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# Hamilton's theory of turns revisited

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**Abstract.** We present a new approach to Hamilton's theory of turns for the groups $SO(3)$ and $SU(2)$ which renders their properties, in particular their composition law, nearly trivial and immediately evident upon inspection. We show that the entire construction can be based on binary rotations rather than mirror reflections.

**Keywords.** Turns; groups $SO(3)$ and $SU(2)$; composition law for turns; rotations; reflections; binary rotations.

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## 1. Introduction

Hamilton's theory of turns, arising out of his profound and more general theory of quaternions, gives a beautiful geometric visualization of the elements of the groups $SO(3)$ and $SU(2)$, and of their noncommutative composition laws [1]. Thus, group elements can be pictured as equivalence classes of directed great circle arcs on a two-dimensional sphere of turns, the equivalence being with respect to sliding an arc along its great circle. The group composition law is expressed by the tail-to-head 'addition' of turns, reminiscent of the parallelogram law of addition of free vectors in the context of the Abelian group of translations in a Euclidean space.

The important difference between the two situations is that while the propositions of Euclidean geometry are scale-invariant, those pertaining to the sphere $\mathbb{S}^2$ are not. For instance, the unit sphere $\mathbb{S}^2$ does not support any notion of similar triangles; indeed, the angles of a spherical triangle fully quantify also the area through the spherical excess. In the present context these are hints of the noncommutativity of the composition law for turns. An important consequence of this lack of scale invariance is this: an $SO(3)$ rotation of amount $\alpha$ gets represented by a (geodesic) arc of length $\alpha/2$; no other multiple of $\alpha$ will return the correct multiplication law for the group in terms of geometric composition of these arcs.

An easily accessible and readable account of the theory of turns is given in the well-known monograph of Biedenharn and Louck on angular momentum in quantum physics [2]. Applications to classical polarization optics [3], to geometric phases for two-level systems [4], and generalizations to the noncompact groups $SL(2,\mathbb{R}) \sim SU(1,1) \sim Sp(2,\mathbb{R})$ [5,6] and $SL(2,\mathbb{C})$ [7] which are double covers of the Lorentz groups $SO(2,1)$ and $SO(3,1)$ respectively, have been developed elsewhere.

The purpose of this paper is to present a treatment of turns which renders their origin and properties extremely elementary, indeed to such an extent that further simplification would seem essentially impossible. The main property that is exploited is the fact that every plane rotation can be expressed (not uniquely) as the product of two reflections, and the amount of freedom available in doing so (this freedom or nonuniqueness turns out to be essential for the theory). It is then shown that the origin of the turn concept can be traced to this geometrical fact, and that it can be easily generalized to all proper rotations of $\mathbb{R}^3$. As a result of the formulae that one is led to, the (noncommutative) rule for composing turns is found to require no calculations at all as it is immediately evident upon inspection.

Section 2 settles notations for describing elements of the two groups $SO(3)$ and $SU(2)$ in a manner that matches the two-to-one homomorphism from the latter to the former. Section 3 considers first the representation of elements in the $SO(2)$ subgroup of $SO(3)$ as a product of two plane reflections, and then generalizes to all elements of $SO(3)$. As is known, the concept of turns originates from such representations. However, to render the group composition law as a trivial geometrical operation with turns requires that we re-express reflections in planes, which are improper rotations, in terms of reflections through lines, which are proper rotations. These are the so-called binary rotations, and the details are given in §4. The case of $SU(2)$ is taken up in §5, and §6 contains some concluding remarks.

## 2. Notational preliminaries, the groups $SO(3)$ and $SU(2)$

As is well known, $SO(3)$ and $SU(2)$ are locally isomorphic compact connected three-parameter Lie groups, which are moreover simple at the level of their (common) Lie algebra. We describe the elements of both groups using the well-known axis-angle parameters. Denoting axes by unit vectors $\widehat{\mathbf{n}}, \widehat{\mathbf{n}}', \ldots \in \mathbb{S}^2$ and angles of (right-handed) rotations by $\alpha, \beta, \ldots$, we can describe the (define representation of the) real proper orthogonal rotation group $SO(3)$ in three Euclidean dimensions as follows:

$$\begin{aligned} SO(3) &= \{\mathcal{R} = 3\times 3 \text{ real matrix} \,|\, \mathcal{R}^T\mathcal{R} = \mathbb{I}_{3\times 3},\ \det\mathcal{R} = 1\} \\ &= \{\mathcal{R}(\widehat{\mathbf{n}};\alpha) \,|\, \widehat{\mathbf{n}} \in \mathbb{S}^2,\ \alpha \in [0,\pi]\}, \end{aligned}$$

$$\mathcal{R}(\widehat{\mathbf{n}};\alpha) = (\mathcal{R}_{jk}(\widehat{\mathbf{n}};\alpha)), \quad j,k = 1,\ 2,\ 3,$$

$$\begin{aligned} \mathcal{R}_{jk}(\widehat{\mathbf{n}};\alpha) &= \delta_{jk}\cos\alpha + n_j n_k(1-\cos\alpha) - \epsilon_{jkl}\, n_l \sin\alpha \\ &= (\mathrm{e}^{-i\alpha\widehat{\mathbf{n}}\cdot\mathbf{J}})_{jk}, \quad (J_l)_{jk} = -i\epsilon_{jkl}. \end{aligned} \tag{2.1}$$

Group composition is given by matrix multiplication. While the angle $\alpha$ in $\mathcal{R}(\widehat{\mathbf{n}};\alpha)$ and $\mathcal{R}_{jk}(\widehat{\mathbf{n}};\alpha)$ can take any real value, it is adequate to limit it to the range $[0,\pi]$ on account of the easily verified relations

$$\begin{aligned}&\mathcal{R}(\widehat{\mathbf{n}};\alpha+2\pi)=\mathcal{R}(\widehat{\mathbf{n}};\alpha),\\&\mathcal{R}(\widehat{\mathbf{n}};\alpha)=\mathcal{R}(-\widehat{\mathbf{n}};2\pi-\alpha),\end{aligned}\tag{2.2}$$

provided we allow all $\widehat{\mathbf{n}}\in\mathbb{S}^2$. In this way there is a unique set of axis angle coordinates for each $\mathcal{R}\in SO(3)$, except for elements belonging to a subset of measure zero. These are the elements $\mathcal{R}\in SO(3)$ corresponding to right-handed rotations of amount $\pi$ about all possible axes $\widehat{\mathbf{n}}\in S^2$, sometimes called binary rotations. The nonuniqueness of parameters for such elements arises from the relations

$$\begin{aligned}&\mathcal{R}_{jk}(\widehat{\mathbf{n}};\pi)=2n_jn_k-\delta_{jk},\\&\mathcal{R}(\widehat{\mathbf{n}};\pi)=\mathcal{R}(-\widehat{\mathbf{n}};\pi),\end{aligned}\tag{2.3}$$

a reflection of the nontrivial global (topological) structure of $SO(3)$. Clearly, binary rotations are square roots of the identity:

$$\mathcal{R}(\widehat{\mathbf{n}};\pi)\mathcal{R}(\widehat{\mathbf{n}};\pi)=\mathbb{I}_{3\times3}.\tag{2.4}$$

The $SO(3)$ composition rule, i.e., the explicit determination of $\widehat{\mathbf{n}}'',\alpha''$ in terms of $\widehat{\mathbf{n}},\alpha$ and $\widehat{\mathbf{n}}',\alpha'$ is such that

$$\mathcal{R}(\widehat{\mathbf{n}}';\alpha')\mathcal{R}(\widehat{\mathbf{n}};\alpha)=\mathcal{R}(\widehat{\mathbf{n}}'';\alpha''),\tag{2.5}$$

was evidently first obtained by Rodrigues in 1840 using the geometry of spherical triangles on $S^2$ [8].

The analogous definitions for the unitary unimodular group $SU(2)$ in two complex dimensions are

$$\begin{aligned}SU(2)&=\{\mathcal{U}=2\times2\text{ complex matrix}\,|\,\mathcal{U}^\dagger\mathcal{U}=\mathbb{I}_{2\times2},\ \det\mathcal{U}=\infty\}\\&=\{\mathcal{U}(\widehat{\mathbf{n}};\alpha)\,|\,\widehat{\mathbf{n}}\in\mathbb{S}^2,\ \alpha\in[0,2\pi]\},\end{aligned}$$

$$\mathcal{U}(\widehat{\mathbf{n}};\alpha)=\mathrm{e}^{-i\alpha\widehat{\mathbf{n}}\cdot\boldsymbol{\sigma}/2}=\cos\frac{\alpha}{2}-i\widehat{\mathbf{n}}\cdot\boldsymbol{\sigma}\sin\frac{\alpha}{2}.\tag{2.6}$$

Here the $\boldsymbol{\sigma}$'s are the standard triplet of Pauli matrices. The range of the angle $\alpha$ is now $[0,2\pi]$ in contrast to the $SO(3)$ case, on account of the following replacements:

$$\begin{aligned}&\mathcal{U}(\widehat{\mathbf{n}};\alpha+2\pi)=-\mathcal{U}(\widehat{\mathbf{n}};\alpha),\\&\mathcal{U}(\widehat{\mathbf{n}};\alpha+4\pi)=\mathcal{U}(\widehat{\mathbf{n}};\alpha),\\&\mathcal{U}(\widehat{\mathbf{n}};2\pi)=-\mathbb{I}_{2\times2},\end{aligned}\tag{2.7}$$

for eqs (2.2). The Rodrigues formulae mentioned in eq. (2.5) hold again, with suitable extensions, for the $SU(2)$ composition law

$$\mathcal{U}(\widehat{\mathbf{n}}';\alpha')\mathcal{U}(\widehat{\mathbf{n}};\alpha)=\mathcal{U}(\widehat{\mathbf{n}}'';\alpha'').\tag{2.8}$$

$SU(2)$ is a two-fold cover of $SO(3)$. The corresponding homomorphism $\phi$ 'preserves' parameters in the sense that, consistent with eqs (2.2) and (2.7), we have

$$\phi:SU(2)\to SO(3):\ \phi(\mathcal{U}(\widehat{\mathbf{n}};\alpha))=\mathcal{R}(\widehat{\mathbf{n}};\alpha).\tag{2.9}$$

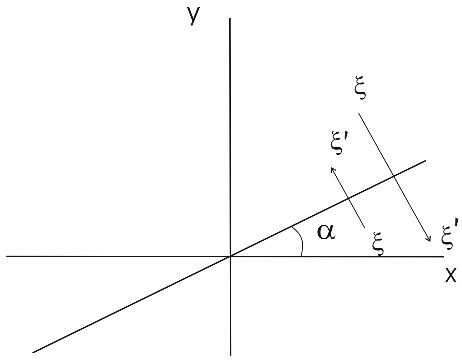


**Figure 1.** Showing the reflection $\mathcal{P}_0(\alpha)$.

## 3. Rotations, reflections and the origin of turns

Rotations about the $z$-axis, when $\widehat{\mathbf{n}} = \widehat{\mathbf{e}}_3$, form an $SO(2)$ subgroup of $SO(3)$:

$$\begin{aligned}
&SO(2; \widehat{\mathbf{e}}_3) = \{\mathcal{R}(\widehat{\mathbf{e}}_3; \alpha) \,|\, \alpha \in [0, 2\pi]\} \subset SO(3), \\
&\mathcal{R}(\widehat{\mathbf{e}}_3; \alpha')\mathcal{R}(\widehat{\mathbf{e}}_3; \alpha) = \mathcal{R}(\widehat{\mathbf{e}}_3; \alpha''), \\
&\alpha'' = \alpha' + \alpha \bmod 2\pi.
\end{aligned} \tag{3.1}$$

Notice that since the axis $\widehat{\mathbf{e}}_3$ is kept fixed, the range of $\alpha$ here is $[0, 2\pi]$ and not $[0, \pi]$ as in eq. (2.1). The action on the $x$ and $y$ coordinates in the plane perpendicular to $\widehat{\mathbf{e}}_3$ is most compactly expressed in complex form:

$$\begin{aligned}
&\xi = x + iy : \mathcal{R}(\widehat{\mathbf{e}}_3, \alpha)\xi = \xi' = \mathrm{e}^{i\alpha}\xi, \\
&\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned} \tag{3.2}$$

Let us now introduce the improper operation $\mathcal{P}_0(\alpha)$ which is reflection within the $x$–$y$ plane about the line passing through the origin $x = y = 0$ and making an angle $\alpha$ in the positive sense with the $x$-axis as shown in figure 1.

In equations we have

$$\begin{aligned}
&\mathcal{P}_0(\alpha)\xi = \xi' = \mathrm{e}^{2i\alpha}\xi^*, \\
&\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\end{aligned} \tag{3.3}$$

The $2 \times 2$ matrix $\mathcal{P}_0(\alpha)$ obeys

$$\begin{aligned}
&\mathcal{P}_0(\alpha)^T = \mathcal{P}_0(\alpha) = \mathcal{P}_0(\alpha + \pi), \\
&\mathcal{P}_0(\alpha)^2 = \mathbb{I}_{2\times 2}, \\
&\det \mathcal{P}_0(\alpha) = -1,
\end{aligned} \tag{3.4}$$

and so we can limit $\alpha$ here to $[0, \pi)$.

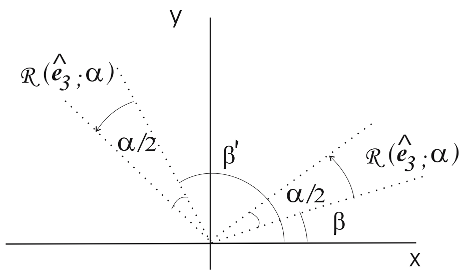


**Figure 2.** Showing the turn associated with a rotation in the $x$–$y$ plane.

The resultant of two such reflections about two generally different lines is a proper rotation:

$$\mathcal{P}_0(\beta)\mathcal{P}_0(\alpha)\xi = \mathcal{P}_0(\beta)\mathrm{e}^{2i\alpha}\xi^* = \mathrm{e}^{2i(\beta-\alpha)}\xi. \tag{3.5}$$

That is,

$$\mathcal{P}_0(\beta)\mathcal{P}_0(\alpha) = \mathcal{R}(\widehat{\mathbf{e}}_3; 2(\beta-\alpha)).$$

That is,

$$\mathcal{R}(\widehat{\mathbf{e}}_3;\alpha) = \mathcal{P}_0(\beta+\alpha/2)\mathcal{P}_0(\beta), \quad \text{any } \beta\in[0,2\pi). \tag{3.5}$$

We can represent this pictorially via the diagram in figure 2. It is clear that these reflections, for different values of $\alpha$, do not commute. It is for this reason that $O(2)$ is non-Abelian even though $SO(2)$ is an Abelian group.

Remembering that $\beta$ is arbitrary, we are immediately led to the representation of the plane rotation $\mathcal{R}(\widehat{\mathbf{e}}_3;\alpha)$ as a turn: a directed (counter-clockwise) arc of length $\alpha/2$ located anywhere on the unit circle in the $x$–$y$ plane. Again using the 'sliding' freedom – equivalence relation among arcs – we recover the composition law (3.1),

$$\begin{aligned}\mathcal{R}(\widehat{\mathbf{e}}_3;\alpha')\mathcal{R}(\widehat{\mathbf{e}}_3;\alpha) &= \mathcal{P}_0(\beta'+\alpha'/2)\mathcal{P}_0(\beta')\mathcal{P}_0(\beta+\alpha/2)\mathcal{P}_0(\beta),\\ &\qquad \text{any } \beta,\beta',\\ &= \mathcal{P}_0(\beta'+\alpha'/2)\mathcal{P}_0(\beta'-\alpha/2),\\ &\qquad \beta+\alpha/2=\beta', \quad \text{any } \beta',\\ &= \mathcal{R}(\widehat{\mathbf{e}}_3;\alpha+\alpha'). \end{aligned} \tag{3.6}$$

The geometrical reflection construction of $\mathcal{R}(\widehat{\mathbf{e}}_3;\alpha)$ thus leads immediately to the turns picture for such rotations.

We will now generalize these considerations for rotations on a fixed plane to the full group of $SO(3)$ rotations, but it is useful to keep the following in mind. Since $SO(2)$ is Abelian (and continuous), the double or indeed any multiple cover of $SO(2)$ remains isomorphic to $SO(2)$, displaying a kind of scale invariance. However, when we pass on from $SO(2)$ to the full non-Abelian group $SO(3)$, this feature is lost, just as the scale invariance of Euclidean space is absent on the unit sphere $\mathbb{S}^2$.

The extension to general $\mathcal{R}(\widehat{\mathbf{n}};\alpha)$ is straightforward. For any mutually orthogonal vectors $\widehat{\mathbf{n}},\widehat{\mathbf{n}}_1\in\mathbb{S}^2$ we define

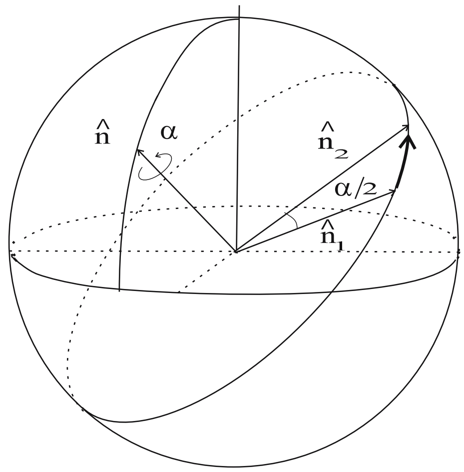


**Figure 3.** Showing the turn picture of a general rotation $\mathcal{R}(\widehat{\mathbf{n}};\alpha)$.

$$\begin{aligned}\mathcal{P}(\widehat{\mathbf{n}};\widehat{\mathbf{n}}_1) &= \text{reflection, in plane perpendicular to } \widehat{\mathbf{n}}, \text{ in line } \widehat{\mathbf{n}}_1\\ &= \text{reflection about the plane spanned by } \widehat{\mathbf{n}},\ \widehat{\mathbf{n}}_1.\end{aligned} \tag{3.7}$$

Then as in eq. (3.5) we easily obtain

$$\begin{aligned}&\mathcal{R}(\widehat{\mathbf{n}};\alpha) = \mathcal{P}(\widehat{\mathbf{n}};\widehat{\mathbf{n}}_2)\mathcal{P}(\widehat{\mathbf{n}};\widehat{\mathbf{n}}_1),\\ &\widehat{\mathbf{n}}_1\cdot\widehat{\mathbf{n}}_2 = \cos\frac{\alpha}{2},\quad \widehat{\mathbf{n}}_1\wedge\widehat{\mathbf{n}}_2 = \widehat{\mathbf{n}}\sin\frac{\alpha}{2}.\end{aligned} \tag{3.8}$$

Clearly $\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2$ can be any two vectors perpendicular to $\widehat{\mathbf{n}}$, enclosing angle $\alpha/2$, such that $\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2, \widehat{\mathbf{n}}$ form a right-handed system. This gives the turns picture for the three-dimensional rotation $\mathcal{R}(\widehat{\mathbf{n}};\alpha)$:

$$\begin{aligned}\mathcal{R}(\widehat{\mathbf{n}};\alpha) = \ &\text{any directed (by right-hand rule) arc of `length' } \alpha/2\\ &\text{along great circle on } \mathbb{S}^2 \text{ perpendicular to } \widehat{\mathbf{n}}.\end{aligned} \tag{3.9}$$

This is pictorially depicted in figure 3.

However, this expression for a general $\mathcal{R}$ as a product of two $\mathcal{P}$'s is not yet in a form convenient for reading off a composition law for turns. For this we need to go back from $\mathcal{P}$'s to $\mathcal{R}$'s.

## 4. The composition law for turns

Let us go back to the reflection $\mathcal{P}_0(\alpha)$ of eqs (3.3), (3.4) within the $x$–$y$ plane. With the understanding that $z$ is invariant, we can extend the $2\times 2$ matrix of $\mathcal{P}_0(\alpha)$ to a $3\times 3$ matrix retaining the same symbol for simplicity:

$$\mathcal{P}_0(\alpha) = \begin{pmatrix}\cos 2\alpha & \sin 2\alpha & 0\\ \sin 2\alpha & -\cos 2\alpha & 0\\ 0 & 0 & 1\end{pmatrix},$$

$$\mathcal{P}_0(\alpha)^T = \mathcal{P}_0(\alpha), \quad \mathcal{P}_0(\alpha)^2 = \mathbb{I}_{3\times 3},$$
$$\det \mathcal{P}_0(\alpha) = -1. \tag{4.1}$$

Therefore, since we are now in odd dimension, we see that $-\mathcal{P}_0(\alpha)$ is a symmetric element of $SO(3)$, a proper binary rotation:

$$-\mathcal{P}_0(\alpha) = \mathcal{R}(\pm\sin\alpha, \mp\cos\alpha, 0; \pi) \tag{4.2}$$

both sign choices being allowed for binary rotations. For a general $\mathcal{P}(\widehat{\mathbf{n}}; \widehat{\mathbf{n}}_1)$ defined in eq. (3.7) we can then write the $3 \times 3$ matrix result as

$$\mathcal{P}(\widehat{\mathbf{n}}; \widehat{\mathbf{n}}_1) = -\mathcal{R}(\pm\widehat{\mathbf{n}} \wedge \widehat{\mathbf{n}}_1; \pi). \tag{4.3}$$

(We emphasize that the geometric meaning of the left-hand side is that it is a reflection in three dimensions in the plane perpendicular to $\widehat{\mathbf{n}} \wedge \widehat{\mathbf{n}}_1$, i.e., in the plane containing $\widehat{\mathbf{n}}$ and $\widehat{\mathbf{n}}_1$. Thus points in this plane are unaffected. A binary rotation $\mathcal{R}(\cdot, \pi)$ on the other hand is a reflection in three dimensions through a line, not in a plane.)

Putting this back into eq. (3.8) we have

$$\begin{aligned} \mathcal{R}(\widehat{\mathbf{n}}; \alpha) &= \mathcal{P}(\widehat{\mathbf{n}}; \widehat{\mathbf{n}}_2)\mathcal{P}(\widehat{\mathbf{n}}; \widehat{\mathbf{n}}_1) \\ &= \mathcal{R}(\pm\widehat{\mathbf{n}} \wedge \widehat{\mathbf{n}}_2; \pi)\mathcal{R}(\pm\widehat{\mathbf{n}} \wedge \widehat{\mathbf{n}}_1; \pi), \\ \widehat{\mathbf{n}}_1 \cdot \widehat{\mathbf{n}}_2 &= \cos\alpha/2, \quad \widehat{\mathbf{n}}_1 \wedge \widehat{\mathbf{n}}_2 = \widehat{\mathbf{n}}\sin\alpha/2. \end{aligned} \tag{4.4}$$

Given $\widehat{\mathbf{n}}$ and $\alpha$ to begin with, all four choices of signs are permitted, and $\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2$ can be chosen freely subject to the conditions given. From the geometry involved, we see that we can simplify this to say:

$$\begin{aligned} \mathcal{R}(\widehat{\mathbf{n}}; \alpha) &= \mathcal{R}(\pm\widehat{\mathbf{n}}_2; \pi)\mathcal{R}(\pm\widehat{\mathbf{n}}_1; \pi), \quad \text{any signs}, \\ \widehat{\mathbf{n}}_1 \cdot \widehat{\mathbf{n}}_2 &= \cos\alpha/2, \quad \widehat{\mathbf{n}}_1 \wedge \widehat{\mathbf{n}}_2 = \widehat{\mathbf{n}}\sin\alpha/2. \end{aligned} \tag{4.5}$$

The turn representing the element on the left runs from $\widehat{\mathbf{n}}_1$ towards $\widehat{\mathbf{n}}_2$.

Now the composition rule is immediate. Start with elements $\mathcal{R}(\widehat{\mathbf{n}}; \alpha)$, $\mathcal{R}(\widehat{\mathbf{n}}'; \alpha')$ and assume $\widehat{\mathbf{n}} \neq \widehat{\mathbf{n}}'$ for definiteness. In the sense of eq. (4.5), let the pair $\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2$ go with the first element, and $\widehat{\mathbf{n}}_3, \widehat{\mathbf{n}}_4$ with the second. Since the two great circles definitely intersect, we use the sliding freedom to arrange $\widehat{\mathbf{n}}_2 = \widehat{\mathbf{n}}_3$ and then get

$$\begin{aligned} \mathcal{R}(\widehat{\mathbf{n}}'; \alpha')\mathcal{R}(\widehat{\mathbf{n}}; \alpha) &= \mathcal{R}(\widehat{\mathbf{n}}_4; \pi)\mathcal{R}(\widehat{\mathbf{n}}_3; \pi)\mathcal{R}(\widehat{\mathbf{n}}_2; \pi)\mathcal{R}(\widehat{\mathbf{n}}_1; \pi) \\ &= \mathcal{R}(\widehat{\mathbf{n}}_4; \pi)\mathcal{R}(\widehat{\mathbf{n}}_1; \pi), \quad \widehat{\mathbf{n}}_2 = \widehat{\mathbf{n}}_3, \\ &= \mathcal{R}(\widehat{\mathbf{n}}''; \alpha'') \end{aligned} \tag{4.6}$$

the 'product' turn runs from $\widehat{\mathbf{n}}_1$ to $\widehat{\mathbf{n}}_4$.

## 5. The $SU(2)$ case

We can develop a similar argument now, relying more on algebraic relations than pure geometry. We begin with the general result

$$(\mathbf{a}\cdot\boldsymbol{\sigma})\,(\mathbf{b}\cdot\boldsymbol{\sigma}) = \mathbf{a}\cdot\mathbf{b} + i\mathbf{a}\wedge\mathbf{b}\cdot\boldsymbol{\sigma} \tag{5.1}$$

involving the Pauli matrices. Now, for a general element $\mathcal{U}(\widehat{\mathbf{n}};\alpha) \in SU(2)$ choose $\widehat{\mathbf{n}}_1$ and $\widehat{\mathbf{n}}_2$ as indicated in eqs (3.8) and (4.4), remembering however the extended range of $\alpha$. Then we find

$$\mathcal{U}(\widehat{\mathbf{n}};\alpha) = -\mathcal{U}(\widehat{\mathbf{n}}_2;\pi)\,\mathcal{U}(\widehat{\mathbf{n}}_1;\pi) \tag{5.2}$$

which leads to the turns picture. The elements $\mathcal{U}(\widehat{\mathbf{n}}';\pi) = -i\widehat{\mathbf{n}}'\cdot\boldsymbol{\sigma}$ play the role of the earlier binary rotations, with the important difference that their squares are $-\mathbb{I}_{2\times 2}$ as seen in the last line of eqs (2.7). Now for the product rule:

$$\begin{aligned}\mathcal{U}(\widehat{\mathbf{n}}';\alpha')\mathcal{U}(\widehat{\mathbf{n}};\alpha) &= \mathcal{U}(\widehat{\mathbf{n}}_4;\pi)\mathcal{U}(\widehat{\mathbf{n}}_3;\pi)\mathcal{U}(\widehat{\mathbf{n}}_2;\pi)\mathcal{U}(\widehat{\mathbf{n}}_1;\pi)\\ &= -\mathcal{U}(\widehat{\mathbf{n}}_4;\pi)\mathcal{U}(\widehat{\mathbf{n}}_1;\pi), \quad \widehat{\mathbf{n}}_2 = \widehat{\mathbf{n}}_3\\ &= \mathcal{U}(\widehat{\mathbf{n}}'';\alpha'').\end{aligned} \tag{5.3}$$

The sliding freedom has been used to arrange $\widehat{\mathbf{n}}_2 = \widehat{\mathbf{n}}_3$, and the turn for the product element then runs from $\widehat{\mathbf{n}}_1$ towards $\widehat{\mathbf{n}}_4$. As in the $SO(3)$ case, the noncommutative 'addition' of turns involves no calculations at all.

## 6. Concluding remarks

The main step leading to trivialization of the composition law for $SO(3)$ turns is the expression (3.5) [equivalently (4.5)] of a general element $\mathcal{R}(\widehat{\mathbf{n}};\alpha)$ in factored form, one factor each coming from the tail and the head of the turn. (This also means that the great circle arc from $\widehat{\mathbf{n}}_1$ to $\widehat{\mathbf{n}}_2$ is drawn by us to help in visualization, as eq. (4.5) does not by itself require that it be drawn.) The equation is also valid as a relation between abstract group elements, i.e., it expresses a property of the group in itself though we have obtained it through the defining three-dimensional representation.

That every $SO(3)$ rotation is a (nonunique) ordered pair of reflections about planes is known. That the same can be realized as an ordered pair of binary rotations has the advantage that we stay within the $SO(3)$ group without having to make a 'virtual transition' to the $O(3)$ group. A binary rotation in three dimensions is the reflection through an axis, but this reflection (unlike reflection in a plane) is an $SO(3)$ element.

In the case of $SU(2)$ relation (5.2) holding in its defining representation, the representation at the level of abstract group elements involves viewing the right-hand side as a product of three group elements; the negative sign on the right-hand side stands for $-\mathbb{I}_{2\times 2}$ in the defining representation, and so for the nontrivial second element in the centre $\mathbb{Z}_2$ of $SU(2)$ in the abstract. However the fact that this is in the centre, and the property $\mathcal{U}(\widehat{\mathbf{n}};\pi)^2 = -\mathbb{I}_{2\times 2}$ mentioned after eq. (5.2), together ensure the proof of the turns composition law (5.3) remains completely trivial.

We may point out that binary rotations $\mathcal{R}(\widehat{\mathbf{n}};\pi)$ in $SO(3)$ and the special elements $\mathcal{U}(\widehat{\mathbf{n}};\pi)$ in $SU(2)$ play similar roles in making composition law for turns trivial in each case. According to eq. (2.9), the former are the results of the homomorphism $\phi\colon SU(2) \to SO(3)$ applied to the latter.

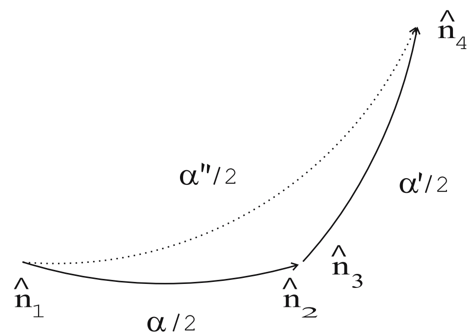


**Figure 4.** Showing that the length of the composed turn can be as large as the sum of the lengths of the component turns.

The important difference between $SO(3)$ and $SU(2)$ needs mention. In the case of $SO(3)$, every turn has 'arc length' not exceeding $\pi/2$; while in $SU(2)$ we encounter turns of arc length up to $\pi$. When we use the turns composition rule for two $SO(3)$ elements, $\mathcal{R}(\widehat{\mathbf{n}};\alpha)$ and $\mathcal{R}(\widehat{\mathbf{n}}';\alpha')$, it can happen that even though $\alpha/2, \alpha'/2$ are both less than or at most $\pi/2$, their 'resultant' $\alpha''/2$ could exceed $\pi/2$ as indicated in figure 4.

It is true that $\alpha''/2$, being the length of the geodesic from $\widehat{\mathbf{n}}_1$ to $\widehat{\mathbf{n}}_4$, is less than or equal to $\alpha/2+\alpha'/2$, that is, $\alpha''/2 \le \pi$; but this allows for $\alpha''/2 > \pi/2$, i.e, we could have $\pi/2 < \alpha''/2 \le \pi$. However, in that case we can argue as follows:

$$\frac{\alpha''}{2} > \frac{\pi}{2} \Rightarrow \frac{(2\pi-\alpha'')}{2} < \frac{\pi}{2}$$

and then appealing to $\mathcal{R}(\widehat{\mathbf{n}};\alpha'') = \mathcal{R}(-\widehat{\mathbf{n}};2\pi-\alpha'')$ we can represent the 'product' turn also by an arc of length not more than $\pi/2$, but with reversed sense. Such a problem is absent in the $SU(2)$ case.

This step which may be needed in the $SO(3)$ case motivates the following additional remarks. As is well known, $SU(2)$ and $SO(3)$ share a common Lie algebra as they are locally isomorphic. Nevertheless it is $SU(2)$ that is specially associated with this Lie algebra, in the sense that it is the unique simply connected Lie group arising from this Lie algebra. Globally as a manifold $SU(2)$ is the same as $\mathbb{S}^3$, while $SO(3)$ is $\mathbb{S}^3$ modulo the identification of 'diametrically opposite' (or antipodal) points. (Thus $SU(2)$ is the double and universal covering group of $SO(3)$.) As a consequence, any (irreducible) matrix representation of the Lie algebra always exponentiates to an (irreducible) representation of $SU(2)$, which is faithful in only 'half' the cases. It is only in the non-faithful cases that we have an $SO(3)$ representation. These facts ultimately underlie the comments made above in regard to turns for $SO(3)$.

Given any two points $\widehat{\mathbf{n}}_1$, $\widehat{\mathbf{n}}_2 \in \mathbb{S}^2$, we can always understand the angle between them, written as $\alpha/2$ as in §3–5 above, to be in the range $[0,\,\pi]$: $\widehat{\mathbf{n}}_2 = \pm\widehat{\mathbf{n}}_1$ correspond to $\alpha/2 = 0,\,\pi$ respectively, otherwise $0 < \alpha/2 < \pi$. Now if we read eq. (4.5) from left to right, i.e., we start with some $SO(3)$ element $\mathcal{R}(\widehat{\mathbf{n}};\alpha)$ where $\widehat{\mathbf{n}} \in S^2$, $\alpha \in [0,\,\pi]$ and find a pair $\widehat{\mathbf{n}}_1$, $\widehat{\mathbf{n}}_2$ for the right-hand side, by our construction we will have $0 \le \alpha/2 \le \pi/2$. But if we read this equation 'backwards' and start with a general pair of points $\widehat{\mathbf{n}}_1$, $\widehat{\mathbf{n}}_2$ on $\mathbb{S}^2$; in case $0 \leq \alpha/2 \leq \pi/2$ we have the same situation as before; but in case $\pi/2 < \alpha/2 \leq \pi$, we replace $\widehat{\mathbf{n}}_2$ by $\widehat{\mathbf{n}}_2' = -\widehat{\mathbf{n}}_2$ and $\mathcal{R}(\widehat{\mathbf{n}}_2;\pi)$ by $\mathcal{R}(\widehat{\mathbf{n}}_2';\pi)$, and have the angle between $\widehat{\mathbf{n}}_1$ and $\widehat{\mathbf{n}}_2'$ back in the range $[0, \pi/2]$. This sign freedom is already explicitly stated in eq. (4.5).

We can summarize by saying that turns are naturally or intrinsically associated with $SU(2)$, there being no restrictions in the choices of $\widehat{\mathbf{n}}_1, \widehat{\mathbf{n}}_2$ in eq. (5.2); the angle $\alpha/2$ between them can be anywhere in $[0, \pi]$ as permitted by $S^2$. Such restrictions show up only when we use turns to represent $SO(3)$ elements, so they must be carried along.

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