NOTE ON THE LOCALLY PRODUCT AND ALMOST LOCALLY PRODUCT STRUCTURES

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Abstract

This paper is concerned with a study of some of the properties of locally product and almost locally product structures on a differentiable manifold X_n of class C^k. Every locally product space has certain almost locally product structures which transform the local tangent space to X_n at an arbitrary point P in a set fashion: this is studied in Theorem (2.2). Theorem (2.3) considers the nature of transformations that exist between two co-ordinate systems at a point whenever an almost locally product structure has the same local representation in each of these co-ordinate systems. A necessary and sufficient condition for X_n to be a locally product manifold is obtained in terms of the pseudo-group of co-ordinate transformations on X_n and the subpseudo-groups [*cf.*, Theoren (2.1)]. Sectiom 3 is entirely devoted to the study of integrable almost locally product structures.

§ 0. Notation and Conventions

In this section, we shall explain the symbolism used in the paper. Sometimes the symbol/s used is/are explained in the proper context.

The Greek indices, used both as sub- and superscripts, have always the same range of values $(1, 2, 3, \ldots, n)$. Only lower case Greek indices are used as sub- and superscripts for geometrical quantities. The lower case Latin (script) indices i, j, k take values in the range $\underline{1}, \underline{2}, \underline{3}, \ldots, \underline{n}$; however, in Section 1, we have used them to distinguish between co-ordinate maps. There will not be any confusion about this ambiguity if we keep in mind that in the former case, the indices are *always* placed above or below the kernel letter of a geometric quantity while in the latter case they occur as subscripts. The lower case Latin (script) indices a, b, c, always have the range $\underset{\cdot}{1}, \underset{\cdot}{2}, \underset{\cdot}{3}, \ldots, \underset{\cdot}{p}$; they occur as (i) subscripts of Greek indices or as (ii) distinguishing marks placed above the kernel of a geometric quantity. The indexed

Greek sub- or superscripts λ_a, μ_a, etc., take values in the range $m_{a-1}+1$, $m_{a-1}+2, \ldots, m_a$, where we have $m_{a-1} = n_1 + n_2 + \ldots + n_{a-1}$ for each value of a, $m_0 = 0$ and $m_p = n$. The summation convention applies only to Greek indices, whether further indexed by a, b or c, or not. The Latin indices *do not* follow the summation convention.

The symbol $\otimes$ is used to denote the tensor product of two geometrical quantities. It is also used to denote the tensor products of linear spaces.

§ 1. Locally Product and Almost Locally Product Structures

In this section we shall study some definitions basic to our work. The locally product and the almost locally product structures will be defined globally.

Definition (1.1 *a*).—Let X_n denote an n-dimensional real differentiable manifold of class C^k ($k \geqslant 2$) and let it be possible to cover X_n by a system of co-ordinate neighbourhoods $(U_i, \phi_i)_{i \in I}$ such that the following conditions hold:

(i) The indexing set I is countable and $X_n = \bigcup_{i \in I} (U_i)$.

(ii) For any two co-ordinate neighbourhoods (U_i, ϕ_i) and (U_j, ϕ_j) which are not disjoint, *i.e.*, $U_i \cap U_j \neq O$, the map $\phi_j \cdot \phi_i^{-1}: \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$ is a diffeomorphism of class C^k. This diffeomorphism has a non-singular Jacobian matrix of the form

$$(J\phi_j \cdot \phi_i^{-1}) = \text{diag.}\left((M_{n_1}, M_{n_2}, M_{n_3}, \ldots, M_{n_p})\right). \tag{1.1}$$

[Here M_{n_a} is a real non-singular matrix with n_a rows and columns for each $a = 1, 2, 3, \ldots, p$ and $\sum_{1}^{p} (n_a) = n$.] Then we say that the system of co-ordinate neighbourhoods $(U_i, \phi_i)_{i \in I}$ defines a *locally product structure* of order p and characteristic $(n_1, n_2, n_3, \ldots, n_p)$. The differentiable manifold endowed with such a structure will be called a *locally product manifold,* its order and characteristic being the same as those of the locally product structure.

A co-ordinate system belonging to a locally product structure has pleasant properties. We use such a co-ordinate system in the following.

Definition (1.1 *b*).—Let a differentiable manifold admit a locally product structure. Any co-ordinate system belonging to such a structure is called a *canonical co-ordinate system.*

The concept of a canonical co-ordinate system is due to T. Y. Thomas[6] (*see* § 4). In the next section it will be shown that there exists in every locally product space a quadratic tensor $T_\lambda{}^\mu$ of rank n which satisfies the following two properties: (i) In a canonical co-ordinate system, the tensor T has constant components, and (ii) it satisfies the following equation at each point of the manifold:

$$T_\lambda{}^\mu T_\mu{}^\nu = \delta_\lambda{}^\nu. \tag{1.2}$$

In the sense of property (i), the tensor T is canonical to a locally product manifold. (1.2) will now be used to define a new type of structure on a differentiable manifold:

Definition (1.2).—Let a tensor field $T_\lambda{}^\mu$ of rank n satisfying (1.2) be defined at each point of a differentiable manifold. Then such a tensor field is called an *almost locally product structure* on the manifold.

Every differentiable manifold X_n admits an infinity of almost locally product structures. For there always exists an infinite number of sets of n linearly independent local cross-sections $(\underset{i}{u})$ of the tangent bundle $T(X_n)$ and the corresponding sets of n linearly independent (dual) local cross-sections $(\overset{i}{u})$ of the cotangent bundle $T^*(X_n)$. If P is a generic point of X_n and (x) is a local coordinate system at P, then $\underset{i}{u}$ and $\overset{i}{u}$ have the following local representations:

$$\underset{i}{u} = \underset{i}{u}^\lambda \left(\frac{\partial}{\partial x^\lambda}\right) \tag{1.3 a}$$

$$\overset{i}{u} = \overset{i}{u}_\lambda (dx^\lambda). \tag{1.3 b}$$

Since $\underset{i}{u}$ and $\overset{i}{u}$ are dual local cross-sections, we have at P,

$$\underset{i}{u}^\lambda \overset{j}{u}_\lambda = \delta_i{}^j \tag{1.3 c}$$

$$\sum_{1}^{n} (\overset{i}{u}_\lambda \underset{i}{u}^\mu) = \delta_\lambda{}^\mu. \tag{1.3 d}$$

It is easy to see that $\Sigma\, \eta_i\, (\underset{i}{u} \otimes \overset{i}{u})$ is a local section of the tensor bundle $(T \otimes T^*)(X_n)$, where the fibre above an arbitrary point P of X_n is the vector space $T_n(P)$

$\otimes T_n{}^*(P)$ $(\eta_i = \pm 1)$. As a global consequence of (1.3 *cd*), we obtain at once the fact that the square of $\sum_{1}^{n} \eta_i (u \otimes \overset{i}{u})$ is the unit cross-section I of $(T \otimes T^*)(X_n)$. Thus we see that there are an infinity of almost locally product structures on a differentiable manifold X_n.

We end this section with a few bibliographical comments. The concept of a locally product structure has been studied by F. A. Ficken,[1] J. A. Schouten[5] (pp. 285–287), T. Y. Thomas,[6] K. Yano[7] (Chapter X), etc. A product space has been utilized by V. Hlavaty in his work on the Unified Field Theory of Einstein[4] (*cf.* § 6, Chapter III).

In the present work, the definitions of locally product and almost locally product structures are along the lines followed by Alfred Frölicher who defined the complex and almost complex structures globally. He also brought forth the close relation between these two structures (*cf.* A. Frölicher,[2] pp. 53–55). However, the generality in our definition has been motivated by the results of V. Hlavatý on almost complex spaces (*cf.* V. Hlavaty[3]). K. Yano deals extensively with the locally product and almost locally product spaces in his book[7] (*see* Chapters X and XI; here the reader could find further references to the work in this direction).

§ 2. Properties of Locally Product and Almost Locally Product Structures

Let X_n be a locally product manifold of class C^k, order p and characteristic $(n_1, n_2, n_3, \ldots, n_p)$. The locally product structure on the manifold splits the local tangent space $T_n(P)$ at a generic point P of the manifold into p distinct subspaces T_{n_a} of dimension n_a. The Jacobian matrix of co-ordinate transformations at P have the form (1.1) whenever the local co-ordinates are canonical to the structure. Then in such a case the Jacobian acts on $T_n(P)$ so that M_{n_a}, and none else, operates on the subspace T_{n_a}. Further, the totality of all canonical co-ordinate systems are such that for any two members (U, x) and (V, y), $U \cap V \neq O$, $x^{n_a} \epsilon C^k [U \cap V]$ and are independent, for $b \neq a$, of y^{n_b}. Hence if the pseudo-group of co-ordinate transformations for X_n is Γ_n, it contains the direct product $(\Gamma_{n_1} \times \Gamma_{n_2} \times \ldots \times \Gamma_{n_p})$ of the p sub-pseudo-groups Γ_{n_a}. Conversely, it is obvious that X_n is a locally product manifold whenever the pseudo-group Γ_n of co-ordinate transformations of X_n contains the direct product of P sub-pseudo-groups Γ_{n_a}. Then the order and characteristic of the locally product structure on X_n are respectively p and $(n_1, n_2, n_3, \ldots, n_p)$. Summarizing these results, we have the following:

Theorem (2.1).—*A necessary and sufficient condition for a differentiable manifold X_n to admit a locally product structure is: The pseudo-group Γ_n of co-ordinate transformations contains the direct product of the* p *subpseudo-groups Γ_{n_a}. If these conditions are satisfied, the locally product structure is of order* p *and characteristic $(n_1, n_2, n_3, \ldots, n_p)$.*

Now we shall define a canonical base for the local (co-) tangent space $(T_n^*(P))$ $T_n(P)$ at a generic point P of a locally product space. A similar base has already been used in our work (*see* the discussion immediately preceding the bibliographical comments of § 1).

Definition (2.1).—Let X_n be a locally product manifold of class C^k, order p and characteristic $(n_1, n_2, n_3, \ldots, n_p)$. If (x) is a local canonical co-ordinate system at a generic point P of the manifold, then

$$\left(\frac{\partial}{\partial x^{\lambda_1}}, \frac{\partial}{\partial x^{\lambda_2}}, \frac{\partial}{\partial x^{\lambda_3}}, \ldots, \frac{\partial}{\partial x^{\lambda_p}}\right)$$

constitutes a base for the local tangent space $T_n(P)$ at P. Similarly $(dx^{\lambda_1}, dx^{\lambda_2}, \ldots, dx^{\lambda_p})$ constitutes a base for the local cotangent space $T_n^*(P)$. These two bases are called *local canonical bases* for $T_n(P)$ and $T_n^*(P)$.

Let X_n be a locally product manifold of class C^k, order p and characteristic $(n_1, n_2, n_3, \ldots, n_p)$. Define the linear map $J: T_n(P) \to T_n(P)$ as follows: On a canonical base $(\partial/\partial x^\lambda)$ at P,

$$J\left(\frac{\partial}{\partial x^{\lambda_a}}\right) = \eta_a\left(\frac{\partial}{\partial x^{\lambda_a}}\right) (\eta_a = \pm 1). \tag{2.1 a}$$

It is easily seen that J is an endomorphism of $T_n(P)$. It is a simple matter to verify that J induces on $T_n^*(P)$ a unique endomorphism J^* such that the following hold:

$$J^*(dx^{\lambda_a}) = \eta_a dx^{\lambda_a}. \tag{2.1 b}$$

Now let (x) and (y) be two canonical co-ordinate systems. Affect the co-ordinate transformation $(x) \leftrightarrow (y)$. Then $((\partial y^\mu/\partial x^\lambda))$ and $((\partial x^\mu/\partial y^\lambda))$ are of the form (1: 1).

$$\left(\left(\frac{\partial y^\mu}{\partial x^\lambda}\right)\right) = \text{diag.}\left((M_{n_1}, M_{n_2}, M_{n_3}, \ldots, M_{n_a})\right) \tag{2.2 a}$$

$$\left(\left(\frac{\partial x^\mu}{\partial y^\lambda}\right)\right) = \text{diag.}\left((N_{n_1}, N_{n_2}, N_{n_3}, \ldots, N_{n_p})\right) \tag{2.2 b}$$

where

$$M_{n_a} = \left(\left(\frac{\partial y^{\mu_a}}{\partial x^{\lambda_a}}\right)\right) \tag{2.2 c}$$

$$N_{n_a} = \left(\left(\frac{\partial x^{\mu_a}}{\partial y^{\lambda_a}}\right)\right) \tag{2.2 d}$$

are $n_a \times n_a$ matrices.

The canonical bases for $T_n(P)$ and $T_n^*(P)$ transform according to the following:

$$\frac{\partial}{\partial y^{\lambda_a}} = \left(\frac{\partial x^{\mu_a}}{\partial y^{\lambda_a}}\right)\left(\frac{\partial}{\partial x^{\mu_a}}\right) \tag{2.2 e}$$

$$\frac{\partial}{\partial x^{\lambda_a}} = \left(\frac{\partial y^{\mu_a}}{\partial x^{\lambda_a}}\right)\left(\frac{\partial}{\partial y^{\mu_a}}\right) \tag{2.2 f}$$

$$dx^{\mu_a} = \left(\frac{\partial x^{\mu_a}}{\partial y^{\lambda_a}}\right) dy^{\lambda_a} \tag{2.2 g}$$

$$dy^{\mu_a} = \left(\frac{\partial y^{\mu_a}}{\partial x^{\lambda_a}}\right) dx^{\lambda_a}. \tag{2.2 h}$$

Because of (2.1 *ab*), (2.2 *e–h*) and the linearity of J and J* we have the following equations:

$$\eta_a \frac{\partial}{\partial x^a} = J\left(\frac{\partial}{\partial x^{\lambda_a}}\right) = \left(\frac{\partial y^{\mu_a}}{\partial x^{\lambda_a}}\right) J\left(\frac{\partial}{\partial y^{\mu_a}}\right), \tag{2.3 a}$$

$$\eta_a dx^{\lambda_a} = J^*(dx^{\lambda_a}) = \left(\frac{\partial x^{\lambda_a}}{\partial y^{\mu_a}}\right) J^*(dy^{\mu_a}). \tag{2.3 b}$$

Using (2.2 *fg*) in 2.3 *ab*) we obtain

$$\left(\frac{\partial y^{\mu_a}}{\partial x^{\lambda_a}}\right)\left[J\left(\frac{\partial}{\partial y^{\mu_a}}\right) - \eta_a\left(\frac{\partial}{\partial y^{\mu_a}}\right)\right] = 0, \tag{2.3 c}$$

$$\left(\frac{\partial x^{\mu_a}}{\partial y^{\lambda_a}}\right)[J^*(dy^{\lambda_a}) - \eta_a(dy^{\lambda_a})] = 0. \tag{2.3 d}$$

But the coefficient matrices in (2.3 *cd*) are non-singular matrices with n_a rows and columns. Therefore we have

$$J\left(\frac{\partial}{\partial y^{\mu_a}}\right) = \eta_a\left(\frac{\partial}{\partial y^{\mu_a}}\right) \tag{2.3 e}$$

$$J^*(dy^{\mu_a}) = \eta_a(dy^{\mu_a}). \tag{2.3 f}$$

Thus we see that J (J*) has a matrix representation of the form

$$((\mathrm{J})) = \text{diag.}\left((\eta_1 \delta_{\lambda_1}{}^{\mu_1}, \eta_2 \delta_{\lambda_2}{}^{\mu_2}, \ldots, \eta_p \delta_{\lambda_p}{}^{\mu_p})\right) \left(= ((\mathrm{J}^*))\right) \tag{2.4}$$

in any canonical co-ordinate system, and therefore it is a tensor whose square is the unit tensor. Consequently, J (J*) is an almost locally product structure on the given manifold. Further it may be noted that, in fact, J and J* are identical, and that there are precisely p linearly independent quadratic tensors $\overset{a}{\mathrm{J}}$ such that in canonical co-ordinates $\overset{a}{\mathrm{J}}$ has the form (2.4) with $\eta_a = -1$ and $\eta_b = +1$ $(a \neq b)$. Any other almost locally product structure which has the form (2′.4) in canonical co-ordinate systems on the manifold can be written as a product of $\overset{a}{\mathrm{J}}$'s.

Summing up these results, we have the following:

Theorem (2.2).—*Every locally product structure of class* C^k, *order* p *and characteristic* $(n_1, n_2, n_3, \ldots, n_p)$ *admits* p *linearly independent almost locally product structures* $\overset{a}{J}$ *which have the matrix representation in canonical co-ordinate systems given by* (2.4) *with* $\eta_a = -1$ *and* $\eta_b = +1$ $(a \neq b)$. *Any other almost locally product structure on the manifold which has the form* (2.4) *in canonical co-ordinate systems can be expressed as the product of the* $\overset{a}{J}$*'s.*

The results of Theorem (2.2) motivate the following definition:

Definition (2.2).—Any almost locally product structure on a locally product manifold of class C^k, order p and characteristic $(n_1, n_2, n_3, \ldots, n_p)$ having the form (2.4) in canonical co-ordinate system is called a *canonical almost locally product structure.*

Let J be an endomorphism of the local tangent space T_n (P) such that it has the same form (2.4) in two local co-ordinate systems (x) and (y) at P. If there is a change in co-ordinate system from (x) to (y), then the bases $(\partial/\partial x^{\mu})$ and $(\partial/\partial y^{\mu})$ for T_n (P) in the two co-ordinate systems are related by the following equations:

$$\left(\frac{\partial}{\partial y^{\mu_a}}\right) = \left(\frac{\partial}{\partial x^{\lambda}}\right)\left(\frac{\partial x^{\lambda}}{\partial y^{\mu_a}}\right). \tag{2.5 a}$$

Since J is linear and it is of the form (2.4) in the co-ordinate systems (x) and (y), we obtain the following after some simplifications:

$$\eta_a \left(\frac{\partial}{\partial y^{\lambda_a}}\right) = \left(\frac{\partial x^{\mu_a}}{\partial y^{\lambda_a}}\right) \eta_a \left(\frac{\partial}{\partial x^{\mu_a}}\right) + \sum_{b \neq a} \left(\frac{\partial x^{\mu_b}}{\partial y^{\lambda_a}}\right) \eta_b \left(\frac{\partial}{\partial x^{\mu_b}}\right). \tag{2.5 b}$$

Using (2.5 *a*) in the left-hand side of (2.5 *b*) and simplifying, we obtain the following equations:

$$\sum \left(\frac{\partial x^{\mu_b}}{\partial y^{\lambda_a}}\right)(\eta_b - \eta_a)\left(\frac{\partial}{\partial x^{\mu_b}}\right) = 0. \tag{2.5 c}$$

But the vectors of the base for T_n (P) in either co-ordinate system are linearly independent, and therefore we have the following equations:

$$\left(\frac{\partial x^{\mu_b}}{\partial y^{\lambda_a}}\right)(\eta_b - \eta_a) = 0. \tag{2.5 d}$$

The equations (2.5 *d*) are symmetric in *a* and *b*. Further, we have

$$\left(\frac{\partial x^{\mu_b}}{\partial y^{\lambda_a}}\right) = 0, \quad \text{for} \quad \eta_a \neq \eta_b. \tag{2.5 e}$$

Let us now denote by $\overset{a}{J}$ the endomorphism which has the form (2.4) in the two co-ordinate systems (x) and (y), where $\eta_a \neq \eta_b$ for $b \neq a$. If there are p such $\overset{a}{J}$ ($a = 1, 2, \ldots, p$), then these $\overset{a}{J}$'s are linearly independent. Further, the equations (2.5 *e*) hold for *all* a and b ($a \neq b$). Therefore the Jacobian matrix of the co-ordinate transformation (y) $\rightarrow$ (x) has the form (1.1).

These results are summarized by the following:

Theorem (2.3).—*Let* X_n *be a* C^k *manifold and let P be a generic point of* X_n. *Whenever there exist* p *endomorphisms* $\overset{a}{J}$ *of* T_n (P), *the endomorphism* $\overset{a}{J}$ *having a matrix representation of the form* (2.4) *with* $\eta_a \neq \eta_b$ *for* $a \neq b$ *in two co-ordinate systems* (x) *and* (y) *at P, then the Jacobian matrix of either of the co-ordinate transformations* $(x) \leftrightarrow (y)$ *is of the form* (1.1).

We end this section by stating a theorem about the algebraic structure of the set of all canonical almost locally product structures of a locally product space. The proof is left to the reader.

Theorem (2.4).—*The set of all canonical almost locally product structures on a locally product manifold of order* p *and characteristic* (n_1, n_2, n_3, $\ldots$, n_p) *forms an abelian group of finite order. The order of each element of the group is two.*

§ 3. INTEGRABLE ALMOST LOCALLY PRODUCT STRUCTURES

In the previous section we saw that there always existed almost locally product structures which were canonical in the sense of Definition (2.2) to a given locally product structure on the differentiable manifold X_n. The converse of this is not always true. In fact, we shall see in this section that if the Nijenhuis tensor of the almost locally product structure does not vanish identically, then the latter cannot be canonical to any locally product structure.

Definition (3.1).—Let $J_\lambda{}^\mu$ be an almost locally product structure on a differentiable manifold X_n. J will be said to be *integrable* whenever there exists a locally product structure on X_n such that J is canonical to it.

The Nijenhuis tensor belonging to the tensor $J_\lambda{}^\mu$ is defined as follows:

$$N_{\lambda\mu}{}^\nu \stackrel{\text{def.}}{=} 2\,(\partial_\gamma J_{[\lambda}{}^\nu)\, J_{\mu]}{}^\gamma + 2\,(\partial_{[\lambda} J_{\mu]}{}^\gamma)\, J_\gamma{}^\nu. \tag{3.1}$$

If the almost locally product structure J is canonical to a locally product structure on X_n, then it has constant components in a canonical co-ordinate system. Therefore its Nijenhuis tensor vanishes; as a result, we have the following:

Theorem (3.1).—*A necessary condition for an almost locally product structure J on X_n to be canonical to a locally product structure on X_n is: The Nijenhuis tensor* (3.1) *of J vanishes.*

In Section 2 we saw that the totality of all canonical almost product structures on a locally product manifold was a group. In the following we shall find conditions in order that a collection of almost locally product structures may be a group.

Let p linearly independent real almost locally product structures $\overset{a}{J}_\lambda{}^\mu$ be defined on X_n. Consider the two sets of p tensor fields $(\overset{a}{A}_\lambda{}^\mu)$ and $(\overset{a}{B}_\lambda{}^\mu)$ defined by the equations:

$$\overset{a}{A}_\lambda{}^\mu \stackrel{\text{def.}}{=} \tfrac{1}{2}\,(\delta_\lambda{}^\mu + \overset{a}{J}_\lambda{}^\mu) \tag{3.2 a}$$

$$\overset{a}{B}_\lambda{}^\mu \stackrel{\text{def.}}{=} \tfrac{1}{2}\,(\delta_\lambda{}^\mu - \overset{a}{J}_\lambda{}^\mu). \tag{3.2 b}$$

It may be verified that $\overset{a}{A}_\lambda{}^\mu$ and $\overset{a}{B}_\lambda{}^\mu$ are complimentary projection tensors on X_n since they satisfy the following:

$$\text{(i)}\ \overset{a}{B}_\lambda{}^\gamma\, \overset{a}{B}_\gamma{}^\mu = \overset{a}{B}_\lambda{}^\mu; \quad \text{(ii)}\ \overset{a}{A}_\lambda{}^\gamma\, \overset{a}{A}_\gamma{}^\mu = \overset{a}{A}_\lambda{}^\mu; \quad \text{(iii)}\ \overset{a}{A}_\lambda{}^\mu + \overset{a}{B}_\lambda{}^\mu = \delta_\lambda{}^\mu. \tag{3.2 c}$$

Let the rank of $((\overset{a}{\mathrm{B}}))$ be n_a so that the rank of $((\overset{a}{\mathrm{A}}))$ is $n - n_a$. Further suppose that the tensors $\overset{a}{\mathrm{B}}$ $(a = 1, 2, \ldots, p)$ satisfy the following equations:

$$\overset{a}{\mathrm{B}}_\lambda{}^\mu \overset{b}{\mathrm{B}}_\mu{}^\nu = 0 \text{ for all } a, b = 1, 2, 3, \ldots, p \text{ and } a \neq b. \tag{3.2 d}$$

Immediate consequences of (3.2 *abd*) are

$$\overset{b}{\mathrm{J}}_\lambda{}^\gamma \overset{a}{\mathrm{J}}_\gamma{}^\mu = (\overset{a}{\mathrm{J}}_\lambda{}^\mu + \overset{b}{\mathrm{J}}_\lambda{}^\mu - \delta_\lambda{}^\mu) = \overset{a}{\mathrm{J}}_\lambda{}^\gamma \overset{b}{\mathrm{J}}_\gamma{}^\mu \tag{3.3 a}$$

$$\overset{a}{\mathrm{A}}_\gamma{}^\gamma \overset{b}{\mathrm{A}}_\gamma{}^\mu = \tfrac{1}{2} (\overset{a}{\mathrm{J}}_\lambda{}^\mu + \overset{b}{\mathrm{J}}_\lambda{}^\mu). \tag{3.3 b}$$

Using the formulae (3.3) it can be easily verified that any product of the $\overset{a}{\mathrm{J}}$'s is an almost locally product structure. These results lead us to the following:

Theorem (3.2).—*The totality of all quadratic tensors which are products of the* p *almost locally product structures* $\overset{a}{J}$ *on* X_n *forms a finite abelian group G whenever* (*i*) *the* $\overset{a}{J}$ *are linearly independent, and* (*ii*) *the corresponding* $\overset{a}{B}$*'s defined by* (3.2 *b*) *satisfy* (3.2 *d*). *Every element of G is an almost locally product structure on* X_n.

Let us denote by $\overset{a}{\mathrm{N}}_{\lambda\mu}{}^\nu$, $'\overset{a}{\mathrm{N}}_{\lambda\mu}{}^\nu$, $''\overset{a}{\mathrm{N}}_{\lambda\mu}{}^\nu$ the Nijenhuis tensors corresponding to $\overset{a}{\mathrm{J}}$, $\overset{a}{\mathrm{A}}$, $\overset{a}{\mathrm{B}}$ respectively. The following relation among the different Nijenhuis tensors may be easily derived:

$$\overset{a}{\mathrm{N}} = 4\,'\overset{a}{\mathrm{N}} = 4\,''\overset{a}{\mathrm{N}}. \tag{3.5a}$$

Thus the three Nijenhuis tensors either vanish or do not vanish simultaneously. Let then the tensor $\overset{a}{\mathrm{N}}$ for a paricular value of a vanish identically. We shall now show that the structure $\overset{a}{\mathrm{J}}$ is integrable [*c.f.*, Definition (3.1)]. Since $\overset{a}{\mathrm{J}}$ is an almost locally product structure on the manifold, it is at once obvious that there are precisely n eigenvalues ± 1. We have assumed that the rank of the tensor $\overset{a}{\mathrm{B}}$ is n_a and therefore the eigenvalue $+1$ for $\overset{a}{\mathrm{J}}$ is repeated $(n - n_a)$ times and the eigenvalue -1 is repeated n_a times. Let $\overset{i_a}{u}{}^\mu$ denote the n_a eigenvectors of $\overset{a}{\mathrm{J}}$ belonging to the eigenvalue -1. We shall denote by $\underset{i_b}{u^\mu}$ the eigenvectors of $\overset{a}{J}$ belonging to the eigenvalue $+1$. It may be easily seen that $\underset{i_c}{u^\mu}$ for each value of $c = 1, 2, \ldots, p$ is an eigenvector of $\overset{c}{B}$ corresponding to the eigenvalue $+1$, which is repeated n_c times. Further the vectors $\underset{i_b}{u^\mu}$ $(b \neq c)$ are eigenvectors of $\overset{c}{B}$ corresponding to the eigenvalue 0. Let $\overset{i_c}{u_\mu}$ denote the n vectors dual to the system of vectors $\underset{i_c}{u^\mu}$ for $c = 1, 2, \ldots, p$. Then we have the following equations:

$$\overset{c}{B}_\lambda{}^\mu \underset{i_c}{u^\lambda} = \underset{i_c}{u^\mu} \quad (c = 1, 2, 3, \ldots, p) \tag{3.6 a}$$

$$\overset{b}{B}_\lambda{}^\mu \underset{i_c}{u^\lambda} = 0 \quad (b, c = 1, 2, 3, \ldots, p;\ b \neq c) \tag{3.6 b}$$

$$\overset{i_c}{u_\mu} \underset{j_c}{u^\mu} = \delta_{j_c}^{i_c} \quad \text{and} \quad \sum_1^n \overset{i}{(u_\lambda} \underset{i}{u^\mu)} = \delta_\lambda^\mu \tag{3.6 c}$$

$$\overset{i_b}{u_\mu} \underset{i_c}{u^\mu} = 0 \quad (b, c = 1, 2, 3, \ldots, p;\ b \neq c). \tag{3.6 d}$$

Since the Nijenhuis tensor of $\overset{a}{J}$ vanishes and since the vectors $\overset{i_b}{u_\mu}$ are covariant eigenvectors of $\overset{a}{J}_\lambda{}^\mu$ $(b \neq a)$ corresponding to the eigenvalue $+1$, we have the following equation:

$$\overset{a}{N}_{\lambda\mu}{}^\nu \overset{i_b}{u_\nu} = 2\overset{i_b}{u_\nu} (\partial_\gamma \overset{a}{J}_{[\lambda}{}^\nu) \overset{a}{J}_{\mu]}{}^\gamma + 2 (\partial_{[\lambda} \overset{a}{J}_{\mu]}{}^\gamma) \overset{i_b}{u_\gamma} = 0. \tag{3.7}$$

The Nijenhuis tensor $''\overset{a}{N}$ of $\overset{a}{B}$ also vanishes in the present case. Therefore the following equation, similar to (3.7), must hold:

$$''\overset{a}{N}_{\lambda\mu}{}^\nu \overset{i_b}{u_\nu} = 2\overset{i_b}{u_\nu} (\partial_\gamma \overset{a}{B}_{[\lambda}{}^\nu) \overset{a}{B}_{\mu]}{}^\gamma = 0. \tag{3.8}$$

Using (3.2 b) and (3.7) we obtain

$$\overset{i_b}{u_\nu} (\partial_{[\lambda} \overset{a}{B}_{\mu]}{}^\nu) = \overset{i_b}{u_\nu} (\partial_\gamma \overset{a}{J}_{[\lambda}{}^\nu) \overset{a}{J}_{\mu]}{}^\gamma. \tag{3.9}$$

Since the vectors $\underset{i_a}{u^\nu}$ $(\overset{i_a}{u_\lambda})$ are eigenvectors of $\overset{a}{B}_\lambda{}^\mu$ corresponding to the eigenvalue $+1$ and the vectors $\underset{i_b}{u^\nu}$ $(\overset{i_b}{u_\lambda})$ (for $b \neq a$) are eigenvectors of the same tensor corresponding to the eigenvalue 0, the following relation must hold:

$$\overset{a}{B}_{\lambda}{}^{\nu} = \Sigma_{i_a} (\overset{i_a}{u}_{\lambda} \underset{i_a}{u}^{\nu}). \qquad (3.10)$$

Using (3.6 *d*) and (3.10) in (3.9), we obtain the following equation:

$$\overset{a}{B}_{[\lambda}{}^{\nu} \partial_{\mu]} (\overset{i_b}{u}_{\nu}) = \overset{a}{J}_{[\lambda}{}^{\nu} \overset{a}{J}_{\mu]}{}^{\gamma} \partial_{\nu} \overset{i_b}{u}_{\gamma}. \qquad (3.11\,a)$$

On the other hand, the equations (3.8) themselves yield the equations:

$$\underset{k_a}{u}^{\lambda} \underset{j_a}{u}^{\mu} \partial_{[\lambda} \overset{i_b}{u}_{\mu]} = 0, \quad (a \neq b = 1, 2, 3, \ldots, p). \qquad (3.11\,b)$$

But (3.11 *b*) are the complete integrability conditions for the system of partial differential equations

$$\underset{i_a}{u}^{\lambda} \partial_{\lambda} f = 0, \qquad (3.11\,c)$$

and the T_{n_a}- field whose connecting quantities are $(\underset{i_a}{u}^{\mu}, \overset{i_b}{u}_{\lambda})$ is X_{n_a}-forming (*see* J. A. Schouten,[5] p. 81). Further there exist $n - n_a$ number of functions f which are linearly independent solutions of (3.11 *c*). These solutions are analytic and constant along the X_{n_a}'s which are themselves a normal family.

If the Nijenhuis tensors $\overset{a}{N}$ of $\overset{a}{J}$ for each $a = 1, 2, 3, \ldots, p$ vanishes, there exist n analytic functions $\overset{i}{f}$ $(i = 1, 2, 3, \ldots, n)$ such that n_a of them vary and the rest remain constant along the X_{n_a} for each value of a. These functions may be chosen as the co-ordinate functions on the manifold X_n. Let now $\overset{j}{g}$ $(j = 1, 2, 3, \ldots, n)$ be n analytic functions whose arguments are $\overset{i}{f}$ $(i = 1, 2, \ldots, n)$, and let the functions $\overset{i_a}{f}$ vary along X_{n_a} while the rest remain constant for each a. If the functions $\overset{j}{g}$ also have the same property along the normal X_{n_a}'s, it can be verified that the Jacobian of transformation of the set of functions $\overset{i}{f}$ into the set of functions $\overset{j}{g}$ is of the form (1.1). Conversely, if the Jacobian matrix of transformation of the set of functions $\overset{i}{f}$ into the set of functions $\overset{j}{g}$ is of the form (1.1), then $\overset{j_a}{g}$ $(j_a = m_{a-1} + 1,$

$m_{a-1}+2, m_{a-1}+3, \ldots, m_a$) for $a = 1, 2, 3, \ldots, p$ vary along X_n while the rest remain constant. In such a case, we have a locally product maifold of order p and characteristic $(n_1, n_2, \ldots, n_p)$.

We end this discussion by stating the results in the form of:

Theorem (3.3).—*Let* X_n *be endowed with* p *linearly independent almost locally product structures* $\overset{a}{J}$ *such that the projection tensors* $\overset{a}{B}$ *defined as in* (3.2 *b*) *for each* a = 1, 2, ..., p *satisfy the relation* (3.2 *d*). *Then a sufficient condition for the* $\overset{a}{J}$*'s to be canonical to a locally product structure of order* p *and characteristic* $(n_1, n_2, \ldots, n_p)$ *is: The Nijenhuis tensors* $\overset{a}{N}_{\lambda\mu}{}^{\nu}$ *of* $\overset{a}{J}$ *vanish.*

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REFERENCES

1. Ficken, F. A. .. "The Riemannian and affine differential geometry of product spaces," *Annals of Mathematics*, 1939, **40**, 892–913.
2. Alfred Frolicher .. "Zur Differentialgeometrie der Komplexen und Fast-Komplexen Structuren," *Mathematische Annalen*, 1955, **129**, 50–95.
3. Václav Hlavaty .. "An almost complex space of order p and characteristic $(n_1, n_2, \ldots n_p)$," *Tensor*, 1960, **10**, 90–124.
4. —— .. *Geometry of Einstein's Unified Field Theory*, P. Noordhoff, Ltd., Groningen, Holland, 1957.
5. Schouten, J. A. .. *Ricci Calculus*, Second Edition, Springer Verlag, Berlin, 1954.
6. Tracy Y. Thomas .. "Decomposition of Riemann spaces in the large," *Monatshefte für Mathematik und Physik*, 1939, Band, **49**, 388–418.
7. Kentaro Yano .. *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon Press, Macmillan and Co., New York, 1965.