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Effect of viscosity on internal waves from a source in a wall

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ABSTRACT

The solution for a line source of oscillatory strength kept at the origin in a wall bounding a semi-infinite viscous imcompressible stratified fluid is presented in an integral form. The behaviour of the flow at far field and near field is studied by an asymptotic expansion procedure. The streamlines for different parameters are drawn and discussed. The real characteristic straight lines present in the inviscid problem are modified by the viscosity and the solutions obtained are valid even at the resonance frequency.

1. INTRODUCTION

IT is well known that the internal waves play an important role in many atmosphere and geophysical phenomena. The internal waves can be produced in a stably stratified fluid due to the forced motion of bodies when the forcing frequency of the body is less than the Brunt Väisäla frequency N of the fluid. The linear theory for inviscid stratified fluids predicts the presence of internal waves propagating along the straight lines inclined at an angle $\sin^{-1}(\omega/N)$ to the horizontal. Mowbray and Rarity[1] have confirmed the existence of these waves in a uniformly stratified salt solution by the 'Schlieren' photographs of the phase configuration of the waves. The two-dimensional internal waves generated by travelling oscillating bodies have been studied by Lighthill,[2] Stevenson and Thomas,[3] and Rehm and Radt.[4] The internal waves emitted by stationary vibrating bodies in an inviscid stratified fluid have been investigated by Hurley.[5,6] So far, very few have studied the effect of viscosity on internal gravity waves (Yanowitch[7] and Thomas and Stevenson)[8]. An attempt is made in this note to study the viscosity effects on a simple model generating internal gravity waves.

In this paper, we study the effect of viscosity on the internal waves generated by a line source of sinusoidally pulsating strength. The linearized equations are solved and the solution is presented in an integral form. The usual radiation condition which is necessary to pick a correct solution in an inviscid problem is not required in the present case. The correct solution satisfying the radiation condition is produced in the limit of viscosity tending to zero. The far field and near field behaviour of the flow are discussed. The streamlines for various values of the parameters are drawn by evaluating the integral solution by numerical quadrature formula. The real characteristic discontinuity lines present in the inviscid problem are seen to be modified by the viscosity.

2. Formulation of the Problem

The linearized equations governing the two dimensional propagation of small disturbances in a stable, density stratified incompressible viscous fluid are (Phillips)[9]

$$\rho_0 \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial x} + \mu \nabla^2 u, \tag{1}$$

$$\rho_0 \frac{\partial v}{\partial t} = - \frac{\partial p}{\partial y} - \rho g + \mu \nabla^2 v, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + v \frac{d\rho_0}{dy} = 0 \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{dy^2}, \tag{4}$$

where x and y are horizontal and vertical coordinates. The quantities ρ_0, ρ, p, u, v, g, μ are respectively, the equilibrium density, perturbation density, perturbation pressure, horizontal velocity, vertical velocity, accleration due to gravity and the coefficient of viscosity (a constant). The line source of sinusoidally pulsating strength is located at the origin $x = y = 0$ and the plane $y = 0$ is a rigid wall and the fluid occupies $y \geqslant 0$. The equilibrium density distribution is given by $\rho_0 = \rho'_0 \exp(-\beta y)$ where ρ'_0 and β are constants.

We introduce the stream function $\psi'(x, y, t)$ such that the velocity components are

$$u = - \frac{\partial \psi'}{\partial y}, \qquad v = \frac{\partial \psi'}{\partial x}. \tag{5}$$

Eliminating p from the above equations and using the Boussinesq approximation (the density variation on the inertia terms is negligible), the stream function ψ' satisfies

$$\nabla^2 \psi'_{tt} + N^2 \psi'_{xx} = \nu \nabla^4 \psi'_t \tag{6}$$

where $N = (-g/\rho_0 \, . \, d\rho_0/dy)^{1/2}$ is the Brunt-Väisäla frequency which is a constant and $\nu = \mu/\rho'_0$ is also a constant. Since the line source is oscillating with frequency ω, we assume the resulting solution as

$$\psi'(x, y, t) = \text{Real part of } \{\psi(x, y)\, e^{i\omega t}\}, \tag{7}$$

where $\psi(x, y)$ is a complex valued function and it satisfies

$$\nabla^4 \psi = a^2 (\nabla^2 \psi) - k^2 \psi_{xx}), \tag{8}$$

where

$$a^2 = i\omega/\nu \qquad \text{and} \qquad k^2 = N^2/\omega^2. \tag{9}$$

Equation (8) is elliptic for all finite a and k. For an inviscid fluid $\nu \to 0$ $(a \to \infty)$, eq. (8) becomes

$$(1 - k^2)\, \psi_{xx} + \psi_{yy} = 0, \tag{10}$$

which is elliptic for $k < 1$, hyperbolic for $k > 1$ and parabolic for $k = 1$. In the hyperbolic case, there exist real characteristics which are straight lines inclining at an angle $\sin^{-1} (k^{-1})$ to the horizontal.

The boundary conditions on the velocites

$$u, \quad v \to 0 \quad \text{as} \quad y \to \infty \tag{11}$$

and

$$u = v = 0 \quad \text{on} \quad y = 0_+, \quad (x \neq 0), \tag{12}$$

can be written in terms of $\psi(x, y)$ as

$$\psi_x, \quad \psi_y \to 0 \quad \text{as} \quad y \to \infty \tag{13}$$

and

$$\psi_x = 0, \qquad \psi_y = 0 \quad \text{on} \quad y = 0_+, \quad (x \neq 0). \tag{14}$$

From first of the conditions (14), it is clear that $\psi =$ constant on $y = 0_+$, $x > 0$, with a similar condition (but a different constant) on $y = 0_+$, $x < 0$. In fact, the difference between the constant values of ψ on $x > 0$ and $x < 0$ is just the magnitude of the oscillatory flux generated by the source at the origin. Since the problem is linear, without loss of generality, we choose these constants at will and hence we take

$$\psi = \pm 1 \qquad \text{on} \qquad y = 0_{+} \; (x \gtrless 0). \tag{15}$$

Since the flow should be symmetrical with respect to plane $x = 0$, we have

$$\psi = \nabla^2 \psi = 0, \qquad \text{on } x = 0. \tag{16}$$

We wish to confine our attention to the first quadrant $x > 0$, $y > 0$ and if (16) is used, then we need only part of (15), namely

$$\psi = 1 \quad \text{on} \quad y = 0_{+} \; (x > 0). \tag{17}$$

Now the problem is to solve for ψ the eq. (8) subject to the boundary conditions (13), (14), (16) and (17).

3. SOLUTION FOR THE STREAM FUNCTION AS AN INTEGRAL

By taking Fourier-sine transform of (8) with respect to x, we get,

$$\frac{d^4 \bar{\psi}}{dy^4} - (2\lambda^2 + a^2) \frac{d^2 \bar{\psi}}{dy^2} + \lambda^2 \{\lambda^2 + a^2 (1 - k^2)\} \bar{\psi} = 0, \tag{18}$$

where

$$\bar{\psi} (\lambda, y) = \int_0^\infty \psi (x, y) \sin \lambda x \, dx. \tag{19}$$

The condition (16) is used while taking the above transform. The solution of (18) satisfying (13) is

$$\bar{\psi} = A (\lambda) e^{-\beta_1 y} + B (\lambda) e^{-\beta_2 y}, \text{ Real part of } (\beta_1, \beta_2) > 0 \tag{20}$$

where

$$\beta_1 = \{\lambda^2 + \tfrac{1}{2} a^2 (1 + \sqrt{1 + k_1^2})\}^{1/2}, \qquad k_1^2 = 4 \lambda^2 k^2 / a^2, \tag{21}$$

$$\beta_2 = \{\lambda^2 + \tfrac{1}{2} a^2 (1 - \sqrt{1 + k_1^2})\}^{1/2}. \tag{22}$$

By the second condition of (14), we have

$$B (\lambda) = - (\beta_1 / \beta_2) \cdot A (\lambda). \tag{23}$$

Using the inversion formula for Fourier-sine transform, ψ is given by

$$\psi = \frac{2}{\pi} \int_0^\infty \frac{A (\lambda)}{\beta_2} \{\beta_2 e^{-\beta_1 y} - \beta_1 e^{-\beta_2 y}\} \sin \lambda x \, d\lambda. \tag{24}$$

The only condition left to be satisfied by ψ is (17) and it is used to determine $A(\lambda)$. By (17)

$$\psi(x, 0_+) = \frac{2}{\pi} \int_0^\infty \frac{A(\lambda)}{\beta_2} (\beta_2 - \beta_1) \sin \lambda x \, d\lambda,$$

$$= 1. \tag{25}$$

Therefore, we have

$$A(\lambda) = \frac{\beta_2}{\lambda(\beta_2 - \beta_1)} = -\frac{\beta_1(\beta_1 + \beta_2)}{\lambda a^2 \sqrt{1 + k_1^2}}. \tag{26}$$

Thus the resulting expression for the stream function is:

$$\psi = -\frac{2}{\pi a^2} \int_0^\infty \frac{\beta_1 + \beta_2}{\sqrt{(1 + k_1^2)}} (\beta_2 e^{-\beta_1 y} - \beta_1 e^{-\beta_2 y} \frac{\sin \lambda x}{\lambda} d\lambda. \tag{27}$$

Another interesting form of ψ can be got from (27) after some manipulation as

$$\psi = \frac{2}{\pi} \int_0^\infty \frac{e^{-\beta_2 y}}{\lambda} \sin \lambda x \, d\lambda - \frac{2}{\pi} \int_0^\infty \frac{\beta_2(\beta_1 + \beta_2)}{\lambda \sqrt{(1 + k_1^2)}}$$

$$\times (e^{-\beta_1 y} - e^{-\beta_2 y}) \sin \lambda x \, d\lambda. \tag{28}$$

The results for homogeneous fluid can be obtained in the limit $N \to 0$ $(k \to 0)$. Taking this limit of the eq. (27), we get,

$$\psi = -\frac{2}{\pi a^2} \int_0^\infty \frac{(\lambda + \beta)}{\lambda} (\lambda e^{-\beta y} - \beta e^{-\lambda y}) \sin \lambda x \, d\lambda. \tag{29}$$

where $\beta^2 = \lambda^2 + a^2$ and this coincides with the expression for ψ given by Tuck.[10] The velocity components can be obtained from the expression (27) or (28) using (5). As the exact evaluation of the integrals given in (27) and (28) is not possible, we have investigated the asymptotic properties of the flow at far-field and near-field in the following sections. Further the values of ψ are obtained easily enough by numerical evaluation of the integrals. The results obtained by numerical evaluation of the integral are used to draw the streamlines and they are discussed in section 6.

4. FAR-FIELD BEHAVIOUR OF THE FLOW

The flow far away from the wall should resemble the flow due to a source in an inviscid fluid. That is at a great distance from the wall the effect of viscosity is negligible. We obtain a valid asymptotic expansion, by expanding all the quantities in the integrals for small λ which broadly correspond to large y. Thus, for small λ, we have

$$\beta_1 \sim \alpha + \frac{(1+k^2)\,\lambda^2}{2\alpha}, \qquad \beta_2 \sim \eta\lambda + \frac{k^4\,\lambda^3}{2\alpha^2\,\eta} \tag{30}$$

and the integral occurring in eq. (28) for ψ becomes

$$\psi \sim \frac{2}{\pi} I_1 - \frac{2}{\pi\alpha^2} I_2, \tag{31}$$

where

$$I_1 = \int_0^\infty \exp\left[-\{\eta\lambda + k^4\,\lambda^3/2\,\alpha^2\,\eta\}\,y\right].\,\frac{\sin\lambda x}{\lambda}\,d\lambda, \tag{32}$$

$$I_2 = \int_0^\infty \frac{B_1(\alpha,\lambda,k)}{\lambda}\Big[\exp\Big(-\Big\{\alpha + \frac{(1+k^2)\,\lambda^2}{2\alpha}\Big\}\,y\Big) - \exp\Big(-\Big\{\eta\lambda + \frac{k^4\,\lambda^3}{2\alpha^2\,\eta}\Big\}\,y\Big)\Big]\sin\lambda x\,d\lambda, \tag{33}$$

$$B_1(\alpha,\lambda,k) = \alpha\eta + \lambda\eta^2 + \frac{\lambda^2}{2\alpha\eta}(1-2k^2)^2 + \frac{\lambda^3}{\alpha^2}(3k^4 - 2k^2),$$

$$\eta = (1-k^2)^{1/2} \quad \text{and} \quad i\zeta = i\,(k^2-1)^{1/2} = \eta \quad \text{for} \quad k > 1. \tag{34}$$

The terms of order λ^4 are neglected in the above expansions.

The solutions for an inviscid fluid are obtained from (31) by taking the limit as $\alpha \to \infty$ and the expressions for ψ in this case are given by

$$\psi = \frac{2}{\pi}\int_0^\infty e^{-\eta\lambda y}\,\frac{\sin\lambda x}{\lambda}\,d\lambda, \tag{35}$$

$$\psi = 1 - \frac{2}{\pi}\tan^{-1}(\eta y/x), \qquad \text{for} \qquad k < 1, \tag{36}$$

$$\left.\begin{aligned} &= 1 - \frac{i}{\pi}\log\left(\frac{x+\zeta y}{x-\zeta y}\right) && x > \zeta y,\\ &= \infty, && x = \zeta y,\\ &= -\frac{i}{\pi}\log\left(\frac{\zeta y + x}{\zeta y - x}\right) && x < \zeta y \end{aligned}\right\}\ \text{for } k > 1. \tag{37}$$

The solutions (37) for $k > 1$ are similar to the solution for a source in an unbounded fluid given by Hurley,[5] and they clearly show that the field quantities are discontinuous across the two straight lines $x = \pm \zeta y$. The inviscid equations cannot be solved directly to get (37) without the application of a radiation condition. Thus the limiting process of getting inviscid solution from viscous solution avoids the application of a radiation condition.

The integrals in (32) and (33) can be evaluated exactly from the results given in Magnus *et al.*[11] in terms of Lommel and hypergeometric functions. In evaluating the integrals (32) and (33) for $k > 1$, one gets essentially the integrals of the type

$$J_n = \int_0^\infty \xi^n \exp(i\delta\xi - \epsilon\xi^3) \sin \gamma\xi d\xi, \quad \text{Real part of } \epsilon > 0,$$

$$n \,(\text{integer}) \geq -1, \tag{38}$$

where γ, ϵ are constants. We evaluate the integral in (38) when $n = 0$ and for the other values of n it can be evaluated by integrating or differentiating with respect to the parameter δ. For $n = 0$ (38) is written in the form

$$J_0 = \frac{1}{2i} \int_0^\infty \{\exp[i(\delta + \gamma)] - \exp[i(\delta - \gamma)]\} \exp(-\epsilon\xi^3)\, d\xi. \tag{39}$$

Using the results of Ref. 11 we get

$$J_0 = \frac{1}{2}\left[\frac{1}{\delta + \gamma} v_1 S_{0,1/3}(v_1) - \frac{1}{\delta - \gamma} u_1 S_{0,1/3}(u_1)\right], \tag{40}$$

where $S_{0,1/3}$ is a Lommel function,

$$v_1 = 2\left\{\frac{-i(\delta + \gamma)}{3\epsilon^{1/3}}\right\}^{3/2} \quad \text{and} \quad u_1 = 2\left\{\frac{-i(\delta - \gamma)}{3\epsilon^{1/3}}\right\}^{3/2}. \tag{41}$$

The expression for J_0 in (40) remains finite when $\delta = \pm \gamma$ for $\epsilon > 0$. But in the limit $\epsilon \to 0$, J_0 becomes infinite when $\delta = \pm \gamma$ as

$$\lim_{v_1 \to \infty} \{v_1 S_{0,1/3}(v_1)\} = 1 \quad \text{and} \quad \lim_{u_1 \to \infty} \{u_1 S_{0,1/3}(u_1)\} = 1. \tag{42}$$

We notice from (40) that the integrals in (32) and (33) will no longer be discontinuous on $x = \pm \zeta y$ even when $k \geq 1$, when the terms of order λ^3 in the exponentials are retained. Thus, the real characteristics in the inviscid fluid get modified by viscosity and they actually appear as free shear layers in the fluid. Further, the solution for ψ remains finite in the entire flow field even at the resonance frequency $k = 1$.

Evaluating the integrals in (32) and (33) keeping only the important terms for large y, the approximate expression for ψ is obtained for $k < 1$ as

$$\psi = \left[1 - \frac{2\theta_1}{\pi}\right] + \left[\frac{2\eta}{\pi a}\frac{\cos\theta_1}{r_1}\right] + \left[\frac{2\eta^2}{\pi a^2}\frac{\sin 2\theta_1}{r_1^2}\right]$$

$$- \left[\frac{2(1-2k^2)^2}{\pi a^3 \eta}\frac{\cos 3\theta_1}{r_1^3}\right] + 0(r_1^{-4})$$

$$+ \text{exponentially small terms,} \tag{43}$$

where

$$x = r_1 \cos\theta_1 \qquad y = r_1 \sin\theta_1. \tag{44}$$

In the expression (43) the leading term is interpreted as the solution due to a source, the second term due to a dipole, the third term due to a quadrupole and so on.

The real valued quantity ψ' given in (7) becomes

$$\psi'(x, y, t) = \cos\omega t\left[1 - \frac{2\theta_1}{\pi} + \frac{2}{\pi}\left(\frac{\nu}{2\omega}\right)^{1/2}\frac{\eta\cos\theta_1}{r_1} + 0(r_1^{-3})\right]$$

$$+ \sin\omega t\left[\frac{2}{\pi}\left(\frac{\nu}{2\omega}\right)^{1/2}\frac{\eta\cos\theta_1}{r_1} + \eta^2\frac{2\nu}{\pi\omega}\frac{\sin 2\theta_1}{r_1^2} + 0(r_1^{-3})\right]. \tag{45}$$

We obtain the results for homogeneous fluids from (45) in the limit $k \to 0$. The streamline pattern for $k < 1$ is qualitatively similar to that of the homogeneous fluids which has been discussed by Tuck.[10]

At 0° phase $(\omega t = 0, 2), \ldots)$, the expression for assuming $\eta^2\nu/2\omega$ small, is given by

$$\psi'(x, y, 0) \sim 1 - \frac{2}{\pi}\tan^{-1}\left[\frac{1}{x}\left\{y - \left(\frac{\nu\eta^2}{2\omega}\right)^{1/2}\right\}\right] + 0(r_1^{-3}). \tag{46}$$

This shows that the flow at infinity is that due to a source at $\{0, (\eta^2\nu/2\omega)^{1/2}\}$. The streamlines are ultimately radial lines emanating from this point. Similarly at 90° phase $(t = \pi/2, 5\pi/2, \ldots)$

$$\psi'(x, y, \pi/2) \sim \frac{2\eta}{\pi}\left(\frac{\nu}{2\omega}\right)^{1/2}\frac{\cos\theta_1}{r_1} + \frac{2\nu\eta^2}{\pi\omega}\frac{\sin 2\theta_1}{r_1^2} + 0(r_1^{-3}) \tag{47}$$

and the streamlines approximate to ellipses. The computed streamlines shown in the figures for the case $k < 1$ confirm the asymptotic analysis described above, Where as for $k \geq 1$, we observe quite different features: for $k > 1$ evaluating the integrals by a formal procedure the expression for ψ' takes the form:

$$\psi' = \cos \omega t\left[1 + \frac{2}{\pi}\left(\frac{\nu}{2\omega}\right)^{1/2} \frac{\zeta \cos \phi \sec 2\phi}{r_2} + 0\,(r_2^{-2})\right]$$

$$- \sin \omega t\left[\frac{2}{\pi}\tanh^{-1}\left(\frac{\zeta y}{x}\right) + \frac{2}{\pi}\left(\frac{\nu}{2\omega}\right)^{1/2} \frac{\zeta \cos \phi \sec 2\phi}{r_2}\right.$$

$$\left. + 0\,(r_2^{-3})\right], \qquad \text{for } x > \zeta y, \tag{48}$$

$$= \cos \omega t\left[\frac{2}{\pi}\left(\frac{\nu}{2\omega}\right)^{1/2} \frac{\zeta \cos \phi \sec 2\phi}{r_2} + 0\,(r_2^{-2})\right]$$

$$- \sin \omega t\left[\frac{2}{\pi}\tanh^{-1}\left(\frac{x}{\zeta y}\right) + \frac{2}{\pi}\left(\frac{\nu}{2\omega}\right)^{1/2} \frac{\zeta \cos \phi \sec 2\phi}{r_2}\right.$$

$$\left. + 0\,(r_2^{-3})\right], \qquad \text{for } x < \zeta y, \tag{49}$$

where

$$x = r_2 \cos \phi \qquad \text{and} \qquad y = r_2 \sin \phi. \tag{50}$$

At 0° phase, the streamline pattern for far-field is given by

$$\frac{\cos \phi \sec 2\phi}{r_2} = \text{constant}, \tag{51}$$

which represents hyperbolas. At 90° phase the streamlines are approximately given by

$$\frac{2}{\pi}\tanh^{-1}\left(\frac{x}{\zeta y}\right) + \frac{2}{\pi}\left(\frac{\nu}{2\omega}\right)^{1/2} \frac{\zeta \cos \phi \sec 2\phi}{r_2} = \text{constant}, \tag{52}$$

which are modified hyperbolas. Thus in this case, the streamline pattern is qualitatively different from the case of homogeneous fluids and the case $k < 1$. The lines $x = \pm \zeta y$ are not discontinuous lines in a real fluid, but they are a sort of internal boundary layers where the viscosity effects are important. Actually the term of order λ^3 in the integrals (32) and (33) should not be neglected in the asymptotic analysis given earlier. The computed streamlines shown in figures for $k > 1$ do not show any discontinuities along the lines $x = \pm \zeta y$, but the streamline pattern shows the behaviour described in the asymptotic analysis. Similar analysis can be carried out at the resonance frequency.

5. NEAR-FIELD BEHAVIOUR OF THE FLOW

In the near field $r = (x^2 + y^2)^{1/2}$ is small and it is quite difficult to carry out the detailed asymptotic analysis of the integrals. We obtain the first few terms of the stream function by expanding the integrand for large λ, keeping λx, λy bounded. The approximate expression for ψ is given by (31) where

$$I_1 = \int_0^\infty \exp\left(-\left[\lambda - \frac{k\alpha}{2} + \frac{\alpha^2 (2 - k^2)}{8\lambda} \right] y \right) \frac{\sin \lambda x}{\lambda}\, d\lambda, \tag{53}$$

$$I_2 = \int_0^\infty \left[\frac{\alpha}{k} - \frac{\alpha^2}{2\lambda} + \frac{\alpha^3 (2 - k^2)}{4\lambda^2 k} \right] \left[\exp\left(\left\{ \lambda + \frac{k\alpha}{2} + \frac{\alpha^2 (2 - k^2)}{8\lambda} \right\} y \right) - \exp\left(-\left\{ \lambda - \frac{k}{2} + \frac{\alpha^2 (2 - k^2)}{8\lambda} \right\} y \right) \right] \sin \lambda x \, d\pi. \tag{54}$$

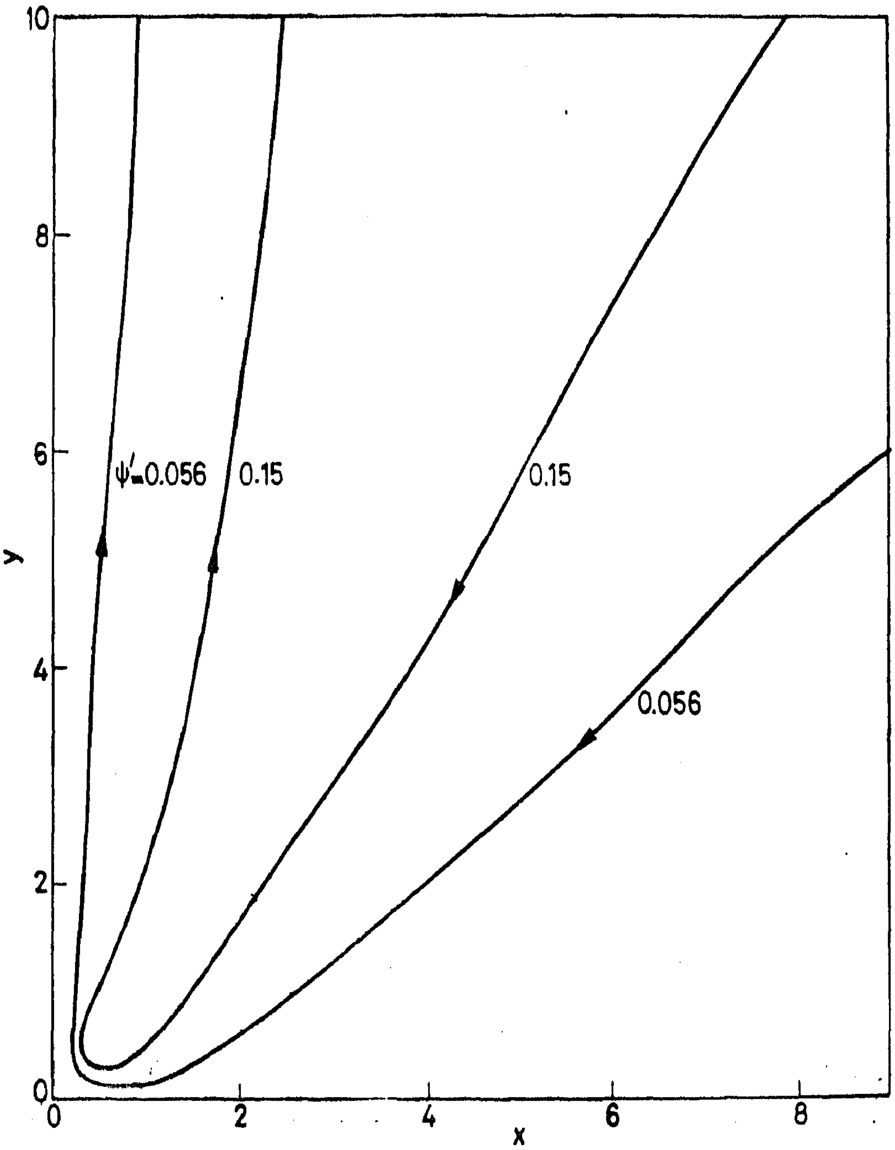

Figure 1. Streamlines for $k = 1$ and $\omega t = \pi/2$,

Evaluating the integrals in (53) and (54) expanding some of the exponential terms and neglecting the terms of order $1/\lambda^2$, we get,

$$\psi \sim e^{[illegible]}\left(1 \; [illegible]\right) \frac{\sinh(\omega k y/2)}{(\omega k y/2)}\left[2r \; [illegible] \; \frac{X}{X^2+y^2} \; [illegible] \; \frac{\omega y}{4}\right.$$

$$\left. [illegible]\, 4k \; [illegible]\, (2 - k^2)\, r_1^2 (\pi/2 - \theta) + \ldots \right]. \qquad (55)$$

where θ = $\tan^{-1}$ [illegible]. Since some of these given above are obtained using the fact that [illegible] 1, one is not justified in taking the limit $k \to 0$. But if we take this limit formally, it is observed the results given (55) coincide with the results (given to the same order of approximation) for homogeneous fluids, Tuck.[10] Further the expression given in (55) is valid for all the values of k greater than, less than or equal to one. One can justify the above asymptotic analysis by a more refined and regorous methods following the analysis given by Tuck.[10]

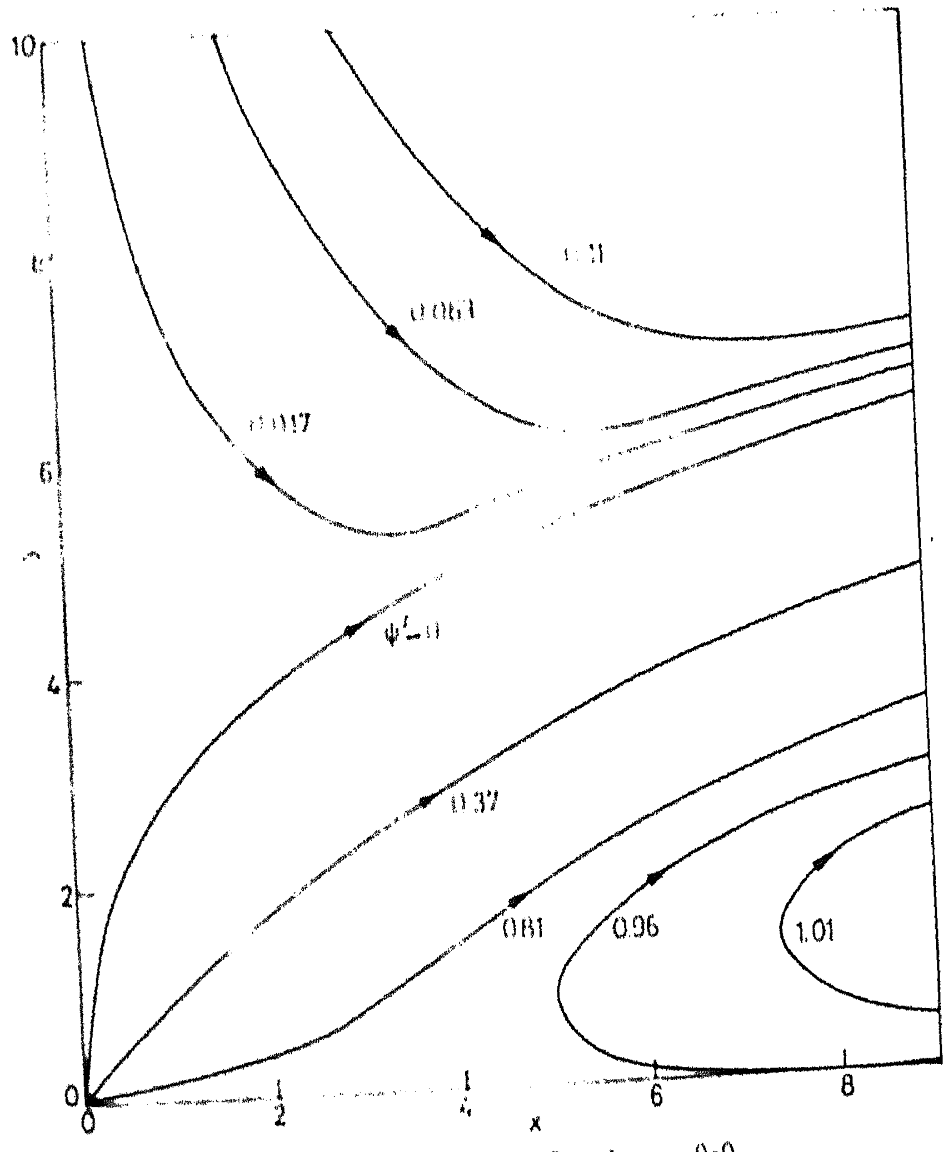

Figure 2. Streamlines for $k = 3$ and $\omega = 0{\cdot}0$.

6. Discussion of the Numerical Results

Introducing the dimensionless variables

$$x^* = x\,(\omega/\nu)^{1/2}, \qquad y^* = y\,(\omega/\nu)^{1/2}, \qquad \lambda^* = \lambda\,(\nu/\omega)^{1/2} \tag{56}$$

in (27) and making use of (7), we get (dropping the asterisks)

$$\psi'(x, y, t) = \frac{2}{\pi} \int_0^\infty \frac{1}{a^2 + b^2} [e^{-a_2 y} \{A_1 \sin(\omega t - b_2 y) + A_2 \cos(\omega t - b_2 y)\} - e^{-a_1 y} \{A_3 \sin(\omega t - b_1 y) + A_4 \cos(\omega t - b_1 y)\}] \frac{\sin \lambda x}{\lambda}\, d\lambda, \tag{57}$$

where

$$a = \frac{1}{\sqrt{2}} \{(1 + 16\,k^4 \lambda^4)^{1/2} + 1\}^{1/2}, \qquad b = -\frac{1}{\sqrt{2}} \times \{(1 + 16\lambda^4 k^4)^{1/2} - 1\}^{1/2} \tag{58}$$

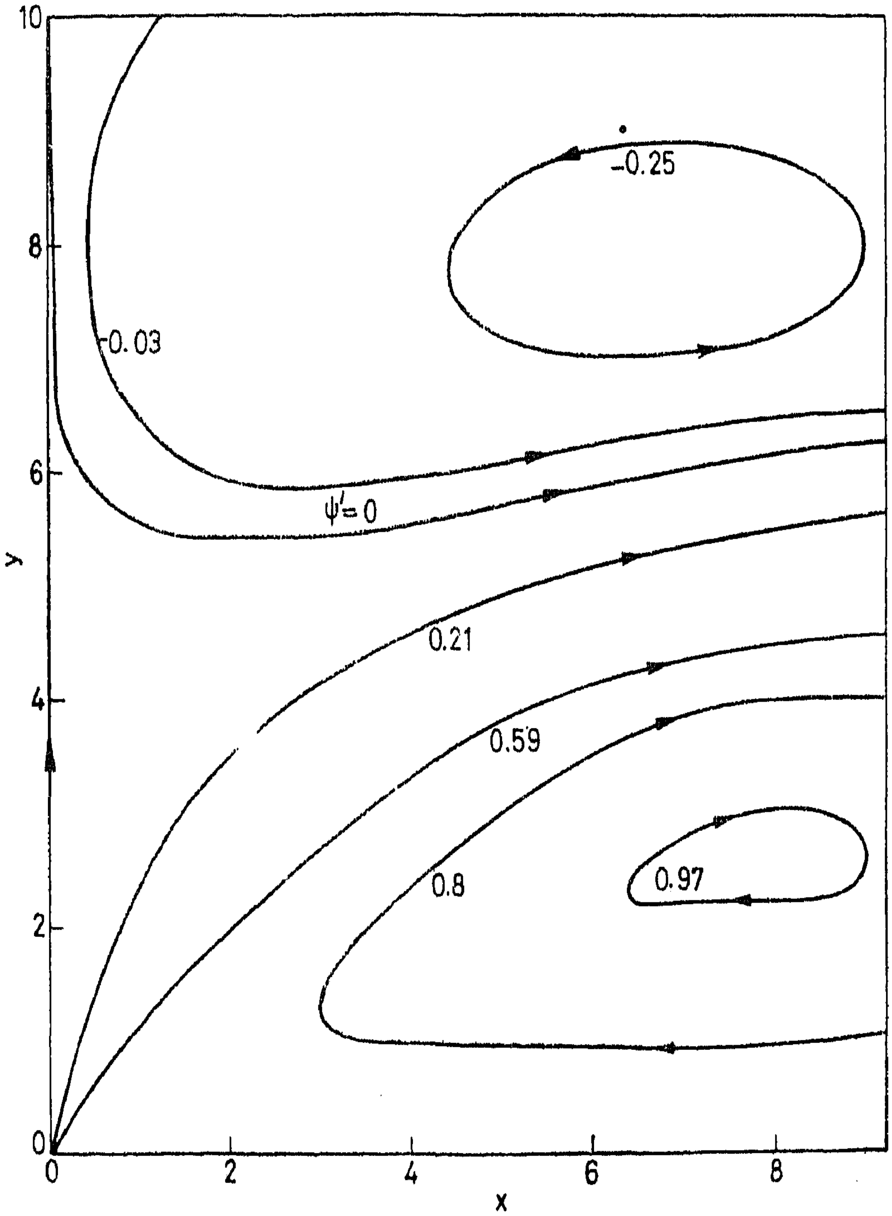

Figure 3. Streamlines for $k = 3$ and $\omega t = \pi/4$.

$$a_1 = \frac{1}{\sqrt{2}}\left\{\left[\left(\lambda^2 - \frac{b}{2}\right)^2 + \frac{1}{4}(1 - a)^2\right]^{1/2} + \left(\lambda^2 - \frac{b}{2}\right)\right\}^{1/2}$$

$$b_1 = \frac{1}{\sqrt{2}}\left\{\left[\left(\lambda^2 - \frac{b}{2}\right)^2 + \frac{1}{4}(1 - a)^2\right]^{1/2} - \left(\lambda^2 - \frac{b}{2}\right)\right\}^{1/2} \tag{59}$$

$$a_2 = \frac{1}{\sqrt{2}}\left\{\left[\left(\lambda^2 + \frac{b}{2}\right)^2 + \frac{1}{4}(1 - a)^2\right]^{1/2} + \left(\lambda^2 + \frac{b}{2}\right)\right\}^{1/2},$$

$$b_2 = \frac{1}{\sqrt{2}}\left\{\left[\left(\lambda^2 + \frac{b}{2}\right)^2 + \frac{1}{4}(1 - a)^2\right]^{1/2} - \left(\lambda^2 + \frac{b}{2}\right)\right\}^{1/2},$$

$$A_1 = (aa_1 - bb_1)(a_1 - a_2) - (ab_1 - a_1b)(b_1 + b_2),$$

$$A_2 = (ab_1 - a_1b)(a_1 - a_2) + (aa_1 - bb_1)(b_1 + b_2), \tag{60}$$

$$A_3 = (aa_2 - bb_2)(a_1 - a_2) - (ab_2 - a_2b)(b_1 + b_2)$$

$$A_4 = (ab_2 - a_2b)(a_1 - a_2) + (aa_2 + bb_2)(b_1 + b_2). \tag{61}$$

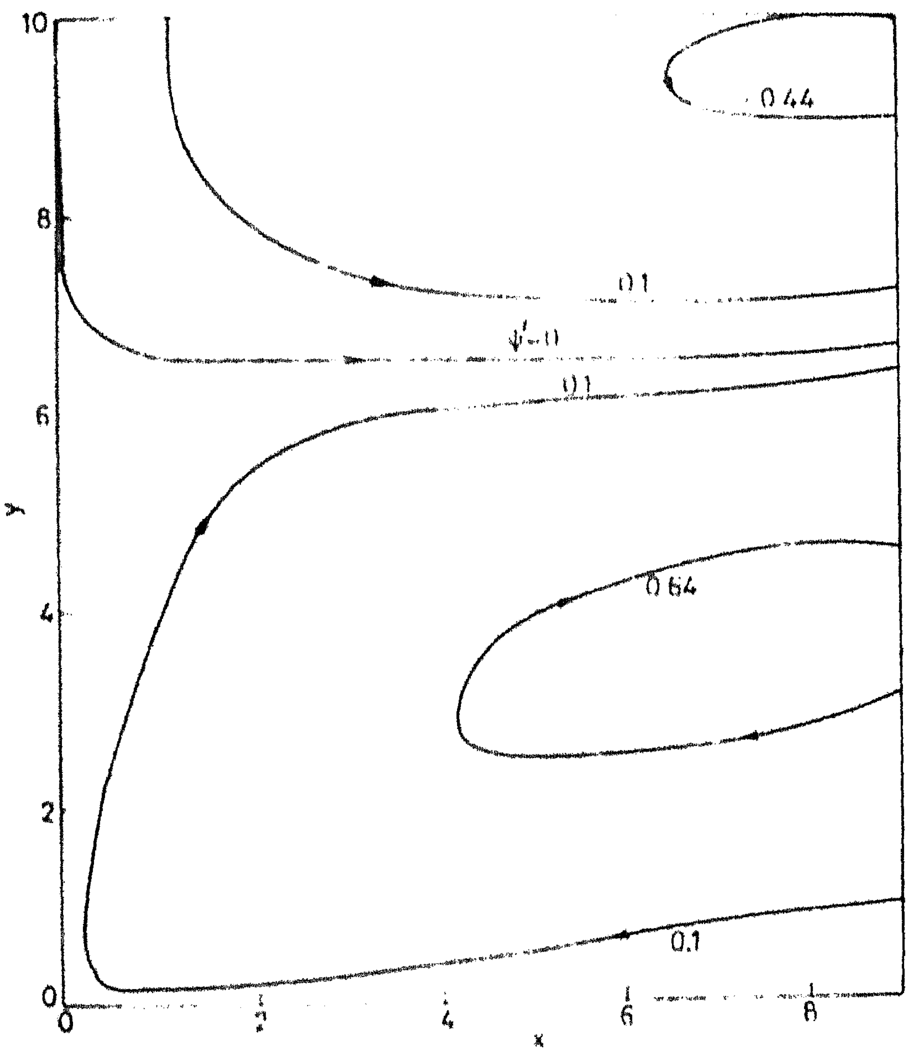

Figure 4. Streamlines for $\lambda = 3$ and $\omega t = \pi/2$.

The integral in (57) is easily evaluated by Simpson's one-third rule on **IBM** 360 Computer for different values of the parameter k at different **phases** and these computed values are used to draw the streamlines.

The streamlines are drawn for different k at different values of ωt lying between 0 and π and this picture repeats itself for other values. **When** $\omega t = 0$, the maximum out-flow occurs and the flow is fully due to a source. When $\omega t = \pi$, the flow is fully sink-like and the streamlines are identical with the streamlines at $\omega t = 0$ with reversed directions. It is noticd that the streamlines in the case $k < 1$ at different ωt are very much similar to those for homogeneous fluids and therefore for a detailed discussion of these graphs one can refer Tuck.[10]

In an inviscid fluid resonance occurs at $\omega = N (k = 1)$ and there do not exist any physically meaningful solution at resonance frequency. When viscosity is present a meaningful solution exists and the streamlines are

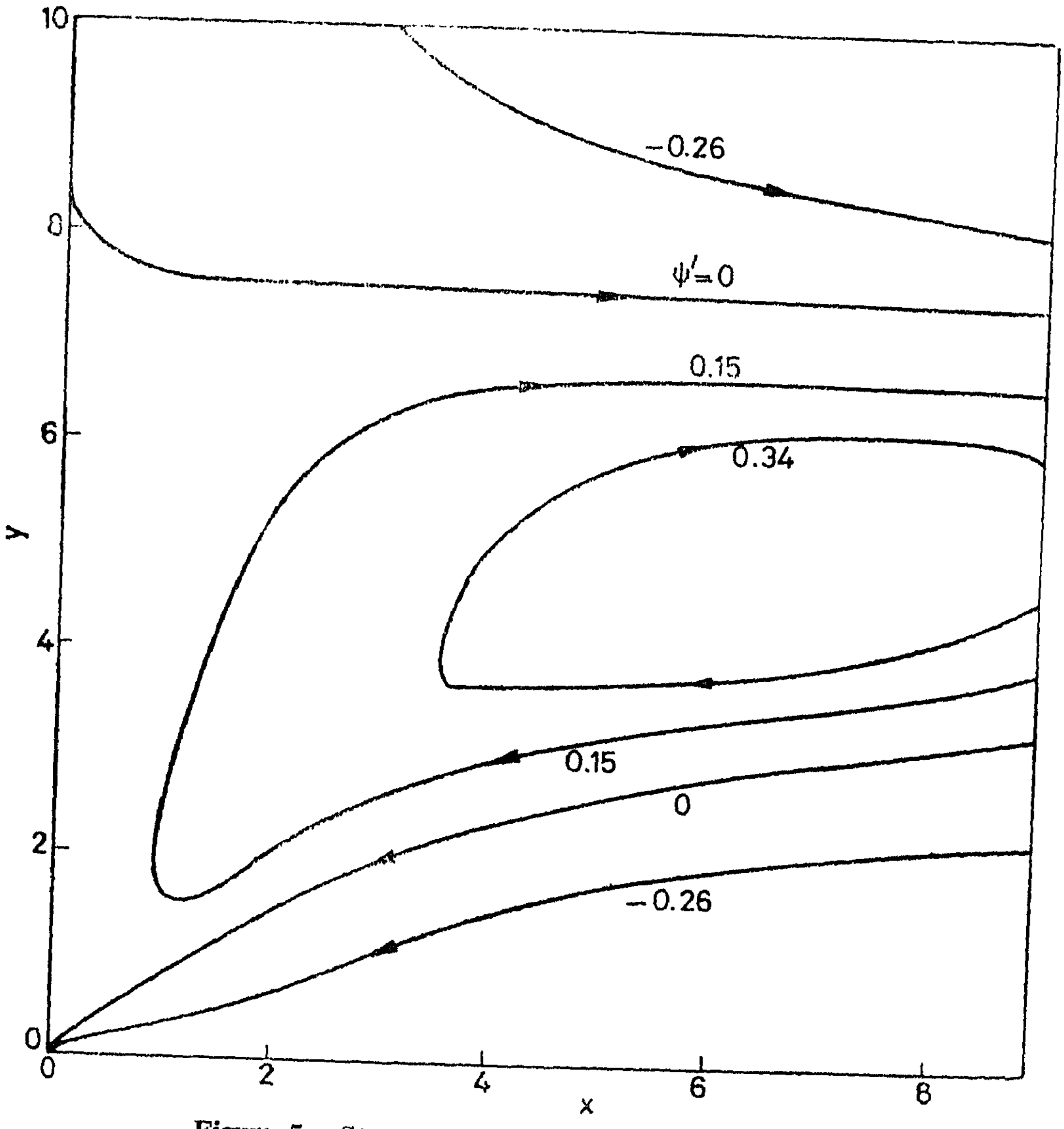

Figure 5. Streamlines for $k = 3$ and $\omega t = 3\pi/4$.

exhibited in figure 1 at phse $\omega t = \pi/2$. Asymptotic analysis in this case gives approximately parabolic streamline pattern which agrees with the computed results.

The streamline pattern for $k > 1$ is qualitatively different from the case $k < 1$. Figures 2 to 5 show the development of the streamline pattern with time when $k = 3$. The fully source-like streamlines are given in figure 2 and they show that there are two regions of closed streamlines one near $y = 0$ and the other near $x = 0$ (y large). Whereas for the case $k < 1$ all the streamlines emanate from the origin and end at infinity. As the strength of the source starts to diminish the two bubble-like regions (regions of closed streamlines) of circulating fluid grow in size (figure 3).

When $\pi/2$ phase (figure 4) is reached, the strength of the source is dropped to zero and no fluid is actually leaving the origin at all. But there are two masses of circulating fluid, the one near $y = 0$ is circulating in the clockwise direction and the other in the anti-clockwise direction.

As the time further increases the flow becomes sink-like and we observe the sreamline $\psi' = 0$ splitting into two lines marked in opposite directions containing in between them a closed circulating fluid (figure 5). At large distances away from the origin the two lines $\psi' = 0$ appear to make an angle $\sin^{-1}(1/3)$ approximately to the horizontal. As the velocity along these lines $\psi' = 0$ are in the opposite directions, it becomes zero in some region between these two lines indicating the existence of a sort of internal boundary. Actually in an inviscid fluid the stream function becomes infinite on the straight lines inclined at an angle $\sin^{-1}(1/k)$ to the horizontal along which the internal waves propagate. These are observed in real fluids as internal boundary layers and the viscosity plays a very important role in that region. Further no such discontinuities can exist in a real fluid. The computed streamline pattern shown in the figures 2 to 5 confirm this result.

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