

On trade-off solution pairs in a special type of transportation problem

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Abstract. In this paper, we have developed an algorithm to determine the trade-off solution pairs in a special type of transportation problem considering two objectives, *viz.*, cost and time. It is assumed that the time of transportation from an origin to a destination depends on the quantity transported. This results in a time objective, which is a piecewise constant increasing function. The cost objective function is taken to be linear. A potential physical situation of this model is given and a numerical example is worked out to illustrate the algorithm.

Keywords. Trade-off solution pairs; transportation problem; time objective.

1. Introduction

The time minimizing transportation problem (TMTP), also known as Bottleneck Transportation Problem, was first studied by Hammer [11, 12]. Garfinkel and Rao [9] proposed a threshold algorithm to solve TMTP. Szwarc [19] refined Hammer's algorithm and also suggested a revision of the proof of a result in Hammer's paper [11]. He also provided a thorough survey of the earlier East European literature on TMTP. Bhatia *et al* [3], Derigs and Zimmermann [6], Finke and Smith [7], Sharma and Swarup [16] provide various algorithms for solving TMTP. Seshan and Tikekar [15] pointed out an error in a step in the algorithm given by Sharma and Swarup [16]. Bhatia [5] has developed a method of solving zero-one time minimization transportation problem. A computational study of this problem was done by Foulds and Gibbons [8]. Isermann [13] presented an algorithm to enumerate all efficient solutions for a linear multiple objective transportation problem.

The fundamental assumption in a TMTP is that the time for transporting a certain amount of goods from a source to a destination is independent of the amount of goods transported. Achary and Seshan [1] replace the assumption by a more general and realistic assumption that the time of transportation over a route depends on the amount of shipment through that route. This assumption leads to a piecewise constant increasing time objective function.

When one considers a transportation problem with two objectives, say, cost and time, one fails to get an optimum solution satisfying both the objectives. In such cases, we see for a sequence of solutions termed as trade-off (efficient, Pareto-optimal, non-dominated) solutions. Methods of obtaining trade-off solutions have been developed by Glickman and Berger [10], Bhatia *et al* [4] and Srinivasan and Thompson [18]. Aneja and Nair [2] developed an algorithm to solve a Bicriteria Transportation Problem having both objective functions linear. Sharma and Swarup [17] provide an

algorithm for obtaining trade-off solutions in a multi-dimensional transportation problem.

In this paper, we develop an algorithm for obtaining trade-off solutions in a special type of transportation problem. We consider the usual linear cost objective along with the time objective considered in Achary and Seshan [1]. Section 2 of this paper deals with the mathematical formulation of the problem and provides a potential physical situation in support of the mathematical model. In § 3, some definitions and the methodology of the algorithm are given followed by an algorithm. A numerical example, illustrating the algorithm, is worked out in § 4.

2. The problem

Consider a transportation problem with m sources and n destinations. Let a_i ($i = 1, 2, \dots, m$) be the availability at source i , b_j ($j = 1, 2, \dots, n$), the demand at destination j and U_{ij} , the capacity restriction over the route (i, j) . The constraints of the problem then are

$$\sum_{j=1}^n x_{ij} = a_i \text{ for all } i. \quad (1)$$

$$\sum_{i=1}^m x_{ij} = b_j \text{ for all } j. \quad (2)$$

$$0 \leq x_{ij} \leq U_{ij} \text{ for all } i, j. \quad (3)$$

The cost objective function is given by

$$\min C(X) = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}. \quad (4)$$

Where c_{ij} is the unit cost of transportation over the route (i, j) . We consider the time objective function

$$\min T(X) = \max \{t_{ij}(x_{ij})\} \quad (5)$$

where $t_{ij}(x_{ij})$ is defined in (6) below.

For each route (i, j) let the interval $[0, U_{ij}]$ be divided into subintervals as follows:

$$0 = \mu_{ij}^0 < \mu_{ij}^1 < \dots < \mu_{ij}^{r(i,j)} = U_{ij}$$

and let $t_{ij}^1 < t_{ij}^2 < \dots < t_{ij}^{r(i,j)}$.

Then

$$t_{ij}(x_{ij}) = \begin{cases} 0 & \text{if } x_{ij} = 0 \\ t_{ij}^k & \text{if } \mu_{ij}^{k-1} < x_{ij} \leq \mu_{ij}^k \\ & (1 \leq k \leq r(i,j)) \end{cases} \quad (6)$$

It is not necessary that all the intervals $[0, U_{ij}]$ are divided into the same number of subintervals. Without loss of generality we can assume that $U_{ij} \leq \min(a_i, b_j)$ for all (i, j) . We shall call the problem defined by (1) through (5) as problem P .

Such a problem can arise in a practical situation described below. Let a single mode of transportation be considered for the shipment of goods from any source i to a destination j with the unit cost of transportation c_{ij} . Suppose that there is a capacity limitation m_{ij} for the amount that can be transported from the i th source to the j th destination in a fixed time interval T_{ij} . Let t_{ij} be the time for transporting up to and including m_{ij} units from the i th source to the j th destination. Then $t_{ij}(x_{ij})$ has the form

$$t_{ij}(x_{ij}) = \begin{cases} 0 & \text{if } x_{ij} = 0 \\ kT_{ij} + t_{ij} & \text{if } km_{ij} < x_{ij} \leq (k+1)m_{ij}, k \geq 0 \end{cases} \quad (7)$$

If $x_{ij} = km_{ij} + y_{ij}$, $0 \leq y_{ij} < m_{ij}$, then transportation of m_{ij} units starts at each of the time points $0, T_{ij}, \dots, (k-1)T_{ij}$ and the transportation of remaining y_{ij} units starts at time kT_{ij} . The cost objective is linear in this case. The function $t_{ij}(x_{ij})$ defined by (6) is a generalisation of $t_{ij}(x_{ij})$ considered in this example.

3. Definitions and methodology

3.1. Definitions

3.1a. *Definition 1:* T is said to be a feasible time for the problem P , if there exists a feasible solution X for the problem P , with $T(X) \leq T$. Otherwise T is said to be an infeasible time for the problem P .

3.1b. *Definition 2:* Let X be a feasible solution for P which is also cost optimal. Let $T(X) = T$ be the feasible time, and $C(X) = C$, the optimal cost. The pair (C, T) is called a solution pair.

3.1c. *Definition 3:* A solution pair (C, T) is said to be an efficient solution pair (trade-off solution pair, Pareto Optimal solution pair, nondominated solution pair) if there exists no other solution pair (C', T') such that (i) $C' < C$ and $T' \leq T$ or (ii) $C' \leq C$ and $T' < T$.

Given a time T , its feasibility can be checked as follows:

Define

$$c'_{ij} = \begin{cases} c_{ij} & \text{if } t_{ij}^k \leq T \text{ for } k \geq 1 \\ M & \text{if } t_{ij}^1 > T \end{cases} \quad (8)$$

$$U'_{ij} = \begin{cases} \mu_{ij}^k & \text{if } t_{ij}^k \leq T < t_{ij}^{k+1}, k \geq 1 \\ U_{ij} & \text{if } t_{ij}^1 > T. \end{cases} \quad (9)$$

Consider the following cost minimising transportation problem (CMTP):

$$\text{Minimize } Z(X) = \sum_{i=1}^m \sum_{j=1}^n c'_{ij} x_{ij}$$

subject to (1), (2) and

$$0 \leq x_{ij} \leq U'_{ij} \quad (10)$$

If the problem has a feasible solution with a finite optimal value Z , then the time T is feasible time. Otherwise it is an infeasible time.

3.2. Methodology of the algorithm

The algorithm starts with a cost optimal solution. $X_1 = (x_{ij})$ for the transportation problem defined by (1) through (4). The feasible time T_1 corresponding to this solution is obtained using (5) and solution pair (C_1, T_1) is recorded. Next, we determine a time $T' = \max_{i,j,k} \{t_{ij}^k | t_{ij}^k < T_1\}$ and check for its feasibility, using the technique discussed in

3.1. The details of solving the bounded transportation problem are given in Murty [14]. The feasible solution obtained with a finite optimal cost may have a feasible time smaller than T' . In that case, the better value is considered and the solution pair is recorded. This step is repeated until the algorithm establishes the infeasibility of a given time. At each iteration, the optimal solution of the previous iteration is taken as a starting solution for arriving at a feasible solution in the current iteration. The sequence of solution pairs obtained is examined for redundant solutions *i.e.* those solution which have the same optimal cost but different feasible times. Among such solution pairs, only those solution pairs having a smaller (better) feasible time are retained and the final sequence of trade-off solution pairs is recorded.

The algorithm terminates in a finite number of steps, since only a finite number of distinct T values are to be checked for their feasibility.

3.3. The Algorithm

STEP 1:

Set $r = 1$. Consider the cost minimising transportation problem in P . Let $X_1 = (x_{ij})$ be an optimal solution and $C(X_1) = C_1$, the optimal value. Find $T_1 = \max \{t_{ij}(x_{ij}) | x_{ij} > 0\}$. Record (C_1, T_1) and go to step 2.

STEP 2:

Set r to $r + 1$. Find $T'_r = \max_{i,j,k} \{t_{ij}^k | t_{ij}^k < T_{r-1}\}$. Define c'_{ij} and U'_{ij} as in (8) and (9). Solve CMTP given in (10). If it has a finite optimal value go to step 3. Otherwise go to step 4.

STEP 3:

Let $X_r = (x_{ij})$ be an optimal solution with $C(X_r) = C_r$ and $T(X_r) = T_r$. Record (C_r, T_r) and go to step 2.

STEP 4:

The time T'_r is an infeasible time. Search the sequence of solution pairs for redundant solutions, if any, and eliminate them from the sequence.

4. Numerical example

Consider a numerical example with the data given in tables 1 and 2. The last column in table 1 gives the availabilities (a_i , $i = 1, 2, 3, 4$) and the last row gives the demands (b_j , $j = 1, 2, \dots, 5$). Every (i, j) th cell contains t_{ij}^k 's and the corresponding intervals ($\mu_{ij}^{k-1}, \mu_{ij}^k$). Table 2 gives the cost matrix (c_{ij}). In tables 3–5, the solutions corresponding to the trade-off solution pairs are shown. The upper bounds of the variables are shown,

Table 1. Basic data for the example.

8(0-25)	6(0-30)	8(0-15)	9(0-20)	11(0-25)	90
10(25-40)	9(30-60)	10(15-30)	11(20-35)	13(25-35)	
12(40-55)	12(60-80)			15(35-50)	
6(0-20)	5(0-25)	6(0-15)	6(0-20)	7(0-20)	35
9(20-35)	8(25-35)	7(15-30)	8(20-35)	9(20-35)	
8(0-25)	7(0-20)	8(0-15)	4(0-20)	15(0-30)	
11(25-40)	9(20-40)	10(15-30)	6(20-35)	17(30-50)	60
13(40-55)	10(40-60)				
6(0-30)	5(0-25)	5(0-15)	13(0-20)	11(0-30)	
9(30-40)	7(25-45)	7(15-30)	15(20-35)	13(30-50)	65
11(40-55)	9(45-65)				
b_j	55	80	30	35	50

Table 2. Cost data for the example.

4	3	7	8	6
2	10	11	5	9
7	10	8	5	6
3	8	1	11	2

Table 3. Iteration 1.

(10)	(80)			
(35)				
			(35)	(25)
(10)		(30)		(25)

Table 4. Iteration 2.

	(55)			(35)	35
(20)				(15)	
	(25)		(35)	M	
(35)		(30)	20		30

Table 5. Iteration 3.

	40	60			25
(5)		60			25
(10)					(25)
(5)	40	(20)		(35)	M
(35)			(30)	M	30

wherever necessary, at the right top corner of the corresponding cells. The values of the basic variables at a solution are bracketed.

ITERATION 1:

The cost minimizing problem is solved. The cost optimal solution X_1 is shown in table 3.

$$C(X_1) = C_1 = 785 \quad T(X_1) = T_1 = \max \{t_{ij}(x_{ij}) | x_{ij} > 0\} = 15.$$

$$(C_1, T_1) = (785, 15).$$

ITERATION 2:

$T'_2 = \max \{t_{ij}^k | t_{ij}^k < 15\} = 13$. The modified CMTP is solved. The cost optimal feasible solution is shown in table 4. $C(X_2) = 1110, T(X_2) = 13$ and the efficient pair is (1110, 13).

ITERATION 3:

$T'_3 = \max \{t_{ij}^k | t_{ij}^k < 13\} = 11$. The optimal solution of the modified CMTP is shown in table 5. $(C_3, T_3) = (1505, 11)$.

ITERATION 4:

$T'_4 = \max \{t_{ij}^k | t_{ij}^k < 11\} = 10$. At this iteration we find that 10 is an infeasible time. There are no redundant solutions. The sequence of efficient solution pairs is (785, 15), (1110, 13) (1505, 11).

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