 154\cos^2\theta)\,D$$

$$+ 24\sin\theta\,(1 - 5\cos^2\theta)]$$

$$\times\left[\frac{1}{\sin\theta}\left(D^2 + \cot\theta\, D + 2 - \frac{1}{\sin^2\theta}\right)\bar{\gamma}(\theta)\right]$$

$$+ (D + \cot\theta)(D - \cot\theta)\,\bar{\gamma}(\theta), \tag{7.2}$$

$$\frac{1}{2N}(D + \cot\theta)\,[\sin^2\theta\,\{4\cos\theta\,(D\omega_0)^2 - \sin\theta\,\omega_0 D\omega_0 - \cos\theta\,\omega_0^2\}]$$

$$= \frac{1}{16H_a^2}[\sin^3\theta\, D^4 + 14\sin^2\theta\cos\theta\, D^3$$

$$- \sin\theta\,(20 - 71\cos^2\theta)\,D^2 - \cos\theta\,(48 - 154\cos^2\theta)\,D$$

$$+ 24\sin\theta\,(1 - 5\cos^2\theta)]$$

$$\times\left[\frac{1}{\sin\theta}\left(D^2 + \cot\theta\, D + 2 - \frac{1}{\sin^2\theta}\right)\breve{\gamma}(\theta)\right]$$

$$+ (D + \cot\theta)(S - \cot\theta)\,\breve{\gamma}(\theta). \tag{7.3}$$

For large H_a we consider solutions of the above equations of the form

$$\gamma(\theta) = \frac{1}{N} \sum_{i=1}^{\infty} \frac{\gamma_i(\theta)}{H_a^{2(i-1)}}, \tag{7.4}$$

$$F(\theta) = \sum_{i=1}^{\infty} \frac{F_i(\theta)}{H_a^{2(i-1)}}, \tag{7.5}$$

with similar expressions for $\bar{\gamma}(\theta)$, $\bar{F}(\theta)$, $\breve{\gamma}(\theta)$ and $\breve{F}(\theta)$.

The equations giving the leading terms $\gamma_1(\theta)$, $\bar{\gamma}_1(\theta)$ and $\breve{\gamma}_1(\theta)$ namely,

$$(D - \cot\theta)\,[(D - 3\cot\theta)\,\gamma_1(\theta) - A_1\omega_0] = 0, \tag{7.6}$$

$$(D + \cot\theta)\,[(D - \cot\theta)\,\bar{\gamma}_1(\theta) - 8A_1^2\cot\theta\,\mathrm{cosec}^3\theta] = 0, \tag{7.7}$$

and

$$(D + \cot\theta)\,[(D - \cot\theta)\,\breve{\gamma}_1(\theta) - 2\sin^2\theta\,\{4\cos\theta\,(D\omega_0)^2 - \sin\theta\,\omega_0\,D\omega_0 - \cos\theta\,\omega_0^2\}] = 0, \tag{7.8}$$

admit the solutions

$$\gamma_1(\theta) = \sin^3\theta\,[b_1^1\cot\theta + b_2^1 + \tfrac{1}{2}\,\omega_0^2], \tag{7.9}$$

$$\bar{\gamma}_1(\theta) = \sin\theta\left[\bar{b}_1^1\cot\theta + \bar{b}_2^1 - \frac{2A_1^2}{\sin^4\theta}\right], \tag{7.10}$$

and

$$\breve{\gamma}_1(\theta) = \sin\theta\left[\breve{b}_1^1\cot\theta + \breve{b}_2^1 - \frac{2A_1^2}{\sin^4\theta} - \frac{\sin^2\theta\,\omega_0^2}{4}\right]. \tag{7.11}$$

Similarly, the equations giving $F_1(\theta)$, $\bar{F}_1(\theta)$ and $\breve{F}_1(\theta)$ are

$$(D^2 - \cot\theta\,D + 6)\,(D^2 - \cot\theta\,D + 20)\,F_1(\theta) = 4b_1^1\sin\theta, \tag{7.12}$$

$$(D^2 - \cot\theta\,D)\,(D^2 - \cot\theta\,D + 6)\,\bar{F}_1(\theta) = \frac{4\bar{b}_1^1}{\sin\theta}, \tag{7.13}$$

and

$$(D^2 - \cot\theta\,D)\,(D^2 - \cot\theta\,D + 6)\,\breve{F}_1(\theta) = \frac{4\breve{b}_1^1}{\sin\theta}, \tag{7.14}$$

which admit the solutions

$$F_1(\theta) = a_1^1(7\cos^5\theta - 10\cos^3\theta + 3\cos\theta)$$
$$+ a_2^1\Big[7\cos^4\theta - \frac{23}{3}\cos^2\theta + \frac{16}{15} + (7\cos^5\theta - 10\cos^3\theta$$
$$+ 3\cos\theta)\log\tan\frac{\theta}{2}\Big] + a_3^1(\cos^3\theta - \cos\theta)$$
$$+ a_4^1\Big[(\cos^3\theta - \cos\theta)\log\tan\frac{\theta}{2} + \cos^2\theta - \frac{2}{3}\Big]$$
$$+ \frac{4b_1^1}{45}\sin^5\theta,$$

$$\bar{F}_1(\theta) = \bar{a}_1^1\cos\theta + \bar{a}_2^1 + \bar{a}_3^1(\cos^3\theta - \cos\theta)$$
$$+ \bar{a}_4^1\Big[\cos^2\theta + (\cos^3\theta - \cos\theta)\log\tan\frac{\theta}{2} - \frac{4}{3}\sin^3\theta\bar{b}_1^1\Big],$$

and

$$\tilde{F}_1(\theta) = \tilde{a}_1^1\cos\theta + \tilde{a}_2^1 + \tilde{a}_3^1(\cos^3\theta - \cos\theta)$$
$$+ \tilde{a}_4^1\Big[\cos^2\theta + (\cos^3\theta - \cos\theta)\log\tan\frac{\theta}{2}\Big] - \frac{4\tilde{b}_1^1}{3}\sin^3\theta.$$

For $i > 1$, we have

$$F_i(\theta) = a_1^i(7\cos^5\theta - 10\cos^3\theta + 3\cos\theta)$$
$$+ a_2^i\Big[7\cos^4\theta - \frac{23}{3}\cos^2\theta + \frac{16}{15} + (7\cos^5\theta - 10\cos^3\theta$$
$$+ 3\cos\theta)\log\tan\frac{\theta}{2}\Big] + a_3^i(\cos^3\theta - \cos\theta)$$
$$+ a_4^i\Big[(\cos^3\theta - \cos\theta)\log\tan\frac{\theta}{2} + \cos^2\theta - \frac{2}{3}\Big]$$
$$+ \frac{4b_1^i}{45}\sin^5\theta + \sin^5\theta\, L^i(\theta),$$

$$\gamma^i(\theta) = \sin^3\theta\Big[b_1^i\cot\theta + b_2^i - \Big\{\frac{1}{16}\sin^2\theta\, D^3 + 16\sin\theta\cos\theta\, D^2$$
$$+ (77 - 63\sin^2\theta)\, D + 147\cot\theta\Big\}\, L^i(\theta)$$
$$- 12\int\frac{L^i(\theta)}{\sin^3\theta}\,d\theta\Big],$$

$$\bar{F}_i(\theta) = \bar{a}_1^i \cos\theta + \bar{a}_2^i + \bar{a}_3^i(\cos^3\theta - \cos\theta)$$
$$+ \bar{a}_4^i\left[\cos^2\theta + (\cos^3\theta - \cos\theta)\log\tan\frac{\theta}{2}\right]$$
$$- \frac{4}{3}\sin^3\theta\bar{b}_1^i + \sin^3\theta\bar{L}^i(\theta),$$

$$\bar{\gamma}_i(\theta) = \sin\theta\left[\bar{b}_1^i\cot\theta + \bar{b}_2^i - \frac{1}{16}\{\sin^2\theta D^3 + 8\sin\theta\cos\theta D^2\right.$$
$$\left. + (15\cos^2\theta - 2)D + \cot\theta\}\bar{L}^i(\theta)\right],$$

$$\tilde{F}_i(\theta) = \tilde{a}_1^i\cos\theta + \tilde{a}_2^i + \tilde{a}_3^i(\cos^3\theta - \cos\theta)$$
$$+ \tilde{a}_4^i\left[\cos^2\theta + (\cos^3\theta - \cos\theta)\log\tan\frac{\theta}{2}\right]$$
$$- \frac{4}{3}\sin^3\theta\tilde{b}_1^i + \sin^3\theta\tilde{L}^i(\theta),$$

and

$$\tilde{\gamma}_i(\theta) = \sin\theta\left[\tilde{b}_1^i\cot\theta + \tilde{b}_2^i - \frac{1}{16}\{\sin^2\theta D^3 + 8\sin\theta\cos\theta D^2\right.$$
$$\left. + (15\cos^2\theta - 2)D + \cot\theta\}\tilde{L}^i(\theta)\right],$$

where

$$DL^i(\theta) = \frac{1}{\sin^3\theta}\left[D^2 + \cot\theta D + 12 - \frac{1}{\sin^2\theta}\right]\gamma_{i-1}(\theta),$$
$$D\bar{L}^i(\theta) = \frac{1}{\sin\theta}\left[D^2 + \cot\theta D + 2 - \frac{1}{\sin^2\theta}\right]\bar{\gamma}_{i-1}$$
$$D\tilde{L}^i(\theta) = \frac{1}{\sin\theta}\left[D^2 + \cot\theta D + 2 - \frac{1}{\sin^2\theta}\right]\tilde{\gamma}_{i-1}(\theta).$$

The constants a's, b's are determined from the boundary conditions (4.7) and (4.8).

Figures 1 and 2 give the stream lines in the following two cases:

Case (a): $R = 0{\cdot}1,\ S = K = 0.$

and

Case (b): $R = S = K = 0{\cdot}1.$

8. CONCLUSIONS

We find that for small axial current the nature of the secondary flow is similar to the one predicted in [1], but if we have large axial current that is high azimuthal magnetic field, this breaking is avoided. It is interesting to note that

(*i*) the effects of cross-viscosity and viscoelasticity are similar inasmuch as they flatten the velocity profile in any meridian plane (Figs. 1, 2). (*ii*) for large angular gap, the flow separation is suppressed at smaller values of Hartmann number (Figs. 1, 3, 4). (*iii*) there is no secondary flow separation in case of Newtonian fluids and for small values of Hartmann number the flow is similar to the one described in [1] but for large values of Hartmann number the sense of flow is reversed.

REFERENCE

1. Bhatnagar, P. L. and Rathna, S. L. *Quar. Journ. Mech. Appl. Maths.*, 1963, **16**, 329.