

Application of Newton's method to a homogenization problem

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Abstract. The homogenization of a family (P_ϵ) of uniformly elliptic semilinear partial differential equations of second order is studied. The main result is that any non-singular solution u of the homogenized problem (P) is the limit of non-singular solutions of (P_ϵ) . The method consists of specifying a function w_ϵ starting from which the Newton iterates converge to a solution u_ϵ of P_ϵ . These solutions u_ϵ converge to the given solution u of (P) .

Keywords. Elliptic ; homogenization ; Newton's method ; semilinear.

1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set with boundary Γ . Consider the following family of second-order differential operators defined in Ω , depending on a parameter ϵ which tends to zero :

$$(1.1) \quad A^\epsilon = - \frac{\partial}{\partial x_i} \left(a_{ij}^\epsilon(x) \frac{\partial}{\partial x_j} \right).$$

(Here, as well as throughout this paper, the summation convention for repeated indices will be assumed.) Various homogenization problems connected with these operators have been considered in the literature. Bensussan *et al* [1] have studied the linear problem $A^\epsilon u_\epsilon = f$ with various boundary conditions. Kesavan [4] has analysed the corresponding eigenvalue problems. In this paper the following semilinear problem will be studied :

$$(1.2) \quad A^\epsilon u_\epsilon = f(u_\epsilon) \text{ in } \Omega,$$

$$(1.3) \quad u_\epsilon = 0 \text{ on } \Gamma,$$

where $f: \mathbf{R} \rightarrow \mathbf{R}$ is a given function.

Indeed the identification of the homogenized problem is not difficult and will not occupy the principal part of this study. In fact Mythily [5] has homogenized (1.2) with a more complicated boundary condition. The more important question, which will be the main preoccupation in this paper, is the following : ' given a solution u of the homogenized problem, does it approximate a solution u_ϵ of the real problem ?' In other words, ' is u the limit of a sequence $\{u_\epsilon\}$ of solutions ?'.

In the case of the linear problem, the uniqueness of the solutions of the various problems concerned trivially answers the above question in the affirmative. In the case of the eigenvalue problem Kesavan [4] has shown that eigenfunctions u of a *simple* eigenvalue of the homogenized problem occur as limits of eigenfunctions u_ϵ of the given family of problems. The case of multiple eigenvalues is still open. In this paper, it will be proved that, under suitable conditions, a *non-singular* solution u of the homogenized problem is the limit of a family $\{u_\epsilon\}$ of solutions of (1.2)–(1.3), which, for sufficiently small ϵ , will also be non-singular.

The main idea of the proof is the following: First a function w_ϵ is defined as the unique solution of the auxiliary linear problem:

$$(1.4) \quad A^\epsilon w_\epsilon = f(u) \text{ in } \Omega,$$

$$(1.5) \quad w_\epsilon = 0 \text{ on } \Gamma,$$

where u is the given non-singular solution of the homogenized problem. Then it is shown that for sufficiently small ϵ , the hypotheses of the Newton-Kantorovich theorem can be verified if w_ϵ is used as the starting vector for the Newton method. Hence the Newton method will converge to a solution u_ϵ which will be unique in a neighbourhood of w_ϵ and of u . Thus $\{u_\epsilon\}$ will converge to u owing to the property of local uniqueness.

In § 2, the important hypotheses are made and some preliminary results are recalled. In § 3, the existence of solutions to the various problems is proved and the homogenized problem is identified. In § 4, the existence of a sequence $\{u_\epsilon\}$ converging to a given solution u is proved. § 5 is reserved for conclusions and various comments.

2. Preliminaries

The following hypotheses are made on the coefficients a_{ij}^ϵ :

(H1) There exists a constant $M > 0$ independent of ϵ such that

$$(2.1) \quad |a_{ij}^\epsilon(x)| \leq M \text{ a.e., } 1 \leq i, j \leq n.$$

(H2) There exists a constant $a_0 > 0$ such that

$$(2.2) \quad \forall \xi = (\xi_i) \in \mathbb{R}^n, a_{ij}^\epsilon(x) \xi_i \xi_j \geq a_0 \xi_i \xi_i \text{ a.e.}$$

Definition 2.1. An operator A of the form,

$$(2.3) \quad A = - \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right),$$

is said to be the *homogenized operator* w.r.t. the family $\{A^\epsilon\}$ if the following holds:

Let $u_\epsilon \rightarrow u$ in $H_0^1(\Omega)$ weakly and $A^\epsilon u_\epsilon \rightarrow g$ in $H^{-1}(\Omega)$ strongly. Then

$$\xi_i^\epsilon = a_{ij}^\epsilon(x) \frac{\partial u_\epsilon}{\partial x_j} \rightarrow a_{ij}(x) \frac{\partial u}{\partial x_j} = \xi_i,$$

in $L^2(\Omega)$ weakly, $1 \leq i \leq n$.

This definition is due to Tartar [8]. As a consequence, given $\{f_\epsilon\}$ in $H^{-1}(\Omega)$ converging strongly to f , and $\{u_\epsilon\}$ the unique solutions of the problems

$$(2.4) \quad \forall v \in H_0^1(\Omega), \quad a_\epsilon(u_\epsilon, v) = \langle f, v \rangle (H^{-1}, H_0^1)$$

where for $w, v \in H_0^1(\Omega)$,

$$(2.5) \quad a_\epsilon(w, v) = \int_{\Omega} a_{ij}^\epsilon(x) \frac{\partial w}{\partial x_j} \frac{\partial v}{\partial x_i} dx,$$

then, $u_\epsilon \rightarrow u$ in $H_0^1(\Omega)$ weakly [and hence in $L^2(\Omega)$ strongly] to the unique solution, u , of the homogenized problem,

$$(2.6) \quad \forall v \in H_0^1(\Omega), \quad a(u, v) = \langle f, v \rangle (H^{-1}, H_0^1)$$

where for $w, v \in H_0^1(\Omega)$,

$$(2.7) \quad a(w, v) = \int_{\Omega} a_{ij}(x) \frac{\partial w}{\partial x_j} \frac{\partial v}{\partial x_i} dx.$$

The existence of such an operator is known (cf. Tartar [7]). The operator is also elliptic with the same constant of ellipticity. The coefficients are bounded. (If they are symmetric, the same bound works.) In case of the a_{ij}^ϵ having a periodic structure the operator A can be explicitly written (cf. Bensoussan *et al* [1]).

Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be a function which verifies the following hypothesis :
 (H3) f is a C^1 function such that both f and f' are Lipschitz continuous with constants K_0 and K_1 respectively. Further, it is assumed that $K_0 < \alpha$ the ellipticity constant of the bilinear forms $a_\epsilon(\cdot, \cdot)$ and $a(\cdot, \cdot)$.

Proposition 2.1. Suppose that the hypothesis (H3) is verified. Then if $u \in L^p(\Omega)$ for some $p \geq 1$, $f(u)$ and $f'(u)$ are also in $L^p(\Omega)$. If $u_\epsilon \rightarrow u$ in $L^p(\Omega)$, then $f(u_\epsilon) \rightarrow f(u)$ and $f'(u_\epsilon) \rightarrow f'(u)$ in $L^p(\Omega)$.

Proof.

$$\int_{\Omega} |f(u(x)) - f(0)|^p dx \leq K_0^p \int_{\Omega} |u(x)|^p dx < +\infty.$$

Since constant functions are in $L^p(\Omega)$ for Ω bounded it follows that $f(u) \in L^p(\Omega)$. Similarly

$$\int_{\Omega} |f(u_\epsilon(x)) - f(u(x))|^p dx \leq K_0^p \int_{\Omega} |u_\epsilon(x) - u(x)|^p dx,$$

which converges to zero. Thus the assertions proved for $f(u)$. The proof for $f'(u)$ is identical.

The problem (1.2)–(1.3) can now be written as follows in its weak form :

(P_ϵ) To find $u_\epsilon \in H_0^1(\Omega)$ such that

$$(2.8) \quad \forall v \in H_0^1(\Omega), \quad a_\epsilon(u_\epsilon, v) = \int_{\Omega} f(u_\epsilon) \cdot v dx.$$

One can define the nonlinear operator

$T_\epsilon: H_0^1(\Omega) \rightarrow H_0^1(\Omega)$ as follows;

$$(2.9) \quad \forall v \in H_0^1(\Omega), \quad a_\epsilon(T_\epsilon(u), v) = \int_\Omega f(u) \cdot v \, dx.$$

Then the problem (P_ϵ) reduces to finding fixed points of T_ϵ or the zeros of F_ϵ where

$$(2.10) \quad F_\epsilon = I - T_\epsilon.$$

Remark 2.1. Since the injection $H_0^1(\Omega) \rightarrow L^2(\Omega)$ is compact, it follows that T_ϵ is compact.

Analogously, the operators T and $F (= I - T)$, associated to the homogenized operator A , can be defined.

Finally, an abstract theorem relating to solution of nonlinear equations in Banach spaces will now be recalled.

Theorem 2.1. Let X and Y be Banach spaces and $F : D \subset X \rightarrow Y$. Suppose that on an open convex set $D_0 \subset D$, F is Fréchet differentiable and

$$(2.11) \quad \forall x, y \in D_0, \quad \|F'(x) - F'(y)\| \leq K \|x - y\|.$$

For some $x_0 \in D_0$, assume that $\Gamma_0 = (F'(x_0))^{-1}$ is defined on all of Y and that $h \equiv \beta K \eta \leq \frac{1}{2}$ where $\|\Gamma_0\| \leq \beta$, and $\|\Gamma_0 F x_0\| \leq \eta$. Set

$$(2.12) \quad t^* = \frac{1}{\beta K} (1 - \sqrt{1 - 2h}), \quad t^{**} = \frac{1}{\beta K} (1 + \sqrt{1 - 2h}).$$

Suppose that $S \equiv \{x \mid \|x - x_0\| \leq t^*\} \subset D_0$.

Then the Newton iterates $\{x_k\}$ given by

$$(2.13) \quad x_{k+1} = x_k - (F'(x_k))^{-1} F(x_k), \quad k = 0, 1, \dots$$

are all well defined, lie in S , and converge to a solution x^* of $F(x) = 0$, which is unique in $D_0 \cap \{x \mid \|x - x_0\| < t^{**}\}$. Moreover, if $h < \frac{1}{2}$ the order of convergence is at least quadratic.

Known as the Newton-Kantorovich Theorem, the above was proved by Kantorovich (cf. [2], [3]). Ortega [6] has given a simpler proof to the same. This theorem will be used to prove the main result in § 4.

3. Existence and homogenization results

Throughout this section we assume that the hypotheses (H1)–(H3) are verified.

Proposition 3.1. There exists a constant $C > 0$, independent of ϵ , such that

$$(3.1) \quad \|u_\epsilon\|_{1, \Omega} \leq C.$$

Proof. Set $v = u_\epsilon$ in (2.8). Then

$$\begin{aligned} \alpha \|u_\epsilon\|_{1, \Omega}^2 &\leq a_\epsilon(u_\epsilon, u_\epsilon) = \int_\Omega f(u_\epsilon) u_\epsilon \, dx \\ &= \int_\Omega (f(u_\epsilon) - f(a)) u_\epsilon \, dx + \int_\Omega f(a) u_\epsilon \, dx, \quad f(a) = 0 \\ &\leq K_0 \int_\Omega |u_\epsilon - a| |u_\epsilon| \, dx + C_1 |u_\epsilon|_{0, \Omega} \\ &\leq K_0 |u_\epsilon|_{0, \Omega}^2 + C_2 |u_\epsilon|_{0, \Omega}, \end{aligned}$$

or

$$\|u_\epsilon\|_{1,\Omega}^2 \leq \frac{C_2}{\alpha - K_0} \|u_\epsilon\|_{0,\Omega} \leq \frac{C_2}{\alpha - K_0} \|u_\epsilon\|_{1,\Omega},$$

which proves (3.1).

Remark 3.1. In the above proof, the hypothesis $K_0 < \alpha$ has been used.

Theorem 3.1. There exists at least one solution, to the problem (P_ϵ) .

Proof. First of all it was remarked in the previous section that T_ϵ is compact. Further if $0 < \lambda < 1$ it can be proved that any solution of the problem

$$(3.2) \quad z_\epsilon = \lambda T_\epsilon z_\epsilon,$$

is such that

$$(3.3) \quad \|z_\epsilon\|_{1,\Omega} \leq C,$$

where $C > 0$ is as in Proposition 3.1. (The proof of this fact is identical to that of the above-mentioned proposition.) Thus by Schaeffer's Theorem T_ϵ admits at least one fixed point in $B(0; C)$, the ball centre 0 and radius C in $H_0^1(\Omega)$.

Theorem 3.2. There is at least one solution to the homogenized problem $(P) : u = Tu$

Proof. The proof is identical to that of the preceding theorem.

Theorem 3.3. Let u_ϵ be a solution to (P_ϵ) . Then there exists a subsequence (again indexed by ϵ) such that $u_\epsilon \rightarrow u$ in $H_0^1(\Omega)$ weakly, u being a solution to the homogenized problem (P) .

Proof. By Proposition 3.1, the u_ϵ are all bounded in $H_0^1(\Omega)$. Thus we can extract a weakly convergent subsequence with limit, say, u . Then $u_\epsilon \rightarrow u$ in $L^2(\Omega)$ strongly. Hence by proposition 2.1, $f(u_\epsilon) \rightarrow f(u)$ in $L^2(\Omega)$ strongly and thus in $H^{-1}(\Omega)$ strongly. Now by the definition of the homogenized operator (cf. definition 2.1) we have

$$(3.4) \quad \forall v \in H_0^1(\Omega), a(u, v) = \int_\Omega f(u) v \, dx,$$

which proves the theorem.

The idea of using Schaeffer's Theorem is quite classical in the existence theory of nonlinear equations (cf. Rabinowitz [7]).

4. The main theorem

Under the hypotheses (H1)–(H3), we already know that the problems (P) and (P_ϵ) admit at least one solution each. We will prove now that if $n \leq 5$, and u a non-singular solution of (P) , among the solution of (P_ϵ) there exists a solution u_ϵ which is non singular, tending to u . This is an important step toward the study of the stability of homogenization for the associated evolution equation.

Proposition 4.1. Let $n \leq 6$. Then T_ϵ (resp. T) is Fréchet differentiable on all of V . The family $\{T_\epsilon, T\}$ is uniformly Lipschitz continuous.

Proof. Since $n \leq 6$, the following inclusion holds :

$$H_0^1(\Omega) \rightarrow L^3(\Omega).$$

Then for $u, v, w \in H_0^1(\Omega)$, the integral

$$\int f'(u) v w \, dx,$$

is well defined and is continuous w.r.t. v and w . It is easy to check that the Fréchet derivative of T_ϵ at u is defined by

$$(4.1) \quad \forall w \in H_0^1(\Omega), a_\epsilon(T'_\epsilon(u) v, w) = \int_\Omega f'(u) v w \, dx,$$

for any $v \in H_0^1(\Omega)$. Now, let $u_1, u_2 \in H_0^1(\Omega)$. Set

$$(4.2) \quad z_i^\epsilon = T'_\epsilon(u_i) v, \quad i = 1, 2,$$

where $v \in H_0^1(\Omega)$. Then

$$\begin{aligned} \alpha \|z_1^\epsilon - z_2^\epsilon\|_{1, \Omega}^2 &\leq a_\epsilon(z_1^\epsilon - z_2^\epsilon, z_1^\epsilon - z_2^\epsilon) \\ &= \int_\Omega (f'(u_1) - f'(u_2))(z_1^\epsilon - z_2^\epsilon) v \, dx \\ &\leq K_1 \int |u_1 - u_2| |z_1^\epsilon - z_2^\epsilon| |v| \, dx \\ &\leq K_1 \|u_1 - u_2\|_{0,3, \Omega} \|z_1^\epsilon - z_2^\epsilon\|_{0,3, \Omega} \|v\|_{0,3, \Omega} \\ &\leq C K_1 \|u_1 - u_2\|_{1, \Omega} \|z_1^\epsilon - z_2^\epsilon\|_{1, \Omega} \|v\|_{1, \Omega}, \end{aligned}$$

$$\text{or} \quad \|(T'_\epsilon(u_1) - T'_\epsilon(u_2)) v\|_{1, \Omega} \leq K \|u_1 - u_2\|_{1, \Omega} \|v\|_{1, \Omega},$$

which gives

$$(4.3) \quad \|T'_\epsilon(u_1) - T'_\epsilon(u_2)\|_{1, \Omega} \leq K \|u_1 - u_2\|_{1, \Omega}$$

K being independent of ϵ . The proof for T is identical.

Proposition 4.2. Let $n \leq 5$. Then for any $u \in H_0^1(\Omega)$, $T'_\epsilon(u)$ (resp. $T'(u)$) is compact.

Proof. For $n \leq 5$, the injection

$$(4.4) \quad H_0^1(\Omega) \rightarrow L^3(\Omega)$$

is compact. The result is a direct consequence of this fact.

Remark 4.1. If $n = 6$, then the inclusion (4.4) is not compact and the above result does not hold.

Henceforth, it will be assumed that u is a given solution of the homogenized problem (P). Let, for each ϵ , $w_\epsilon \in H_0^1(\Omega)$ be defined as the unique solution of the following linear problem.

$$(4.5) \quad \forall v \in H_0^1(\Omega), a_\epsilon(w_\epsilon, v) = \int_\Omega f(u) v \, dx.$$

From the theory of homogenization of linear problems, it follows that $w_\epsilon \rightarrow u$ in $H_0^1(\Omega)$ weakly [and in $L^2(\Omega)$ strongly].

Proposition 4.3. Let $n \leq 5$. Let u be a non-singular solution of problem (P), i.e. $F'(u) = I - T'(u)$ is invertible. Then for sufficiently small ϵ , $F'_\epsilon(w_\epsilon) =$

$I - T'_\epsilon(w_\epsilon)$ is also invertible. Further, there exists a $\beta > 0$, independent of ϵ , such that

$$(4.6) \quad \|(I - T'_\epsilon(w_\epsilon))^{-1}\| \leq \beta.$$

Proof. Assume the first assertion false. Then there exists a sequence $\epsilon_n \rightarrow 0$ such that

$$I - T'_{\epsilon_n}(w_{\epsilon_n}),$$

is singular. (Henceforth, purely as a matter of convenience, the index n will be suppressed.) Since $T'_\epsilon(w_\epsilon)$ is compact, there exists a non-zero function $z_\epsilon \in H_0^1(\Omega)$ such that

$$(4.7) \quad z_\epsilon = T'_\epsilon(w_\epsilon) z_\epsilon.$$

It can be further assumed that

$$(4.8) \quad \|z_\epsilon\|_{1, \Omega} = 1.$$

Now (4.7) can be rewritten as

$$(4.9) \quad \forall w \in H_0^1(\Omega), a_\epsilon(z_\epsilon, w) = \int_\Omega f'(w_\epsilon) z_\epsilon w \, dx.$$

Now for a subsequence (again indexed by ϵ) $z_\epsilon \rightarrow z$ and $w_\epsilon \rightarrow u$ in $H_0^1(\Omega)$ weakly. Consider, for any $w \in H_0^1(\Omega)$,

$$\begin{aligned} & \left| \int_\Omega (f'(w_\epsilon) z_\epsilon - f'(u) z) w \, dx \right| \\ & \leq \int_\Omega |f'(w_\epsilon)| |z_\epsilon - z| |w| \, dx + \int_\Omega |f'(w_\epsilon) - f'(u)| |z| |w| \, dx \\ & \leq C(|z_\epsilon - z|_{0,3,\Omega} + |f'(w_\epsilon) - f'(u)|_{0,3,\Omega}) \|w\|_{1,\Omega}. \end{aligned}$$

Since $z_\epsilon \rightarrow z$ in $L^3(\Omega)$ strongly ($n \leq 5$) and $f'(w_\epsilon) \rightarrow f'(u)$ in $L^3(\Omega)$ strongly [cf. Proposition (2.11)] it follows that

$$f'(w_\epsilon) z_\epsilon \rightarrow f'(u) z,$$

in $H^{-1}(\Omega)$ strongly. Then by the definition of the homogenized operator,

$$(4.10) \quad \forall w \in H_0^1(\Omega), a(z, w) = \int_\Omega f'(u) z w \, dx,$$

or, equivalently,

$$(4.11) \quad z = T'(u) z.$$

By assumption, this is possible only if $z = 0$. Hence $z_\epsilon \rightarrow 0$ in $H_0^1(\Omega)$ weakly ($L^3(\Omega)$ strongly).

Now, choosing $w = z_\epsilon$ in (4.9) and taking into account (4.8), it follows that

$$0 < \alpha \leq \int_\Omega f'(w_\epsilon) z_\epsilon z_\epsilon \, dx \leq |f'(w_\epsilon)|_{0,3,\Omega} |z_\epsilon|_{0,3,\Omega}$$

which converges to zero, thus giving a contradiction. Hence the first assertion is true.

Assume now that (4.6) is false. Then there exists a sequence (denoted, as usual, by ϵ itself) y_ϵ and a sequence v_ϵ in $H_0^1(\Omega)$ such that

$$(4.12) \quad \begin{cases} v_\epsilon = (I - T'_\epsilon(w_\epsilon))^{-1} y_\epsilon, \\ \|v_\epsilon\|_{1, \Omega} = 1, \|y_\epsilon\|_{1, \Omega} \rightarrow 0. \end{cases}$$

Set,

$$(4.13) \quad z_\epsilon = T'_\epsilon(w_\epsilon) v_\epsilon.$$

Then

$$(4.14) \quad y_\epsilon = v_\epsilon - z_\epsilon.$$

Clearly, $\|z_\epsilon\|_{1, \Omega} \leq C$ and hence (for a subsequence) $v_\epsilon \rightarrow v$, $z_\epsilon \rightarrow z$, say, in $H_0^1(\Omega)$ weakly. Since $y_\epsilon \rightarrow 0$ in $H_0^1(\Omega)$ strongly it follows that $z = v$. As before it can be shown that

$$f'(w_\epsilon) v_\epsilon \rightarrow f'(u) v,$$

in $H^{-1}(\Omega)$ strongly. Passing to the limit,

$$(4.15) \quad \forall w \in H_0^1(\Omega), a(z, w) = \int_\Omega f'(u) z w \, dx,$$

which again implies $z = 0 = v$. Once again,

$$(4.16) \quad \alpha \|z_\epsilon\|_{1, \Omega}^2 \leq a_\epsilon(z_\epsilon, z_\epsilon) = \int_\Omega f'(w_\epsilon) v_\epsilon z_\epsilon \, dx.$$

The integral converges to zero since $v_\epsilon \rightarrow 0$, $z_\epsilon \rightarrow 0$ in $L^3(\Omega)$ strongly. Hence it follows from (4.16) that $z_\epsilon \rightarrow 0$ in $H_0^1(\Omega)$ strongly and hence from (4.14) that $v_\epsilon \rightarrow 0$ in $H_0^1(\Omega)$ strongly. But this contradicts the fact that $\|v_\epsilon\|_{1, \Omega} = 1$. Thus, (4.6) is established.

Proposition 4.4.

$$\lim_{\epsilon \rightarrow 0} \|F_\epsilon(w_\epsilon)\|_{1, \Omega} = 0.$$

Proof. Set

$$(4.17) \quad y_\epsilon = T_\epsilon w_\epsilon.$$

$$\text{Then } \|F_\epsilon(w_\epsilon)\|_{1, \Omega} = \|w_\epsilon - y_\epsilon\|_{1, \Omega}.$$

Now by definition,

$$(4.18) \quad \forall w \in H_0^1(\Omega), a_\epsilon(w_\epsilon - y_\epsilon, w) = \int_\Omega (f(u) - f(w_\epsilon)) w \, dx.$$

Thus, it follows that

$$\alpha \|w_\epsilon - y_\epsilon\|_{1, \Omega}^2 \leq K_0 |u - w_\epsilon|_{0, \Omega} |w_\epsilon - y_\epsilon|_{0, \Omega},$$

or, for some $C > 0$, independent of ϵ ,

$$(4.19) \quad \|w_\epsilon - y_\epsilon\|_{1, \Omega} \leq C |u - w_\epsilon|_{0, \Omega},$$

which converges to zero, thus proving the result.

The main theorem can now be stated and proved.