Wigner distributions and quantum mechanics on Lie groups: The case of the regular representation

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We consider the problem of setting up the Wigner distribution for states of a quantum system whose configuration space is a Lie group. The basic properties of Wigner distributions in the familiar Cartesian case are systematically generalized to accommodate new features which arise when the configuration space changes from n-dimensional Euclidean space $\mathcal{R}^n$ to a Lie group G. The notion of canonical momentum is carefully analyzed, and the meanings of marginal probability distributions and their recovery from the Wigner distribution are clarified. For the case of compact G an explicit definition of the Wigner distribution is proposed, possessing all the required properties. Geodesic curves in G which help introduce a notion of the midpoint of two group elements play a central role in the construction.

I. INTRODUCTION

The method of Wigner distributions[1] as a description of states of quantum mechanical systems appeared in 1932, quite early in the history of quantum mechanics. For systems whose kinematics is based upon a set of Heisenberg canonical commutation relations, it gives a way of describing both pure and mixed states in a classical phase space setting, at the level of density operators. Thus it must be sharply distinguished in mathematical structure from the Hilbert space state vector or wave function description of states, which highlights the superposition principle of quantum mechanics. In the Wigner distribution language, this principle is not obvious or manifest, but is somewhat hidden in the formalism. On the other hand, the formation of convex classical statistical mixtures of general states to generate new states becomes much more obvious. Somewhat later it was appreciated that the Wigner distribution way of describing quantum mechanical states is dual to, or is naturally accompanied by, the Weyl ordering rule[2]—a convention by which one can set up a one-to-one correspondence between operators in quantum theory (in the case of the Heisenberg commutation relations) and c-number dynamical phase space variables for the comparison classical system. Thus expectation values for general quantum dynamical variables in general quantum states can be faithfully expressed in the full operator-state vector language, or equally well in

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a completely c-number classical phase space language. In this general scheme, the group $\mathrm{Sp}(2n,R)$ of linear canonical transformations for n Cartesian degrees of freedom, and the related two-fold covering group $\mathrm{Mp}(2n)$, play prominent roles.[3]

An important property of the Wigner distribution for a general quantum state is that while it is a real function on the classical phase space, it is not always pointwise non-negative. Therefore it is usually called a quasi probability distribution, and cannot be interpreted as a phase space probability density in the sense of classical statistical mechanics. However, the Wigner distribution does have the attractive property that the marginal distributions, obtained by integrating away either the momentum or the position variables, do reproduce the correct non-negative position space and momentum space probability distributions respectively as specified by quantum mechanics. This recovery or reproducibility of correct marginal distributions is of course maintained even after unitary action by any $\mathrm{Mp}(2n)$ element.

There have been several attempts[4–15] over the years to generalize the method of Wigner distributions to handle quantum mechanical situations where the basic kinematics is defined, not by Heisenberg-type canonical commutation relations, but by some Lie group which acts as the covariance group of the system of interest. (As will become clear in the sequel, the traditional case is also governed by a group, namely the Abelian group $\mathcal{R}^n$ of translations in n-dimensional Euclidean space.) A commonly studied group is SU(2), in the context of spin systems as well as two-level atoms. One of the important early efforts at providing a general group theoretical setting for the Wigner distribution is due to Stratonovich.[16] In this context we should also mention the comprehensive monograph of Dubin *et al.*[17] It seems to us, however, that in most of these attempts the requirement that certain marginal probability distributions be recovered in a natural way from the Wigner distribution corresponding to a general quantum state, which as mentioned above is an important feature of the usual Cartesian case, is not discussed in a satisfactory manner; in some of these works this important aspect is not considered at all.

The aim of the present paper is to develop from first principles the basic features of quantum kinematics for a system whose configuration space is a general non-Abelian Lie group G rather than a Cartesian space $\mathcal{R}^n$, and then set up a corresponding Wigner distribution formalism which respects the requirement that natural marginal probability distributions are reproduced in a simple manner. This involves several extensions or modifications of the familiar formalism in the Cartesian case. The role of Schrödinger wave functions is of course now played by complex square integrable functions on G, and after normalization each such wave function determines a probability distribution over G. The meaning and definition of canonical momentum variables, and determination of the momentum space probability distribution for a given state, are however nontrivial questions in which the many structural features associated with G play important roles. In particular for a non-Abelian G canonical momenta in quantum theory become noncommuting operators, leading to deep changes in the meaning of momentum eigenstates, momentum eigenvalues and momentum space, etc. It is here that the unitary representation theory of G plays an important role. We show that all these features can be properly taken into account, and a fully satisfactory Wigner distribution can be set up as a function of carefully chosen arguments. This turns out to have simple transformation properties under G action and also to reproduce the marginal position and momentum space probability distributions properly.

It needs to be emphasised that our interest in developing a Wigner distribution formalism for systems whose configuration space is a non-Abelian group is not purely academic. In fact, there are many familiar systems which fall in this category. A general rigid body which has the group SO(3) as its configuration space is a case in point. Another well studied example in this category is the relativistic spherical top[18] whose configuration space is the noncompact group SO(3,1).

The material of this paper is arranged as follows. In Sec. II we recall the main definitions and properties of Wigner distributions in the Cartesian situation. We emphasize several familiar features in this case: the possibility of use of the classical phase space as the domain of definition of Wigner distributions; the roles of the groups $\mathrm{Sp}(2n,R)$ and $\mathrm{Mp}(2n)$; the reality but in general loss of pointwise positivity of Wigner distributions; and the recovery of the coordinate space and momentum space probability distributions for a given state by integrating over half the arguments of the Wigner distribution. Section III describes briefly the properties of Wigner distributions in the angle-angular momentum case.[19] This brings out some new features, namely, loss of the classical phase space as the domain of definition of Wigner distributions, and absence of replacements for the groups $\mathrm{Sp}(2n,R)$ and $\mathrm{Mp}(2n)$, which indicate the type of changes we should expect in the case of a general Lie group. Section IV analyzes in some detail the classical phase space that goes with a non-Abelian Lie group G as configuration space. Both global intrinsic and local coordinate based descriptions are given, and the associated classical Poisson bracket relations are developed and described in several ways. In particular a careful analysis of the concept of classical canonical momenta in this case is provided. The transition to quantum mechanics, based on Schrödinger wave functions over G, is then outlined. It is emphasised that a naive generalization of the usual canonical Heisenberg commutation relations is not possible, and all the concepts of position operators, momentum operators and their commutation relations have to be treated with care. A brief Sec. V indicates the kinds of new features we may expect to appear, based on the results and discussions of Sec. III and IV. In Sec. VI we pose the main problem of defining Wigner distributions in a suitable way, with suitable choice of arguments, subject to the main requirements already mentioned above: reasonable transformation laws under G action, recovery of marginal probability distributions, and capturing the full information contained in a general pure or mixed quantum state. We propose a solution to this problem, possessing all the desired properties. We find that our solution uses in an essential and interesting manner the concept and properties of geodesics in G leading to the notion of a midpoint of two group elements, a key ingredient in our construction. For definiteness we confine ourselves to the case of compact G. Section VII shows how the known results of Secs. II and III, for Cartesian quantum mechanics and for the angle angular-momentum pair, are easily recovered from the general case. They correspond actually to the choices $G=\mathcal{R}^n$ and $G=\mathrm{SO}(2)$, which are both Abelian and, respectively, noncompact and compact. The case of $G=\mathrm{SU}(2)$ is then briefly considered, giving adequate background details so that the structure of the Wigner distribution can be easily appreciated. Some of the important differences compared to the Cartesian case, as well as to earlier approaches, are mentioned. Section VIII contains some concluding remarks. We have included two appendices. Appendix A recollects basic results from the theory of the regular representation of G in the compact case, based essentially on the Peter–Weyl theorem. In addition certain useful operator structures are set up, which help us understand better the construction of Wigner distributions in Sec. VI. Appendix B discusses the question of completeness of the information content in the Wigner distribution set up in Sec. VI, and generalizations of the Weyl exponential operators to the non-Abelian Lie group case.

II. THE WIGNER DISTRIBUTION IN THE CARTESIAN CASE

It is useful to recall briefly the usual definition and the basic properties of the Wigner distribution in the case of Cartesian quantum mechanics, and to highlight those important features which are likely to need generalization when we later take up the treatment of quantum mechanics on a general Lie group.

We consider a quantum system whose kinematics is based on $2n$ Hermitian irreducible Cartesian position and momentum operators $\hat{q}_r,\hat{p}_r, r=1,2,\ldots,n$, obeying the standard Heisenberg commutation relations

$$[\hat{q}_r,\hat{p}_s]=i\hbar\,\delta_{rs}, \quad [\hat{q}_r,\hat{q}_s]=[\hat{p}_r,\hat{p}_s]=0, \quad r,s=1,2,\ldots,n. \tag{2.1}$$

It is useful to express these relations more compactly by defining a $2n$-dimensional column vector with Hermitian operator entries,

$$\hat{\xi}=(\hat{\xi}_a)=(\hat{q}_1\ldots\hat{q}_n \quad \hat{p}_1\ldots\hat{p}_n)^T, \quad a=1,2,\ldots,2n, \tag{2.2}$$

and a real antisymmetric nondegenerate $2n$ dimensional symplectic metric matrix β as

$$\beta=\begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix}. \tag{2.3}$$

Then Eq. (2.1) can be written as

$$[\hat{\xi}_a,\hat{\xi}_b]=i\hbar\beta_{ab}. \tag{2.4}$$

These commutation relations and the Hermiticity properties are preserved when we subject the operators $\hat{\xi}_a$ to a real linear transformation by any matrix of the symplectic group $\mathrm{Sp}(2n,R)$:

$$\mathrm{Sp}(2n,R)=\{S=2n\times 2n \ \text{ real matrix } \ |S\beta S^T=\beta\}, \tag{2.5a}$$

$$S \in \mathrm{Sp}(2n,R): \quad \hat{\xi}_a\rightarrow\hat{\xi}'_a=S_{ab}\hat{\xi}_b,$$

$$[\hat{\xi}'_a,\hat{\xi}'_b]=i\hbar\beta_{ab}. \tag{2.5b}$$

On account of the Stone–von Neumann theorem, such linear transformations must be unitarily induced; i.e., for each $S \in \mathrm{Sp}(2n,R)$, there exists a unitary operator $\overline{\mathrm{U}}(S)$, determined up to a phase, such that

$$\hat{\xi}'_a=S_{ab}\hat{\xi}_b=\overline{\mathrm{U}}(S)^{-1}\hat{\xi}_a\overline{\mathrm{U}}(S). \tag{2.6}$$

These unitary operators give a unitary representation of $\mathrm{Sp}(2n,R)$ up to phases which cannot be totally eliminated, but can at best be reduced to a sign ambiguity:

$$S',S \in \mathrm{Sp}(2n,R): \overline{\mathrm{U}}(S')\overline{\mathrm{U}}(S)=\pm\overline{\mathrm{U}}(S'S). \tag{2.7}$$

This situation may be expressed by the statement that one is actually dealing here with a true representation of the group $\mathrm{Mp}(2n)$ which is a double cover of $\mathrm{Sp}(2n,R)$. These objects will be seen to play important roles in the theory of Wigner distributions in the present case.

Vectors in the Hilbert space $\mathcal{H}$ on which the $\hat{\xi}_a$ are irreducibly represented may be described by their Schrödinger wave functions in the usual manner,

$$|\psi\rangle\in\mathcal{H}:\psi(\underline{q})=\langle\underline{q}|\psi\rangle,$$

$$\langle\underline{q}'|\underline{q}\rangle=\delta^{(n)}(\underline{q}'-\underline{q}), \tag{2.8}$$

$$\langle\psi|\psi\rangle=\|\psi\|^2=\int_{\mathcal{R}^n}\mathrm{d}^n q|\psi(\underline{q})|^2.$$

The (ideal) kets $|\underline{q}\rangle$ are simultaneous eigenvectors of the n commuting position operators $\hat{q}_r, r=1,\ldots,n$. Alternatively we may describe them by their momentum space wave functions $\tilde{\psi}(\underline{p})$ by taking the overlap with simultaneous eigenvectors of the commuting momentum operators $\hat{p}_r, r=1,\ldots,n$:

$$\tilde{\psi}(\underline{p})=\langle\underline{p}|\psi\rangle=\int_{\mathcal{R}^n}\frac{\mathrm{d}^n q}{(2\pi\hbar)^{n/2}}\psi(\underline{q})\exp(-i\ \underline{p}\cdot\underline{q}/\hbar),$$

$$\langle\underline{q}|\underline{p}\rangle=(2\pi\hbar)^{-n/2}\exp(i\ \underline{q}\cdot\underline{p}/\hbar), \tag{2.9}$$

$$\langle\psi|\psi\rangle=\int_{\mathcal{R}^n}\mathrm{d}^n p|\tilde{\psi}(\underline{p})|^2.$$

Given a pure state $|\psi\rangle$ of the above quantum system, the corresponding Wigner distribution is a function $W(\underline{q},\underline{p})$ of $2n$ classical real variables, i.e., a function on $\mathcal{R}^{2n}$. In analogy with Eq. (2.2) we assemble the arguments $q_1\cdots q_n \;\; p_1\cdots p_n$ into a real $2n$-component column vector $\xi=(\xi_a) =(q_1\cdots q_n \;\; p_1\cdots p_n)^T$, and then $W(\xi)$ is defined by a partial Fourier transformation:

$$W(\xi)=(2\pi\hbar)^{-n}\int_{\mathcal{R}^n}\mathrm{d}^n q' \; \psi(\underline{q}-\tfrac{1}{2}\underline{q}')\psi(\underline{q}+\tfrac{1}{2}\underline{q}')^* \exp(i \;\; \underline{p}\cdot\underline{q}'/\hbar). \tag{2.10}$$

Here the dependence of $W(\xi)$ on ψ is left implicit. For a general mixed state we define $W(\xi)$ through the configuration space matrix elements of the density operator $\hat{\rho}$,

$$W(\xi)=(2\pi\hbar)^{-n}\int_{\mathcal{R}^n}\mathrm{d}^n q'\langle\underline{q}-\tfrac{1}{2}\underline{q}'|\hat{\rho}|\underline{q}+\tfrac{1}{2}\underline{q}'\rangle\exp(i \;\; \underline{p}\cdot\underline{q}'/\hbar), \tag{2.11}$$

once again leaving the dependence on $\hat{\rho}$ implicit. It is clear by construction that $W(\xi)$ is a real phase space function. The recovery of the proper non-negative marginal probability distributions is demonstrated by

$$\begin{aligned}\int_{\mathcal{R}^n}\mathrm{d}^n p \; W(\xi)&=|\psi(\underline{q})|^2,\\ \int_{\mathcal{R}^n}\mathrm{d}^n q \; W(\xi)&=|\tilde{\psi}(\underline{p})|^2.\end{aligned} \tag{2.12}$$

On the other hand, if $W(\xi)$ and $W'(\xi)$ correspond, respectively, to $\hat{\rho}$ and $\hat{\rho}'$, it is easily shown that

$$\mathrm{Tr}(\hat{\rho}'\hat{\rho})=(2\pi\hbar)^{-n}\int_{\mathcal{R}^{2n}}\mathrm{d}^{2n}\xi \, W'(\xi)W(\xi)\geqslant 0. \tag{2.13}$$

But since it is easy to construct cases where the trace on the left-hand side actually vanishes, we can expect that in general $W(\xi)$ becomes negative in some regions of $\mathcal{R}^{2n}$. Indeed, the simplest explicit example showing this is the expression for the Wigner function for the first excited state of the harmonic oscillator in one dimension. Taking $n=1$ and setting $\hbar=1$ for simplicity, we have

$$\psi(q)=\frac{\sqrt{2}}{\pi^{1/4}}q \quad e^{-q^2/2}\rightarrow W(\xi)=\frac{2}{\pi}\left(q^2+p^2-\frac{1}{2}\right) \quad e^{-q^2-p^2}. \tag{2.14}$$

In this context it is interesting to recall the following two results (again in one dimension) as indicative of the characteristic features of Wigner distributions.

(i) *Hudson:*[20] For a pure state $\psi(q)$ the Wigner function is pointwise non-negative if and only if $\psi(q)$ [and hence $W(\xi)$ as well] is a (complex) Gaussian.

(ii) *Folland-Sitaram:*[21] If $W(\xi)$ has compact support in $\mathcal{R}^2$, it must vanish identically.

Under the unitary action of $\mathrm{Mp}(2n)$ on $\hat{\rho}$, $W(\xi)$ experiences a simple point transformation

$$\hat{\rho}'=\overline{\mathrm{U}}(S)\hat{\rho}\overline{\mathrm{U}}(S)^{-1}\Leftrightarrow W'(\xi)=W(S^{-1}\xi), \; S\in\mathrm{Sp}(2n,R). \tag{2.15}$$

Thus we have covariance under the group $\mathrm{Sp}(2n,R)$ which is the maximal linear homogeneous group mixing q's and p's. This combined with the results of Eq. (2.12) shows that we recover the correct marginal probability distributions by integrating over half the variables in $W(\xi)$ even after action by any $\mathrm{Sp}(2n,R)$ transformation.

The sense in which the definitions (2.10) and (2.11) of the Wigner distribution are dual to the Weyl ordering rule is as follows. The latter rule associates with each elementary classical exponential a corresponding elementary operator exponential,

$$\exp(i\ \underline{\lambda}\cdot\underline{q}-i\ \underline{\mu}\cdot\underline{p})\rightarrow\exp(i\ \underline{\lambda}\cdot\underline{\hat{q}}-i\ \underline{\mu}\cdot\underline{\hat{p}}), \tag{2.16}$$

where $\underline{\lambda}$ and $\underline{\mu}$ are arbitrary real vectors in $\mathcal{R}^n$; and this is then extended by linearity and Fourier transformation to general classical functions, say

$$f(\underline{q},\underline{p})\equiv f(\xi)\rightarrow\hat{F}. \tag{2.17}$$

Then the dual relationship is expressed by the equality of two ways of computing quantum expectation values,

$$\begin{gathered}\operatorname{Tr}(\hat{\rho}\exp(i\ \underline{\lambda}\cdot\underline{\hat{q}}-i\ \underline{\mu}\cdot\underline{\hat{p}}))=(2\pi\hbar)^{-n}\int_{\mathcal{R}^{2n}}\mathrm{d}^{2n}\xi\, W(\xi)\exp(i\ \underline{\lambda}\cdot\underline{q}-i\ \underline{\mu}\cdot\underline{p}),\\ \operatorname{Tr}(\hat{\rho}\hat{F})=(2\pi\hbar)^{-n}\int_{\mathcal{R}^{2n}}\mathrm{d}^{2n}\xi W(\xi)f(\xi).\end{gathered} \tag{2.18}$$

The definition (2.10) gives $W(\xi)$ for a given pure state $\psi(\underline{q})$. By polarization we can obtain a sesquilinear expression: for any two pure states ψ, φ we can set up a (generally complex) Wigner distribution

$$W_{\psi,\varphi}(\xi)=(2\pi\hbar)^{-n}\int_{\mathcal{R}^n}\mathrm{d}^nq'\ \psi(\underline{q}-\tfrac{1}{2}\underline{q}')\varphi(\underline{q}+\tfrac{1}{2}\underline{q}')^*\exp(i\ \underline{p}\cdot\underline{q}'/\hbar), \tag{2.19}$$

linear in ψ and antilinear in φ. Under complex conjugation we have

$$W_{\psi,\varphi}(\xi)^*=W_{\varphi,\psi}(\xi), \tag{2.20}$$

and both the formula (2.13) and the $\mathrm{Mp}(2n)$ covariance law (2.15) can be easily extended for such objects. For some purposes such expressions may be useful, but we do not make much use of them.

While all of the foregoing is quite familiar, it is useful to make the following additional remarks. It is characteristic of the Heisenberg commutation relations (2.1) that even after quantization, i.e., within quantum mechanics, the possible (sets of simultaneous) eigenvalues of the (commuting) momenta $\hat{p}_r$ by themselves do not suffer any quantization. Thus a general set of eigenvalues $p_r, r=1,\ldots,n$ for $\hat{p}_r$ determines a general point in $\mathcal{R}^n$, just as the eigenvalues q_r of the position operators $\hat{q}_r$ do. It is ultimately this that allows us to describe quantum states for such systems via Wigner distributions over the classical phase space $T^*\mathcal{R}^n\simeq\mathcal{R}^{2n}$, a general point $(\underline{q},\underline{p})$ of which is made up of (nonsimultaneous) eigenvalue sets $\underline{q},\underline{p}$ for the (noncommuting) operator sets $\hat{q}_r,\hat{p}_r$. The appearance and use of the classical phase space here is not as a result of taking the classical or semiclassical limit of the quantum theory, but is a way of expressing the exact content of the quantum theory in a fully c-number language. The role and relevance of the groups $\mathrm{Sp}(2n,R),\mathrm{Mp}(2n)$ in Cartesian quantum mechanics can really be traced back to these facts; it makes sense to form canonical linear combinations of Cartesian $\hat{q}$'s and $\hat{p}$'s. The importance of these remarks is seen from a comparison with the case of an angle-angular momentum pair,[19] and the proper way to set up Wigner distributions in that case. We recall this briefly in the next section, emphasizing the differences compared to the Cartesian situation.

III. THE WIGNER DISTRIBUTION IN THE ANGLE-ANGULAR MOMENTUM CASE

For a classical angle variable $\theta\in(-\pi,\pi)$, the configuration space Q is the circle S^1; so at the classical level the phase space or cotangent bundle is the cylinder $T^*S^1\simeq S^1\times\mathcal{R}$. This contains, in addition to the coordinate θ, a generalized momentum, p_θ say, which can be any real number: $p_\theta\in\mathcal{R}$. Now in the quantum situation we have an angle operator $\hat{\theta}$ with eigenvalues $\theta\in(-\pi,\pi)$, and a conjugate angular momentum operator $\hat{M}$ whose eigenvalues are quantized and are $m=0,\pm 1,\pm 2,...$, i.e., $m\in\mathcal{Z}$ and *not* $m\in\mathcal{R}$. It is unnatural in this case to write down a commutation relation between $\hat{\theta}$ and $\hat{M}$; rather their mutual relationship is best expressed through these eigenvalue and eigenvector statements,

$$\hat{\theta}|\theta\rangle=\theta|\theta\rangle,\quad \theta\in(-\pi,\pi),$$

$$\langle\theta'|\theta\rangle=\delta(\theta'-\theta), \tag{3.1a}$$

$$\hat{M}|m\rangle=m\hbar|m\rangle,\quad m\in\mathcal{Z},$$

$$\langle m'|m\rangle=\delta_{m'm}, \tag{3.1b}$$

$$\langle\theta|m\rangle=(2\pi)^{-1/2}\exp(im\theta), \tag{3.1c}$$

$$\int_{-\pi}^{\pi}\mathrm{d}\theta|\theta\rangle\langle\theta|=\sum_{m\in\mathcal{Z}}|m\rangle\langle m|=1. \tag{3.1d}$$

The Hilbert space $\mathcal{H}$ relevant here is $L^2(-\pi,\pi)\simeq\ell^2$. Now we define the bounded unitary exponentials (Weyl exponentials)

$$U(n)=\exp(in\hat{\theta}),\quad n\in\mathcal{Z},$$

$$V(\tau)=\exp(-i\tau\hat{M}),\quad \tau\in(-\pi,\pi). \tag{3.2}$$

[We do not need to define the more general $U(\sigma),V(\tau)$ for $\sigma,\tau\in\mathcal{R}$.] In contrast to the Cartesian case where both $\hat{q}$ and $\hat{p}$ are unbounded, here only $\hat{M}$ is unbounded. Then, for a given pure state $|\psi\rangle\in\mathcal{H}$ with wave function $\psi(\theta)=\langle\theta|\psi\rangle$, the Wigner distribution is a real function $W(\theta,m)$ of an angle θ and an integer m defined as follows:

$$W(\theta,m)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\tau\,\psi(\theta+\tau/2)\psi(\theta-\tau/2)^*\quad e^{-im\tau}, \tag{3.3}$$

the arguments of ψ and ψ^* always being in the range $(-\pi,\pi)$ via shifts of amounts $\pm 2\pi$. We note that the pair (θ,m) is not a point in the classical phase space T^*S^1, just because the "momentum" eigenvalue m is quantized. The definition (3.3) reproduces the marginals correctly,

$$\int_{-\pi}^{\pi}\mathrm{d}\theta\,W(\theta,m)=|\langle m|\psi\rangle|^2,$$

$$\sum_{m\in\mathcal{Z}}W(\theta,m)=|\langle\theta|\psi\rangle|^2. \tag{3.4}$$

There is an accompanying dual Weyl operator correspondence as well: it takes elementary classical exponentials on $S^1\times\mathcal{Z}$ into specific products of the U's and V's of Eq. (3.2):

$$\exp(in\theta-i\tau m)\rightarrow U(n)V(\tau)e^{-in\tau/2}=V(\tau)U(n)e^{in\tau/2},$$

$$n\in\mathcal{Z},\quad \tau\in(-\pi,\pi), \tag{3.5a}$$

$$\langle\psi|U(n)V(\tau)e^{-in\tau/2}|\psi\rangle=\int_{-\pi}^{\pi}\mathrm{d}\theta\sum_{m\in\mathcal{Z}}W(\theta,m)e^{i(n\theta-\tau m)}. \tag{3.5b}$$

[The operator exponentials in (3.5a) cannot be combined into single exponentials.] It is the case that the operators $U(n)V(\tau)e^{-in\tau/2}$ for all $n\in\mathcal{Z}$ and $\tau\in(-\pi,\pi)$ do form a complete (trace orthonormal) basis for all operators on $\mathcal{H}$; and what the Weyl rule here does is to place any operator $\hat{F}$ on $\mathcal{H}$ in correspondence with a classical function $f(\theta,m)$ on $S^1\times\mathcal{Z}$, not on $T^*S^1 \simeq S^1\times\mathcal{R}$.

One appreciates that here a certain amount of quantization is already incorporated into the classical phase space structure, before the Wigner distribution can be defined in a reasonable way. There is also no room for the groups $\mathrm{Sp}(2,R)$ and $\mathrm{Mp}(2)$. These characteristic differences compared to the Cartesian case will get magnified in the case of a general Lie group.

The replacements for Eqs. (2.11) and (2.13) of the Cartesian case turn out to be as follows:

$$W(\theta,m)=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\tau\langle\theta+\tau/2|\hat{\rho}|\theta-\tau/2\rangle e^{-im\tau},$$
$$\mathrm{Tr}(\hat{\rho}'\hat{\rho})=\sum_{m\in\mathcal{Z}}\int_{-\pi}^{\pi}\mathrm{d}\theta\, W'(\theta,m)W(\theta,m). \tag{3.6}$$

Thus from the latter we can see again that in general $W(\theta,m)$, though real, can become negative for some arguments.

In concluding this section, we mention one interesting case which has no Cartesian analogue. Since $\hat{M}$ has a discrete spectrum, its eigenvectors are normalizable, and in that case we find

$$|\psi\rangle=|m_0\rangle : W(\theta,m)=\frac{1}{2\pi}\delta_{m,m_0}. \tag{3.7}$$

Clearly both of Eqs. (3.4) are satisfied.

IV. CLASSICAL AND QUANTUM MECHANICS ON PHASE SPACE OF A LIE GROUP

As a preliminary step towards setting up the Wigner distribution formalism for quantum systems with kinematics based on a general Lie group, we first briefly recall the important features of the corresponding classical situation.[22]

Let G be a connected Lie group of dimension n, and let us regard it as the configuration space Q of a classical dynamical system. Then the generalized coordinate for the system is a variable element $g\in G$. The corresponding phase space T^*Q is the cotangent bundle T^*G. We can describe T^*G in intrinsic purely geometric terms, which has the advantage of being globally well defined. However from the point of view of facilitating practical calculations in any particular case, and so as to avoid being too cryptic, it is also useful to develop local coordinate based descriptions of T^*G. We outline the former first, and then turn to the latter.

Intrinsic descriptions of T^*G: As is well known, every Lie group is a parallelizable differentiable manifold. Therefore, if we denote the Lie algebra of G by $\underline{G}$, and the dual to $\underline{G}$ by $\underline{G}^*$, it turns out that T^*G is essentially the Cartesian product $G\times\underline{G}^*$. This equivalence can be established in two equally good ways, neither of which is preferred. For definiteness we identify $\underline{G}$ and $\underline{G}^*$ as the tangent and cotangent spaces to G at the identity e,

$$\underline{G}=T_eG,\quad \underline{G}^*=T_e^*G. \tag{4.1}$$

The Lie group G automatically brings with it the set of left translations L_g and the set of right translations R_g. These are mutually commuting realizations of G by mappings of G onto itself. Their definitions and main properties are

$$L_g:g'\in G\rightarrow gg'\in G,$$

$$
\begin{aligned}
&L_{g_1}\circ L_{g_2}=L_{g_1g_2};\\
&R_g:g'\in G\to g'g^{-1}\in G,\\
&R_{g_1}\circ R_{g_2}=R_{g_1g_2};\\
&L_{g_1}\circ R_{g_2}=R_{g_2}\circ L_{g_1}.
\end{aligned}
\tag{4.2}
$$

The corresponding tangent maps and pull backs act as nonsingular linear transformations on the tangent and cotangent spaces respectively at general points of G, according to the following scheme:

$$
\begin{aligned}
&(L_g)_*:T_{g'}G\to T_{gg'}G,\\
&(R_g)_*:T_{g'}G\to T_{g'g^{-1}}G,
\end{aligned}
\tag{4.3a}
$$

$$
\begin{aligned}
&L_g^*:T_{g'}^*G\to T_{g^{-1}g'}^*G,\\
&R_g^*:T_{g'}^*G\to T_{g'g}^*G.
\end{aligned}
\tag{4.3b}
$$

Now introduce dual bases $\{e_r\},\{e^r\},r=1,2,\ldots,n$ for $T_eG,\ T_e^*G$,

$$
\begin{aligned}
&\underset{\sim}{G}=T_eG=\mathrm{Sp}\{e_r\},\quad \underset{\sim}{G}^*=T_e^*G=\mathrm{Sp}\{e^r\},\\
&\langle e^r,e_s\rangle=\delta_s^r,\quad r,s=1,2,\ldots,n.
\end{aligned}
\tag{4.4}
$$

By applying the tangent maps to $\{e_r\}$ at e, we obtain two sets of bases at each T_gG, in fact two bases for general vector fields on G,

$$
\begin{aligned}
&X_r(g)=(R_{g^{-1}})_*(e_r),\\
&\widetilde{X}_r(g)=(L_g)_*(-e_r),\\
&T_gG=\mathrm{Sp}\{X_r(g)\}=\mathrm{Sp}\{\widetilde{X}_r(g)\}.
\end{aligned}
\tag{4.5}
$$

[The negative sign in the second line is to secure common commutation relations in Eq. (4.6) below.] The vector fields $\{X_r\}$ are right invariant and are the generators of the left translations L_g, while the vector fields $\{\widetilde{X}_r\}$ are left invariant and generate the right translations R_g. Each set obeys the commutation relations (commutators among vector fields!) characterizing the Lie algebra $\underset{\sim}{G}$ of G and involving structure constants f_{rs}^t,

$$
\begin{aligned}
&[X_r,X_s]=f_{rs}{}^tX_t,\\
&[\widetilde{X}_r,\widetilde{X}_s]=f_{rs}{}^t\widetilde{X}_t,\\
&[X_r,\widetilde{X}_s]=0.
\end{aligned}
\tag{4.6}
$$

We naturally have two dual bases for the cotangent spaces T_g^*G:

$$
\begin{aligned}
&T_g^*G=\mathrm{Sp}\{\theta^r(g)\}=\mathrm{Sp}\{\widetilde{\theta}^r(g)\},\\
&\langle\theta^r(g),X_s(g)\rangle=\langle\widetilde{\theta}^r(g),\widetilde{X}_s(g)\rangle=\delta_s^r,
\end{aligned}
$$

$$\theta^r(g)=R_g^*(e^r),$$

$$\tilde{\theta}^r(g)=L_{g^{-1}}^*(-e^r). \tag{4.7}$$

In terms of these forms, the commutation relations (4.6) appear as the Maurer–Cartan relations:

$$\begin{aligned} &\mathrm{d}\theta^r+\tfrac{1}{2}f_{st}{}^r\theta^s\wedge\theta^t=0, \\ &\mathrm{d}\tilde{\theta}^r+\tfrac{1}{2}f_{st}{}^r\tilde{\theta}^s\wedge\tilde{\theta}^t=0. \end{aligned} \tag{4.8}$$

At each $g \in G$ the two sets of objects are related by the $n\times n$ matrices $\mathcal{D}(g)=(\mathcal{D}^r_s(g))$ of the adjoint representation of G (superscript = row, subscript = column index):

$$\begin{aligned} &\tilde{X}_r(g)=-\mathcal{D}^s_r(g)X_s(g), \\ &\theta^r(g)=-\mathcal{D}^r_s(g)\tilde{\theta}^s(g). \end{aligned} \tag{4.9}$$

The important point is that all these maps, objects and relationships are globally well defined.

With this geometric preparation, we can easily see in two ways why the phase space T^*G is essentially the product $G\times\underline{G}^*$. A general point in T^*G is a pair (g,ω) where $g\in G$ and $\omega \in T_g^*G$. But we can expand ω in either of the two bases $\{\theta^r(g)\},\{\tilde{\theta}^r(g)\}$ for T_g^*G, and use the expansion coefficients to synthesize elements in $T_e^*G=\underline{G}^*$,

$$\omega=\omega_r\theta^r(g)=R_g^*(\omega_re^r)\in T_g^*G\Leftrightarrow\omega_0=\omega_re^r\in\underline{G}^*, \tag{4.10a}$$

$$\omega=-\tilde{\omega}_r\tilde{\theta}^r(g)=L_{g^{-1}}^*(\tilde{\omega}_re^r)\in T_g^*G\Leftrightarrow\tilde{\omega}_0=\tilde{\omega}_re^r\in\underline{G}^*. \tag{4.10b}$$

Each of these ways of setting up correspondences gives a globally well-defined method of identifying T^*G with $G\times\underline{G}^*$. For given $\omega\in T_g^*G$, ω_0 and $\tilde{\omega}_0$ are related by the coadjoint representation of G, since

$$\tilde{\omega}_r=\mathcal{D}^s_r(g)\,\omega_s\,. \tag{4.11}$$

The above development displays the structure of the classical phase space T^*G in an intrinsic and globally well-defined manner; in particular it brings out the fact that as a bundle over the base G, T^*G is trivial. (In contrast, for example, T^*S^2 is nontrivial). Now, as stated earlier, we link up to local coordinate based descriptions more suited to practical computations and statements of Poisson bracket relations.

Local coordinate descriptions of T^*G: In general the elements of a Lie group G cannot be described with the help of coordinates in a globally smooth manner. In particular this is so if G is compact. One has to work with charts or locally defined coordinates, with well-defined transition rules in overlaps, etc. For simplicity we will work within a single chart over some open neighborhood of the identity; the setting up of a suitable notation to handle a collection of charts is in principle quite straightforward but is omitted.

Let the element $g\in G$ be labeled by n real independent continuous coordinates $q^r, r=1,2,\ldots,n$; as a convention we set $q^r=0$ at e. These q's are numerical generalized coordinates; especially in case G is compact, each of them is expected to be an angle type variable. To the set of coordinates q^r corresponds the element $g(q)\in G$. We identify the basis elements e_r, e^r for T_eG and T_e^*G, Eq. (4.4), as

$$e_r=\left(\frac{\partial}{\partial q^r}\right)_0,\quad e^r=(\mathrm{d}q^r)_0\,. \tag{4.12}$$

For practical convenience it is often useful to work with some faithful matrix representation of G. This has nothing to do with quantization per se, but is just a convenient way of handling G less abstractly than otherwise. In this sense let $A(q)$ be some faithful matrix representation of G; we identify its generator matrices and commutation relations by

$$\begin{aligned} A(\delta q) &\simeq 1 - i\,\delta q^r T_r\,, \\ [T_r, T_s] &= i f_{rs}{}^t T_t\,. \end{aligned} \tag{4.13}$$

The product of two elements $A(q'), A(q)$ is written as

$$A(q')A(q) = A(f(q';q)), \tag{4.14}$$

where the n functions $f^r(q';q)$ of $2n$ real arguments each express the composition law in G. Certain important auxiliary functions play an important role; their definitions and some properties are summarized here,

$$\begin{aligned} \eta^r_s(q) &= \left(\frac{\partial f^r}{\partial q'^s}(q';q) \right)_{q'=0}, \\ \tilde{\eta}^r_s(q) &= \left(\frac{\partial f^r}{\partial q'^s}(q;q') \right)_{q'=0}, \end{aligned} \tag{4.15a}$$

$$\begin{aligned} (\xi^r_s(q)) &= (\eta^r_s(q))^{-1}, \\ (\tilde{\xi}^r_s(q)) &= (\tilde{\eta}^r_s(q))^{-1}, \end{aligned} \tag{4.15b}$$

$$\begin{aligned} f(\delta q;q) &\simeq q + \eta(q)\,\delta q, \\ f(q;\delta q) &\simeq q + \tilde{\eta}(q)\,\delta q, \end{aligned} \tag{4.15c}$$

$$\begin{aligned} \eta^r_s(q)\,\frac{\partial A(q)}{\partial q^r} &= -\,i T_s A(q), \\ \tilde{\eta}^r_s(q)\,\frac{\partial A(q)}{\partial q^r} &= -\,i A(q) T_s\,. \end{aligned} \tag{4.15d}$$

[For matrix operations here, superscripts (subscripts) are row (column) indices.] The vector fields and one forms in Eqs. (4.5) and (4.7) have the following local expressions:

$$\begin{aligned} X_r &= \eta^s_r(q)\,\frac{\partial}{\partial q^s}, \quad \tilde{X}_r = -\,\tilde{\eta}^s_r(q)\,\frac{\partial}{\partial q^s}, \\ \theta^r &= \xi^r_s(q)\mathrm{d}q^s, \quad \tilde{\theta}^r = -\,\tilde{\xi}^r_s(q)\mathrm{d}q^s, \end{aligned} \tag{4.16}$$

and the adjoint representation matrices $\mathcal{D}(g)$ are given by the product

$$\mathcal{D}(g(q)) = \xi(q)\,\tilde{\eta}(q). \tag{4.17}$$

In the sense of classical canonical mechanics when we go to T^*G we have (local) canonically conjugate momentum variables $p_r, r=1,2,\ldots,n$; and the basic classical Poisson bracket (PB) relations are

$$\{q^r, p_s\} = \delta^r_s, \quad \{q^r, q^s\} = \{p_r, p_s\} = 0. \tag{4.18}$$

As for the ranges of these variables, while the structure of G determines the nature of the q^r, it is generally assumed that each p_r ranges independently over the entire real line $\mathcal{R}$. In other words, $T_g^*G \simeq \mathcal{R}^n$ at each $g \in G$.

While both the coordinates q^r and the momenta p_r have so far a local character, it is possible to replace the latter by certain q-dependent linear combinations which are then globally well defined. They express the structure of the phase space T^*G in a much more natural way. We get a clue to their definitions by noticing, upon combining the PB relations

$$\{A(q), p_r\} = \frac{\partial A(q)}{\partial q^r} \tag{4.19}$$

with Eq. (4.15d), that

$$\begin{aligned} &\{A(q), \eta_s^r(q) p_r\} = -iT_s A(q), \\ &\{A(q), -\tilde{\eta}_s^r(q) p_r\} = iA(q) T_s\,. \end{aligned} \tag{4.20}$$

These relations lead us to define generalized canonical momentum like variables $J_r, \tilde{J}_r$ as follows:

$$J_r = \eta_r^s(q) p_s\,, \quad \tilde{J}_r = -\tilde{\eta}_r^s(q) p_s\,. \tag{4.21}$$

The connection between the two sets is

$$\tilde{J}_r = -\mathcal{D}_r^s(g(q)) J_s\,, \tag{4.22}$$

and, consistent with Eqs. (4.13) and (4.15d), their PB relations are

$$\begin{aligned} &\{J_r, J_s\} = f_{rs}{}^t J_t\,, \\ &\{\tilde{J}_r, \tilde{J}_s\} = f_{rs}{}^t \tilde{J}_t\,, \\ &\{J_r, \tilde{J}_s\} = 0. \end{aligned} \tag{4.23}$$

The complete coordinate-based description of the basic PB relations obtaining on T^*G can now be given in many equally good ways, and we list all of them (allowing for some repetition):

$$\{q^r, q^s\} = 0, \tag{4.24a}$$

$$\begin{aligned} &\{q^r, J_s\} = \eta_s^r(q), \\ &\{q^r, \tilde{J}_s\} = -\tilde{\eta}_s^r(q), \\ &\{A(q), J_r\} = -iT_r A(q), \\ &\{A(q), \tilde{J}_r\} = iA(q) T_r\,, \end{aligned} \tag{4.24b}$$

$$\begin{aligned} &\{J_r, J_s\} = f_{rs}{}^t J_t\,, \\ &\{J_r, \tilde{J}_s\} = 0, \\ &\{\tilde{J}_r, \tilde{J}_s\} = f_{rs}{}^t \tilde{J}_t\,. \end{aligned} \tag{4.24c}$$

It is thus best to view the set of J_r (or $\tilde{J}_r$) as the covariant momentum canonically conjugate to the group element $g \in G$ as a generalized coordinate.

At this point, in the present framework, we recognize that the Lie group underlying the kinematic structure of Cartesian quantum mechanics for n degrees of freedom, expressed by the Heisenberg commutation relations (2.1) and (2.3), is the Abelian translation group $G=\mathcal{R}^n$ in n real dimensions. In this case, the coordinates $q^r, r=1,2,\ldots,n$ denoting an element of $\mathcal{R}^n$ are globally well defined, and $T^*G=T^*\mathcal{R}^n\simeq\mathcal{R}^{2n}$, corresponding to the Cartesian phase space q's and p's. Due to the group being Abelian, the structure constants vanish; the $n\times n$ matrices $\eta(q),\xi(q),\tilde{\eta}(q),\tilde{\xi}(q)$ of Eq. (4.15a) and (4.15b) all reduce to the identity matrix; the momenta J_r and $\tilde{J}_r$ essentially coincide as $J_r=-\tilde{J}_r=p_r$; and the PB relations (4.24) reduce to the familiar classical forms for which the Heisenberg relations (2.1) are the quantized version. We have no difficulty in principle in postulating quantum kinematics through these commutation relations.

However the angle-angular momentum case briefly described in Sec. III corresponds to the group $G=\mathrm{U}(1)\simeq\mathrm{SO}(2)$ which is of course also Abelian. But one immediately sees new features emerging. For instance, the angle variable θ is not a globally well-defined coordinate over G. It is also not very satisfactory, due to operator domain problems, to postulate simple minded Heisenberg-type commutation relations between $\hat{\theta}$ and its canonical conjugate $\hat{M}$ in the quantum case. This is over and above the fact that now $\hat{M}$ is quantized. Thus in the $G=\mathrm{SO}(2)$ case, it is better to base the treatment on the set of relations for operators, eigenvalues, and eigenvectors collected in Eqs. (3.1).

Turning to a general Lie group G where the classical PB structure on T^*G is conveyed by any of the forms given in Eqs. (4.24), it should be evident that we should not base the quantum kinematics on a naive set of commutation relations for operator forms of the group coordinates q^r and the momenta $J_r,\tilde{J}_r$. Rather, while the latter can be satisfactorily handled (and this just involves the representation theory of G), the treatment of the abstract group element g as a coordinate operator after quantization has to be handled somewhat differently.

Quantum kinematics for the Lie group case: We now motivate the forms of the replacements for the Heisenberg canonical commutation relations (2.1) and (2.4) when we consider a quantum system whose configuration space Q is a Lie group G. Just as we identified $G=\mathcal{R}^n$ for n-dimensional Cartesian quantum mechanics, where we know that the Schrödinger wave functions are complex valued square integrable functions on $\mathcal{R}^n$, we should now expect that the Schrödinger wave functions should be complex valued square integrable (in a suitable sense) functions on G. The question now is: with what algebraic operator relations do we replace the earlier canonical $\hat{q}-\hat{p}$ commutation relations?

If we try to avoid the use of (local) coordinates for group elements, in the interests of being as intrinsic as possible, we might be tempted to imagine the following: upon quantization, the classical generalized coordinate $g\in G$ is replaced by an "operator $\hat{g}$" for which the possible "eigenvalues" are the classical abstract group elements. However this seems excessively formal. A more reasonable strategy would be to first set up a classical commutative algebra $\mathcal{A}$, say, of all smooth, i.e., $\mathcal{C}^\infty$, real valued functions $f(g)$ on G,

$$\begin{gathered} g \in G\rightarrow f(g) \in \mathcal{R}:f \in \mathcal{A}, \\ f_1,f_2 \in \mathcal{A}\Rightarrow c_1f_1+c_2f_2,f_1f_2 \in \mathcal{A}. \end{gathered} \tag{4.25}$$

Here the c's are real numbers, and the above choice of functions $f\in\mathcal{A}$ captures the differentiable manifold structure of G. The left and right translations L_g,R_g of Eq. (4.2) now act on $\mathcal{A}$ as follows:

$$\begin{gathered} \text{Left action,} \quad g'\in G:f(g)\rightarrow f(g'^{-1}g), \\ \text{Right action,} \quad g'\in G:f(g)\rightarrow f(gg'). \end{gathered} \tag{4.26}$$

Upon quantization we ask for an Abelian operator algebra $\hat{\mathcal{A}}$, say, consisting of Hermitian operators such that in a natural way we ensure

$$
\begin{aligned}
&f\in\mathcal{A}\rightarrow\hat{f}\in\hat{\mathcal{A}},\\
&f_1,f_2 \quad \in\mathcal{A}\Rightarrow c_1 f_1+c_2 f_2\rightarrow c_1\hat{f}_1+c_2\hat{f}_2,\\
&f_1 f_2\rightarrow\hat{f}_1\hat{f}_2.
\end{aligned}
\tag{4.27}
$$

This is the replacement for the $\hat{q}-\hat{q}$ part of the canonical relations (2.1), and is the quantized version of the PB relations $\{q^r,q^s\}=0$ in Eq. (4.24a), in a globally well-defined form.

Turning to the quantization of the remaining PB relations in Eqs. (4.24b) and (4.24c), we can work either with finite group elements or with infinitesimal generators. In the former, we ask for unitary operator families $V(g),\tilde{V}(g)$ realizing the left and right translation groups on G, and producing on $\hat{\mathcal{A}}$ the effects implied by Eq. (4.26),

$$
\begin{aligned}
&f(g)\in\mathcal{A}\rightarrow\hat{f}\in\hat{\mathcal{A}}\Rightarrow\\
&f(g'^{-1}g)\rightarrow V(g')\hat{f}V(g')^{-1},\\
&f(gg')\rightarrow\tilde{V}(g')\hat{f}\tilde{V}(g')^{-1},\ \ g'\in G,
\end{aligned}
\tag{4.28a}
$$

$$
\begin{aligned}
&V(g_1)V(g_2)=V(g_1g_2),\\
&\tilde{V}(g_1)\tilde{V}(g_2)=\tilde{V}(g_1g_2),
\end{aligned}
\tag{4.28b}
$$

$$
V(g_1)\tilde{V}(g_2)=\tilde{V}(g_2)V(g_1). \tag{4.28c}
$$

The operator relations (4.28a) are the quantized and finite forms of the PB relations in (4.24b) involving $\{q^r$ or $A(q),\ J_s$ or $\tilde{J}_s\}$; while the operator relations (4.28b) and (4.28c) are the integrated forms of the result of quantizing the PB relations (4.24c) keeping track of course of the global connectivity properties of G. The latter can also be expressed at the generator level. If the generators of $V(g),\tilde{V}(g)$ are $\hat{J}_r,\hat{\tilde{J}}_r$, respectively, we require them to be Hermitian and to obey

$$
\begin{aligned}
&[\hat{J}_r,\hat{J}_s]=if_{rs}{}^t\hat{J}_t,\\
&[\hat{\tilde{J}}_r,\hat{\tilde{J}}_s]=if_{rs}{}^t\hat{\tilde{J}}_t,\\
&[\hat{J}_r,\hat{\tilde{J}}_s]=0,\\
&\hat{\tilde{J}}_r=-\mathcal{D}_r^s(g)\hat{J}_s.
\end{aligned}
\tag{4.29}
$$

In comparison to the canonical commutation relations (2.1), we see that Eq. (4.28a) correspond to the $\hat{q}-\hat{p}$ part, and Eqs. (4.28b), (4.28c), and (4.29) correspond to the $\hat{p}-\hat{p}$ part, respectively. Thus the complete set of algebraic relations expressing quantum kinematics for quantum mechanics on a Lie group as configuration space are Eqs. (4.27), (4.28a)–(4.28c), (4.29). These have to be realized irreducibly on a suitable Hilbert space.

A clarifying remark may be made at this point. If we were looking only for a unitary representation (UR) or unitary irreducible representation (UIR) of G, the only commutation relations to be satisfied would be those among the Hermitian generators, $\hat{J}_r$ say, of such a UR or UIR. But these comprise only a part—the $\hat{p}-\hat{p}$ part—of the complete set of algebraic relations developed above; and do not include the operators in $\hat{\mathcal{A}}$ which represent smooth functions on G and which capture the notion of position operator in this case. Conversely, a single UIR of G on some Hilbert space, over which the $\hat{J}_r$ act irreducibly, is here the analogue of a single simultaneous (ideal) eigenvector of all the (commuting) momenta $\hat{p}_r$. The latter is always one dimensional because $\mathcal{R}^n$ is Abelian—there is just one (ideal) eigenvector $|\underline{p}\rangle$ of the $\hat{p}_r$ for given eigenvalues p_r. With a general non-Abelian Lie group G, the analogue of a ‘‘momentum eigenstate’’ is a (finite or infinite dimensional) UIR of G.

A natural representation of all the algebraic relations imposed above is via the regular representation of G. We will hereafter always assume that there is a unique (up to a factor) left and right translation invariant volume element $\mathrm{d}g$ on G, the Haar measure, which in the compact case will be normalized so that G has total volume unity,

$$\begin{gathered} f\in\mathcal{A}: \int_G \mathrm{d}g\, f(g) = \int_G \mathrm{d}g\, f(g'^{-1}g) = \int_G \mathrm{d}g\, f(gg'), \\ \int_G \mathrm{d}g = 1 \ \text{ if } \ G \ \text{ compact.} \end{gathered} \tag{4.30}$$

In local coordinates q^r for G, apart from a normalization factor, this volume element involves the determinants of the matrices $\xi(q)$, $\tilde{\xi}(q)$ defined in (4.15b),

$$\mathrm{d}g = \det(\xi(q))\mathrm{d}^n q = \det(\tilde{\xi}(q))\mathrm{d}^n q. \tag{4.31}$$

Then the Hilbert space $\mathcal{H}=L^2(G)$ is defined, in the ‘‘Schrödinger representation,’’ as

$$\mathcal{H} = \left\{ \psi(g)\in\mathcal{C} \middle| \|\psi\|^2 = \int_G \mathrm{d}g |\psi(g)|^2 < \infty \right\}. \tag{4.32}$$

On this space the required operators $\hat{f}\in\hat{\mathcal{A}}, V(g'), \tilde{V}(g')$ are easily defined,

$$\begin{gathered} f(g)\in\mathcal{A}\to\hat{f}\in\hat{\mathcal{A}}: \quad (\hat{f}\psi)(g) = f(g)\psi(g), \\ (V(g')\psi)(g) = \psi(g'^{-1}g), \\ (\tilde{V}(g')\psi)(g) = \psi(gg'). \end{gathered} \tag{4.33}$$

This is indeed an irreducible representation of the complete algebraic system as is shown in Appendix A.

In local coordinates if we write $\psi(g)$ as $\psi(q)$, the generators $\hat{J}_r, \hat{\tilde{J}}_r$ are immediately obtained as

$$\begin{gathered} \hat{J}_r = -i\,\eta_r^s(q)\frac{\partial}{\partial q^s}, \\ \hat{\tilde{J}}_r = i\,\tilde{\eta}_r^s(q)\frac{\partial}{\partial q^s}, \\ V(\delta q) \simeq 1 - i\,\delta q^r \hat{J}_r, \quad \tilde{V}(\delta q) \simeq 1 - i\,\delta q^r \hat{\tilde{J}}_r. \end{gathered} \tag{4.34}$$

Thus these generators are essentially the vector fields $X_r, \tilde{X}_r$ defined earlier in Eqs. (4.5) and (4.16), but now interpreted as Hermitian operators on $L^2(G)$.

It is for the elements $|\psi\rangle$ in the Hilbert space $\mathcal{H}$ of Eq. (4.32) that we wish to set up a Wigner distribution formalism with natural properties.

V. NEW FEATURES TO BE ACCOMMODATED

For Cartesian quantum mechanics we have the well-known Stone–von Neumann theorem which states that up to unitary equivalence there is only one irreducible representation of Hermitian operators $\hat{q}_r, \hat{p}_r$ obeying the Heisenberg relations (2.1). This irreducible representation is of course describable in many ways—position representation with $\hat{q}_r$ diagonal; momentum representation with $\hat{p}_r$ diagonal; Fock basis; coherent states, etc. When $\mathcal{R}^n$ here is replaced by a general Lie group G, we have already appreciated that the basic building block for the quantum theory is *not* a UIR of G, but an irreducible representation of the entire algebraic system consisting of $\hat{\mathcal{A}}, V(\cdot)$ and $\widetilde{V}(\cdot)$. A single UIR of G is too small to support the action of a group element as a generalized coordinate. To achieve this many UIR's of G have to be put together in a careful manner.

As for UIR's of G, we recall several familiar facts. If G is compact, every UIR is finite dimensional. If G is noncompact simple, then every nontrivial UIR is infinite dimensional; every finite-dimensional representation is nonunitary; and in addition there are infinite dimensional nonunitary representations.

We will for the most part and for definiteness consider the case of a compact simple Lie group G. A natural irreducible representation of the entire algebraic structure we are interested in is given, as we have seen, by the regular representation. The main features and auxiliary operators associated with it, and some notations, are given in Appendix A.

In passing we mention the fact that while for compact G every UIR is seen in the regular representation, in the noncompact case there are UIR's (the exceptional series) not contained in the regular representation.

One last general comment is important before proceeding. As we have seen in general the momenta of our problem are noncommuting operators. This is a genuine new feature absent in Cartesian quantum mechanics, and it has significant consequences. We have seen hints of this in the angle-angular momentum case in Sec. III, even though there was only one momentum $\hat{M}$ involved. For general G, the momenta $\hat{J}_r, \hat{\tilde{J}}_r$ cannot all be simultaneously diagonal and their spectra undergo quantization. Therefore the space of arguments of the Wigner distribution has to be carefully chosen; it is definitely not a function on the classical phase space T^*G in general. By the same token, there are in general no analogues to the groups $\mathrm{Sp}(2n,R), \mathrm{Mp}(2n)$ which are so important in the Cartesian case.

VI. THE WIGNER DISTRIBUTION IN THE REGULAR REPRESENTATION

Let $|\psi\rangle \in \mathcal{H} = L^2(G)$ be a normalized state vector. The corresponding "position space" probability density is a probability distribution on the group G given by [cf. Eq. (A5)],

$$|\psi(g)|^2 = |\langle g|\psi\rangle|^2. \tag{6.1}$$

The complementary "momentum space" probability distribution is (assuming G to be compact) a discrete set of probabilities indexed by the quantum numbers JMN and given by [cf. Eq. (A14)]

$$|\psi_{JMN}|^2 = |\langle JMN|\psi\rangle|^2. \tag{6.2}$$

The common normalization states that

$$\|\psi\|^2 = \int_G \mathrm{d}g\, |\psi(g)|^2 = \sum_{JMN} |\psi_{JMN}|^2 = 1. \tag{6.3}$$

At first glance we might suppose that, given $|\psi\rangle$, the corresponding Wigner distribution $W(\cdots)$ should be a real function with g and JMN (coordinates and quantized momenta) as arguments, bilinear in ψ (more precisely involving one ψ factor and one ψ^* factor), such that integration over g yields $|\psi_{JMN}|^2$ while summation over JMN yields $|\psi(g)|^2$. This would be a natural way in which the marginals (6.1) and (6.2) are reproduced. However, we should also require covariance under both (left and right) actions by G on ψ: the choice of the arguments in $W(\cdots)$ should allow for a natural linear transformation law under each of the changes $\psi(g) \rightarrow \psi(g_1^{-1}g)$ and $\psi(g) \rightarrow \psi(gg_2)$ in ψ. Now the momentum space amplitudes ψ_{JMN} of ψ transform linearly as follows [cf. Eq. (A14)]:

$$
\begin{aligned}
&|\psi'\rangle = V(g_1)|\psi\rangle, \quad \psi'(g) = \psi(g_1^{-1}g), \\
&\psi'_{JMN} = \sum_{M'} \mathcal{D}^J_{MM'}(g_1)^* \psi_{JM'N}, \\
&|\psi''\rangle = \tilde{V}(g_2)|\psi\rangle, \quad \psi''(g) = \psi(gg_2), \\
&\psi''_{JMN} = \sum_{N'} \mathcal{D}^J_{N'N}(g_2^{-1})^* \psi_{JMN'}.
\end{aligned}
\tag{6.4}
$$

Thus in each case there is a linear mixing of the components ψ_{JMN} for fixed J at the ψ level. Remembering that $W(\cdots)$ should involve the bilinear expressions $\psi\psi^*$, a little reflection shows that it would be too narrow to imagine that the Wigner distribution should be some real function $W(g;JMN)$: there would be too few momentum space arguments to support the changes (6.4) in ψ in a reasonable manner.

There is another way in which this situation could be described. As we have already pointed out in Sec. V, an essential new feature is that now the analogue of the single momentum eigenket $|\underline{p}\rangle$ in Cartesian quantum mechanics is a multidimensional object, an entire UIR of G; actually in the regular representation even more since both $\hat{J}$'s and $\hat{\tilde{J}}$'s have to be represented. In this sense, with a general Lie group G different from $\mathcal{R}^n$, there is a genuine asymmetry between positions and momenta. While the analogue of position eigenstate remains one dimensional, $|\underline{q}\rangle$ being replaced by $|g\rangle$, the momentum operators constitute the noncommutative algebra of $\hat{J}$'s and $\hat{\tilde{J}}$'s, leading to the quantum numbers JMN where only J remains fixed. (Incidentally the first part of this statement is not in conflict with the fact that G itself may be non-Abelian. In local coordinates q^r for G, the ideal ket $|g\rangle$ may be written as $|\underline{q}\rangle$, and all the q's are simultaneously diagonal.) Out of all the momentum operators, a complete commuting set consists of the (shared) Casimir operators formed out of the $\hat{J}$'s, and separately out of the $\hat{\tilde{J}}$'s, accounting for J in the set JMN; a maximal commuting subset of the $\hat{J}$'s, supplying some of the labels in M; a similar maximal commuting subset of the $\hat{\tilde{J}}$'s supplying the analogous labels in N; and further nonlinear mutually commuting expressions in $\hat{J}$'s (respectively, $\hat{\tilde{J}}$'s) to account for the remaining labels in M (respectively, N). The main point is that in the process of obtaining the marginal distribution (6.2) upon integrating the Wigner distribution with respect to its argument g, we should expect at first to get something like a density matrix within the Jth subspace of momentum labels, and then upon going to the diagonal elements recover the probabilities $|\psi_{JMN}|^2$. The transformation laws (6.4) can already be written in a matrix form (at the level of ψ, not of the density matrix) thus,

$$
\psi^{(J)} = (\psi_{JMN}) : |\psi\rangle \rightarrow V(g_1)\tilde{V}(g_2)|\psi\rangle \Rightarrow \psi^{(J)} \rightarrow \mathcal{D}^J(g_1)^* \psi^{(J)} \mathcal{D}^J(g_2^{-1})^*. \tag{6.5}
$$

Since the individual probabilities $|\psi_{JMN}|^2$ do not transform linearly among themselves under such G actions, but do bring in off-diagonal quantities, the structure of the Wigner distribution will inevitably reflect this fact.

Based on these considerations we now list the basic desired properties for the Wigner distribution $W(\cdots)$ associated with a given normalized $|\psi\rangle \in \mathcal{H}$ (for simplicity the dependence of the former on the latter is left implicit) initially as

$$\psi(g) \in \mathcal{H} \to W(g;JMN \quad M'N'),$$
$$W(g;JMN \quad M'N')^* = W(g;JM'N' \quad MN), \tag{6.6a}$$

$$\int_G \mathrm{d}g \; W(g;JMN \quad MN) = |\psi_{JMN}|^2,$$
$$\sum_{JMN} W(g;JMN \quad MN) = |\psi(g)|^2, \tag{6.6b}$$

$$\psi'(g) = \psi(g_1^{-1}g) \to W'(g;JMN \quad M'N')$$
$$= \sum_{M_1 M_1'} \mathcal{D}^J_{MM_1}(g_1) D^J_{M'M_1'}(g_1)^* W(g_1^{-1}g;JM_1N \quad M_1'N'), \tag{6.6c}$$

$$\psi''(g) = \psi(gg_2) \to W''(g;JMN \quad M'N')$$
$$= \sum_{N_1 N_1'} W(gg_2;JMN_1M'N_1') \mathcal{D}^J_{N_1N}(g_2^{-1}) \mathcal{D}^J_{N_1'N'}(g_2^{-1})^*. \tag{6.6d}$$

One can see that the covariance conditions (6.6c) and (6.6d) are compatible with the transformation laws (6.4) for ψ_{JMN} and the requirement (6.6b) for reproduction of the marginals. Actually one has little option but to extend the requirement in the first of Eq. (6.6b) to read

$$\int_G \mathrm{d}g \; W(g;JMN \quad M'N') = \psi^*_{JMN} \quad \psi_{JM'N'} \,. \tag{6.7}$$

Upon then setting $M' = M, N' = N$ here one recovers the true probabilities $|\psi_{JMN}|^2$. To all of the above we add a natural condition that W be of the general structure $\psi\psi^*$.

We now propose the following form for the Wigner distribution:

$$W(g;JMNM'N') = N_J \int_G \mathrm{d}g' \int_G \mathrm{d}g'' \; \delta(g^{-1}s(g',g'')) \mathcal{D}^J_{MN}(g')\psi(g')^* \mathcal{D}^J_{M'N'}(g'')^* \psi(g''). \tag{6.8}$$

This involves a group element $s(g',g'') \in G$ depending on two arguments also drawn from G, which must have suitable covariance and other properties. The set of conditions (6.6a)–(6.6d), and (6.7) now translates into a set of requirements on this function $s: G \times G \to G$ which are

$$g', g'' \in G \to s(g',g'') \in G,$$
$$s(g',g'') = s(g'',g'),$$
$$s(g',g') = g', \tag{6.9}$$
$$s(g_1 g' g_2^{-1}, g_1 g'' g_2^{-1}) = g_1 s(g',g'') g_2^{-1}.$$

Any choice of a function $s(g',g'')$ obeying these conditions leads to an acceptable definition of a Wigner distribution for quantum mechanics on a (compact) Lie group G.

The second and third lines of Eq. (6.9) suggest that we view $s(g',g'')$ as a kind of symmetric square root of the product of two (generally noncommuting) group elements $g', g'' \in G$. The covariance conditions in the last line help us simplify the problem to the choice of a suitable function $s_0(g')$ of a single argument drawn from G, obeying conditions that ensure (6.9),

$$
\begin{aligned}
s(e,g) &= s_0(g), \\
s(g',g'') &= g' s_0(g'^{-1} g''), \\
s_0(e) &= e, \\
s_0(g^{-1}) &= g^{-1} s_0(g), \\
s_0(g' g g'^{-1}) &= g' s_0(g) g'^{-1}.
\end{aligned} \tag{6.10}
$$

It is a consequence of these conditions on $s_0(g)$ that

$$
s_0(g) g = g s_0(g). \tag{6.11}
$$

We now present a solution to the above problem in the case of a compact simple Lie group G. Any such group carries a unique Riemannian metric defined in terms of the structure constants, and possessing left and right translation invariances. We shall content ourselves with a local coordinate description and use the notations of Eqs. (4.12)–(4.17). Admitting the over use of the letter g, at the identity the metric tensor has components

$$
g_{rs}(0) = -f_{ru}{}^{v} f_{sv}{}^{u}, \tag{6.12}
$$

the negative sign ensuring that the matrix $(g_{rs}(0))$ is positive definite. This tensor is checked to be invariant under the action by the adjoint representation

$$
\mathcal{D}^{u}_{r}(g) \mathcal{D}^{v}_{s}(g) g_{uv}(0) = g_{rs}(0). \tag{6.13}
$$

At a general point $g(q) \in G$ we obtain $g_{rs}(q)$ by shifting $g_{rs}(0)$ as a tensor to $g(q)$ using either left or right translation; on account of (6.13) the two results are the same and we find

$$
g_{rs}(q) = \xi^{u}{}_{r}(q) \xi^{v}{}_{s}(q) g_{uv}(0) = \tilde{\xi}^{u}{}_{r}(q) \tilde{\xi}^{v}{}_{s}(q) g_{uv}(0). \tag{6.14}
$$

Geodesics in G are curves of minimum length with respect to the above Riemannian metric. As is well known, both left and right translations, L_g and R_g, applied pointwise map geodesics onto geodesics. Thus if $g(q(\sigma)) \in G$ is a solution to the variational problem

$$
\delta \int_{\sigma_1}^{\sigma_2} \mathrm{d}\sigma \left(g_{rs}(q(\sigma)) \frac{\mathrm{d}q^{r}(\sigma)}{\mathrm{d}\sigma} \frac{\mathrm{d}q^{s}(\sigma)}{\mathrm{d}\sigma} \right)^{1/2} = 0, \tag{6.15}
$$

where we assume an affine parametrization is chosen so that

$$
g_{rs}(q(\sigma)) \frac{\mathrm{d}q^{r}(\sigma)}{\mathrm{d}\sigma} \frac{\mathrm{d}q^{s}(\sigma)}{\mathrm{d}\sigma} = \text{const}, \tag{6.16}
$$

then both $L_{g_1} g(q(\sigma))$ and $R_{g_2} g(q(\sigma))$ are solutions to the same variational problem.

We now use geodesics in G to construct the function $s_0(g)$. It is a fact that for almost all $g \in G$ (i.e., except for a set of measure zero), there is a unique geodesic [minimizing the functional appearing in Eq. (6.15)] running from the identity e to g. We assume the affine parametrization is normalized so that the geodesic passes through e at $\sigma = 0$ and through g at $\sigma = 1$,

$$
g \in G: g(q(0)) = e, \quad g(q(1)) = g. \tag{6.17}
$$

We then take $s_0(g)$ to be the half-way point reached at $\sigma = 1/2$,

$$
s_0(g) = g(q(1/2)). \tag{6.18}
$$

It is a matter of easy verification that all the conditions (6.10) are indeed obeyed: one has to exploit the natural covariance and other properties of general geodesics. With this we have solved the problem of defining Wigner distributions for quantum mechanics on a (compact) Lie group, possessing all the properties listed in Eqs. (6.6a)–(6.6d), and (6.7). The fact that $s_0(g)$ is defined everywhere except possibly on a set of vanishing measure causes no problems in carrying out integrations over G, or in recovering the marginals.

It may be pointed out that for a general pair of elements $g', g'' \in G$ (except in cases amounting to a set of vanishing measure) there is a unique geodesic running from g' to g'', normalized so that the affine parameter has values $\sigma=0$ and $\sigma=1$ at start and at finish. This geodesic is the result of applying $L_{g'}$ to the geodesic from e to $g'^{-1}g''$, or equally well of applying $R_{g'^{-1}}$ to the one from e to $g''g'^{-1}$. In either view, $s(g',g'')$ is the midpoint of this geodesic, reached at $\sigma=1/2$. Moreover, geodesics passing through the identity e are one-parameter subgroups in G. If we define $s_0(g)$ in Eq. (6.18) to be the square root of the element g, we can write the general quantity $s(g',g'')$ in these suggestive ways,

$$s(g',g'')=g'(g'^{-1}g'')^{1/2}=g''(g''^{-1}g')^{1/2}=(g''g'^{-1})^{1/2}g'=(g'g''^{-1})^{1/2}g''. \tag{6.19}$$

The definition (6.8) of the Wigner distribution associated with a pure state $\psi(g)$ generalizes to a mixed state with density operator $\hat{\rho}$,

$$W(g;JMN\quad M'N')=N_J\int_G \mathrm{d}g' \int_G \mathrm{d}g''\ \delta(g^{-1}s(g',g''))\langle g''|\hat{\rho}|g'\rangle \mathcal{D}^J_{MN}(g')\quad \mathcal{D}^J_{M'N'}(g'')^*,$$

$$\int \mathrm{d}g\ W(g;JMN\quad M'N')=\langle JM'N'|\hat{\rho}|JMN\rangle, \tag{6.20}$$

$$\sum_{JMN} W(g;JMN\quad MN)=\langle g|\hat{\rho}|g\rangle.$$

We now verify that $W(g;JMN\quad M'N')$ is a faithful representation of $\hat{\rho}$ in the sense that it contains complete information concerning $\hat{\rho}$. This will be shown by developing analogues to the previous Eqs. (2.13) and (3.6); in fact we will find two separate analogues.

The Wigner distribution in Eq. (6.20) transforms according to Eqs. (6.6c) and (6.6d) under independent left and right translations. By setting $N=N'$ and then summing over N, we obtain, using (A9), a slightly simpler function, $\widetilde{W}$ say, corresponding to the density operator $\hat{\rho}$,

$$\begin{aligned}\widetilde{W}(g;JMM')&=\sum_N W(g;JMN\quad M'N)\\&=N_J\int_G \mathrm{d}g' \int_G \mathrm{d}g''\ \delta(g^{-1}s(g',g''))\langle g''|\hat{\rho}|g'\rangle \mathcal{D}^J_{MM'}(g'g''^{-1})\quad .\end{aligned} \tag{6.21}$$

This auxiliary function is invariant under right translations except for a change of argument $g \to gg_2$, while under left translations it transforms in a manner similar to Eq. (6.6d). Now consider two density operators $\hat{\rho}_1$ and $\hat{\rho}_2$, with associated functions $\widetilde{W}_1$ and $\widetilde{W}_2$. It can then be shown that we can obtain $\mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2)$ from $\widetilde{W}_1$ and $\widetilde{W}_2$ by summing over all the arguments,

$$\sum_{JMM'} N_J^{-1}\int \mathrm{d}g\ \widetilde{W}_1(g;JMM')\widetilde{W}_2(g;JM'M)=\mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2). \tag{6.22}$$

The proof is presented in Appendix B. Since any density operator $\hat{\rho}_1$ is fully determined by the traces of its products with all other density operators $\hat{\rho}_2$, we can see that even the simpler function $\widetilde{W}(g;JMM')$ fully characterizes $\hat{\rho}$.

Obviously another analogue to Eqs. (2.13) and (3.6) can be obtained by interchanging the roles of left and right translations in the above. If in place of (6.21) we define

$$
\begin{aligned}
\tilde{\tilde{W}}(g;JNN') &= \sum_{M} W(g;JMN \quad MN') \\
&= N_J \int_G \mathrm{d}g' \int_G \mathrm{d}g'' \, \delta(g^{-1}s(g',g''))\langle g''|\hat{\rho}|g'\rangle \mathcal{D}^J_{N'N}(g''^{-1}g'), \qquad (6.23)
\end{aligned}
$$

then for two density operators $\hat{\rho}_1$, $\hat{\rho}_2$ we have

$$
\sum_{JNN'} N_J^{-1} \int \mathrm{d}g \, \tilde{\tilde{W}}_1(g;JNN')\tilde{\tilde{W}}_2(g;JN'N) = \mathrm{Tr}(\hat{\rho}_1\hat{\rho}_2). \qquad (6.24)
$$

The conclusion we can draw, in interesting contrast to the Cartesian and Abelian cases, is this. In order to be able to recover the marginal probability distributions $\langle g|\hat{\rho}|g\rangle$, $\langle JMN|\hat{\rho}|JMN\rangle$ in natural ways and also to have simple transformation behaviors under both left and right translations on G, we need to define the Wigner distribution as in Eqs. (6.8) and (6.20) with independent arguments $g\ J\ M\ N\ M'\ N'$. However, this object captures information contained in $\hat{\rho}$ in an over complete manner, since $\hat{\rho}$ is in fact completely determined in principle already by $\tilde{W}(g;JMM')$ [or $\tilde{\tilde{W}}(g;JNN')$]. All this is traceable to the fact that for non-Abelian G, the UIR's are in general multidimensional, so the concept of momentum eigenstate is also a multidimensional set of vectors.

VII. RECOVERY OF THE CARTESIAN AND ANGLE-ANGULAR MOMENTUM CASES, AND THE SU(2) CASE

We now indicate briefly how the known earlier results of Secs. II and III can be immediately recovered from the definitions of the preceding section. The expression (6.8) for the Wigner distribution $W(g;\ JMNM'N')$ uses the function $s(g',g'')$ depending symmetrically on the group elements g',g'', and is itself a group element obeying the conditions in (6.9). For the case of a compact simple Lie group G with nontrivial Cartan–Killing metric and associated geodesics, we have found a solution for $s(g',g'')$ in terms of the mid point rule. If however G is Abelian we can directly give an elementary solution for $s(g',g'')$ not using the geodesic construction at all.

For Cartesian quantum mechanics we have $G=\mathcal{R}^n$, which is Abelian. Consequently each UIR of G is one-dimensional and corresponds to a definite numerical momentum vector

$$
\underline{q} \in G \to e^{i\underline{q}\cdot\underline{p}/\hbar}, \underline{p} \in \mathcal{R}^n. \qquad (7.1)
$$

We can regard the continuous vector $\underline{p}\in\mathcal{R}^n$ (actually dual to G, the space of characters) as the analogue of the label J of the preceding section, and as each UIR is one dimensional there is no need and no room for the labels $M\ N\ M'\ N'$. If we present the usual definition (2.11) in the form

$$
\begin{aligned}
W(\underline{q},\underline{p}) &= (2\pi\hbar)^{-n} \int_{\mathcal{R}^n} \mathrm{d}^n q' \int_{\mathcal{R}^n} \mathrm{d}^n q'' \, \delta^{(n)}(\underline{q}-\underline{s}(\underline{q}',\underline{q}''))\langle \underline{q}''|\hat{\rho}|\underline{q}'\rangle \\
&\quad \times \exp(i\underline{q}'\cdot\underline{p}/\hbar)\exp(-i\underline{q}''\cdot\underline{p}/\hbar), \\
\underline{s}(\underline{q}',\underline{q}'') &= \tfrac{1}{2}(\underline{q}'+\underline{q}''),
\end{aligned} \qquad (7.2)
$$

we see that all the conditions (6.9) are indeed obeyed and this familiar case is seen to be a special case of our general construction.

The key point is that our construction of the Wigner distribution only depends on finding the group element $s(g',g'')$. We may use the geodesic construction if it is available, but can use any other method if a metric on G and geodesics are not available.

Turning to the compact case $G=\mathrm{SO}(2)$, this is again Abelian, so each UIR is one dimensional,

$$\theta\in G\rightarrow e^{im\theta},\quad m=0,\pm 1,\pm 2,\ldots . \tag{7.3}$$

We can now write the Wigner distribution (3.6) as

$$W(\theta,m)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\theta'\int_{-\pi}^{\pi}\mathrm{d}\theta''\ \delta(\theta-s(\theta',\theta''))\langle\theta''|\hat{\rho}|\theta'\rangle\exp(im\theta')\exp(-im\theta'')\ ,$$

$$s(\theta',\theta'')=\tfrac{1}{2}(\theta'+\theta'')\ \bmod 2\pi, \tag{7.4}$$

and again see that it falls into our general pattern.

Finally we present briefly the structure and some significant features of Wigner functions in the case $G=\mathrm{SU}(2)$, in a sense the simplest yet archetypal compact non-Abelian Lie group. Here the method of geodesics is essential for the construction. We recall very rapidly the basic definitions and notations concerning SU(2), emphasizing the four-dimensional geometric aspects available in this case

The defining representation of SU(2) is via 2×2 unitary unimodular matrices, which leads immediately to the identification of the group manifold with S^3, the real unit sphere in four-dimensional Euclidean space $\mathcal{R}^4$. We shall exploit this way of picturing SU(2). We denote group elements in the abstract by $a,b,a',b',\ldots$, these symbols also standing for points on S^3:

$$\begin{gathered} a=(a_\mu)\in S^3,\quad \mu=0,1,2,3,\\ a_\mu a_\mu=a_0^2+\underline{a}\cdot\underline{a}=1. \end{gathered} \tag{7.5}$$

The spatial part (a_1,a_2,a_3) of (a_μ) is denoted by $\underline{a}$. Inverses and products of group elements are denoted by a^{-1},ab, respectively. (The group element ab is to be carefully distinguished from the four vector inner product $a\cdot b$ which is a real number.) Then in the defining representation the matrix corresponding to $a\in\mathrm{SU}(2)$ is

$$\begin{gathered} u(a)=a_0\cdot I-i\ \ \underline{a}\cdot\underline{\sigma}=\begin{pmatrix}\lambda & \mu\\ -\mu^* & \lambda^*\end{pmatrix},\\ \lambda=a_0-ia_3,\quad \mu=-(a_2+ia_1). \end{gathered} \tag{7.6}$$

Here $\underline{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ are the Pauli matrices. The inverse arises by reversing the sign of $\underline{a}$,

$$u(a)^{-1}=u(a^{-1})=u(a_0,-\underline{a})=a_0\cdot I+i\underline{a}\cdot\underline{\sigma}. \tag{7.7}$$

The group multiplication law is subsumed in the description of left and right translations, each of which is realized by elements of SO(4):

$$u(a)u(b)=u(ab)=u(L(a)b),$$

$$L(a)=\begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3\\ a_1 & a_0 & -a_3 & a_2\\ a_2 & a_3 & a_0 & -a_1\\ a_3 & -a_2 & a_1 & a_0\end{pmatrix}\in\mathrm{SO}(4), \tag{7.8a}$$

$$u(b)u(a^{-1})=u(ba^{-1})=u(R(a)b),$$
$$R(a)=\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ -a_1 & a_0 & -a_3 & a_2 \\ -a_2 & a_3 & a_0 & -a_1 \\ -a_3 & -a_2 & a_1 & a_0 \end{pmatrix} \in \mathrm{SO}(4); \tag{7.8b}$$

$$L(a)L(b)=L(ab), \quad R(a)R(b)=R(ab),$$
$$L(a)R(b)=R(b)L(a). \tag{7.8c}$$

Each of these mutually commuting sets $\{L(a)\},\{R(a)\}$ faithfully represents SU(2) via SO(4) matrices. The only common elements correspond to $a=(\pm 1,\underline{0})$ leading to the two matrices $Z_2 =\{\pm I\}$ within SO(4). This leads to the familiar statement

$$\mathrm{SO}(4)=\mathrm{SU}(2)\times\mathrm{SU}(2)/Z_2. \tag{7.9}$$

We mention these details since the general covariance requirements in Eqs. (6.6c) and (6.6d) require them.

The relation to the Euler angles parametrization is given by

$$u(a)=e^{-(i/2)\,\alpha\sigma_3}e^{-(i/2)\,\beta\sigma_2}e^{-(i/2)\,\gamma\sigma_3},$$
$$a_0-ia_3=\cos\beta/2\;\; e^{-i(\alpha+\gamma)/2},$$
$$a_2+ia_1=\sin\beta/2\;\; e^{i(\gamma-\alpha)/2}, \tag{7.10}$$
$$0\leqslant\alpha\leqslant 2\pi, \quad 0\leqslant\beta\leqslant\pi, \quad 0\leqslant\gamma\leqslant 4\pi.$$

We can regard α, β, γ as angular coordinates over S^3, though because of the occurrence of half angles they are not quite the natural generalization of spherical polar angles from S^2 to S^3. The invariant line element $(\mathrm{d}s)^2$ on S^3, the invariant normalized volume element da on SU(2), and the element of solid angle $\mathrm{d}\Omega(a)$ on S^3 can all be easily worked out,

$$\begin{aligned}(\mathrm{d}s)^2&=\mathrm{d}a_\mu\,\mathrm{d}a_\mu=|\mathrm{d}(a_0-ia_3)|^2+|\mathrm{d}(a_2+ia_1)|^2\\ &=\tfrac{1}{4}((\mathrm{d}\alpha)^2+(\mathrm{d}\beta)^2+(\mathrm{d}\gamma)^2+2\cos\beta\;\;\mathrm{d}\alpha\;\;\mathrm{d}\gamma),\end{aligned} \tag{7.11a}$$

$$\mathrm{d}a=\frac{1}{2\pi^2}\mathrm{d}\Omega(a)=\frac{1}{16\pi^2}\mathrm{d}\alpha\sin\beta\,\mathrm{d}\beta\,\mathrm{d}\gamma. \tag{7.11b}$$

It is clear that the above line element on S^3 is the one induced from the Euclidean line element in $\mathcal{R}^4$, hence the corresponding geodesics are great circle arcs. Such arcs are carried into one another by both left and right SU(2) translations—the SO(4) invariance of $(\mathrm{d}s)^2$ along with Eq. (7.9) make this obvious. Therefore, if a,b are any two points of S^3 [any two elements of SU(2)] which are not diagonally opposite one another $[u(a)\neq -u(b)]$, the (shorter) geodesic connecting them is the affinely parametrized curve

$$a(\theta)=a\cos\theta+(b-a\;\;a\cdot b)\sin\theta/\sqrt{1-(a\cdot b)^2},$$
$$0\leqslant\theta\leqslant\theta_0=\cos^{-1}(a\cdot b)\in[0,\pi). \tag{7.12}$$

Along this geodesic we have $(\mathrm{d}s)^2=(\mathrm{d}\theta)^2$, and the midpoint is given by

$$a\left(\frac{\theta_0}{2}\right)=(a+b)/\sqrt{2(1+a\cdot b)} \tag{7.13}$$

which is geometrically obvious.

The Dirac delta function accompanying the volume element da on SU(2) may be written as $\delta(a,b)$ involving two group elements, or in a more compact form as $\delta(a^{-1}b)$. Its properties are summarized by

$$\int_{\mathrm{SU(2)}} \mathrm{d}b\ \delta(a,b)\quad f(b)\equiv\int_{S^3}\frac{\mathrm{d}\Omega(b)}{2\pi^2}\,\delta(a,b)\quad f(b)=f(a),$$

i.e.,

$$\int_{S^3}\ \mathrm{d}\Omega(b)\quad\delta(a,b)\quad f(b)=2\pi^2 f(a) \tag{7.14}$$

for suitable test functions $f(b)$. We can equally well regard $\delta(a,b)$ as a delta function on SU(2) or on S^3. In terms of Euler angles we have

$$\begin{gathered} a\rightarrow(\alpha,\beta,\gamma),\quad b\rightarrow(\alpha',\beta',\gamma'),\\ \delta(a,b)=16\pi^2\,\delta(\alpha'-\alpha)\,\delta(\beta'-\beta)\,\delta(\gamma'-\gamma)/\sin\beta. \end{gathered} \tag{7.15}$$

The last item in this resume concerns the matrices $D^j_{mm'}(a)$ representing SU(2) elements in the various UIR's. The ranges of the UIR label j and magnetic quantum numbers m,m' are, as usual, $j=0,1/2,1,3/2,\ldots,\quad m,m'=j,j-1,\ldots,-j$. Then with canonical basis vectors $|jm\rangle$ in the jth UIR and Hermitian generators J_1,J_2,J_3 we have from the quantum theory of angular momentum,[23]

$$D^j_{mm'}(a)=\langle jm|e^{-i\alpha J_3}e^{-i\beta J_2}e^{-i\gamma J_3}|jm'\rangle=e^{-im\alpha-im'\gamma}d^j_{mm'}(\beta),$$

$$\begin{aligned} d^j_{mm'}(\beta)&=\langle jm|e^{-i\beta J_2}|jm'\rangle\\ &=\sqrt{\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}}\left(\sin\frac{\beta}{2}\right)^{m'-m}\left(\cos\frac{\beta}{2}\right)^{m'+m}P^{(m'-m,m'+m)}_{j-m'}(\cos\beta), \end{aligned} \tag{7.16}$$

where the P's are the Jacobi polynomials. The orthogonality and completeness properties of these D-functions are

$$\begin{gathered} \int_{\mathrm{SU(2)}}\mathrm{d}a\,D^j_{mm'}(a)^*D^{j'}_{m''m'''}(a)=(2j+1)^{-1}\quad\delta_{jj'}\delta_{mm''}\delta_{m'm'''},\\ \sum_{j=0,1/2,1,\ldots}\ \sum_{m,m'=-j}^{j}(2j+1)D^j_{mm'}(a)\quad D^j_{mm'}(b)^*=\delta(a,b)=\delta(a^{-1}b). \end{gathered} \tag{7.17}$$

With these details in place we can proceed to the definition of the Wigner distribution. The Hilbert space of Schrödinger wave functions is

$$\mathcal{H}=L^2(\mathrm{SU(2)})=\left\{\psi(a)\in C\,\middle|\,a\in\mathrm{SU(2)},\|\psi\|^2=\int_{\mathrm{SU(2)}}\ \mathrm{d}a\quad|\psi(a)|^2<\infty\right\}. \tag{7.18}$$

Given $\psi\in\mathcal{H}$, the corresponding Wigner distribution is obtained by specializing Eqs. (6.8), (6.10), and (6.18) to this case and using Eq. (7.13) above,

$$W(a;jmn \;\; m'n') = \frac{(2j+1)}{4\pi^4} \int_{S^3} \mathrm{d}\Omega(a') \int_{S^3} \mathrm{d}\Omega(a'') \;\; \delta\left(a, \frac{a'+a''}{\sqrt{2(1+a'\cdot a'')}} \right)$$
$$\times D^j_{mn}(a')\psi(a')^* \;\; D^j_{m'n'}(a'')^* \;\; \psi(a''). \tag{7.19}$$

The occurrence of the midpoint of the geodesic from a' to a'' within the delta function is to be noted. We see immediately that the marginals are properly reproduced,

$$\int \mathrm{d}a \, W(a;jmn \;\; m'n') = \psi_{jm'n'}\psi^*_{jmn},$$

$$\psi_{jmn} = \frac{\sqrt{2j+1}}{2\pi^2} \int \mathrm{d}\Omega(a) \;\; D^j_{mn}(a)^* \;\; \psi(a), \tag{7.20a}$$

$$\sum_{jmn} W(a;jmn \;\; mn) = |\psi(a)|^2. \tag{7.20b}$$

Since the integrations involved in Eq. (7.19) are nontrivial, we limit ourselves to pointing out some qualitative features of the SU(2) Wigner distribution (7.19) which distinguish it from the Cartesian case as well as from earlier treatments of this problem.

(a) The appearance of all the arguments $a \;\; jmn \;\; m'n'$ in the Wigner distribution is essential to be able to satisfy the covariance laws (6.6c) and (6.6d) under independent left and right SU(2) translations, and to reproduce the configuration space and momentum space marginal probability distributions as in Eq. (7.20). In this respect the situation is markedly different from earlier approaches to the SU(2) Wigner distribution problem,[4] where attention was limited to states within some fixed (finite dimensional) UIR of SU(2) and the density matrix was expanded in the complete set of unit tensor operators within that UIR.

(b) If we consider as an idealized limit the case of $\psi(a)$ becoming a position eigenstate, the Wigner distribution simplifies as follows:

$$\begin{aligned} &\psi(a) \to \delta(a,a^{(0)}), \\ &W(a;jmn \;\; m'n') = \frac{(2j+1)}{4\pi^4} \delta(a,a^{(0)}) \;\; D^j_{mn}(a^{(0)}) \;\; D^j_{m'n'}(a^{(0)})^*. \end{aligned} \tag{7.21}$$

This retains a dependence on the momentum variables $jmn \;\; m'n'$. This is in contrast to the (one-dimensional) Cartesian case where from Eq. (2.10) we find

$$\psi(q) \to \delta(q-q_0), \quad W(q,p) = \frac{1}{h} \delta(q-q_0), \tag{7.22}$$

showing no p dependence.

(c) Similarly if we consider $\psi(a)$ to be a (normalized) linear combination of $D^{j_0}_{m_0 n_0}(a)$ over $m_0 \;\; n_0$ for some fixed j_0, the Wigner distribution has a nontrivial dependence on all its arguments, and *in particular is generally nonvanishing for* $j \neq j_0$. In the Cartesian case, in contrast, we have, similar to Eq. (7.22),

$$\psi(q) \to \frac{1}{h^{1/2}} e^{ip_0 q}, \quad W(q,p) = \frac{1}{h} \delta(p-p_0), \tag{7.23}$$

concentrated at $p = p_0$ and independent of q.

All these features can be attributed to the non-Abelian nature of SU(2).

VIII. CONCLUDING REMARKS

We have discussed the problem of setting up Wigner distributions for the states of a quantum system whose configuration space is a general non-Abelian Lie group G, and have given a complete solution for the case that G is compact. Many new features compared to the familiar Abelian case where $G=\mathcal{R}^n$ have appeared. For emphasis we repeat some of them here: while the classical phase space T^*G associated to G already brings in interesting structural aspects, in the quantum case the Wigner distribution is not a function defined on the classical T^*G. Instead it is a function of a classical unquantized group element $g\in G$ playing the role of coordinate variable, and quantized momenta consisting of labels $J\ M\ NM'\ N'$ associated with all the UIR's of G. The analogues of the familiar Heisenberg canonical commutation relations are now much more intricate, and the ideas of momentum eigenstates and momentum eigenvalues have to be understood with some care. While the distribution $W(g;\ JMN\ M'N')$ associated with a given $\hat{\rho}$ transforms nicely under left and right group actions, and reproduces the marginal probability distributions satisfactorily, it describes $\hat{\rho}$ in an over complete manner.

The points of view of the present work suggest that we also consider quantum systems whose covariance group is a given Lie group G, even if G is not the configuration space. These arise naturally if the configuration space is a coset space G/H, where H is some Lie subgroup of G. In that case there is only one (say left) action of G on G/H, rather than two independent mutually commuting actions. Action by G remains significant, and we would like to set up Wigner distributions for wave functions belonging to $L^2(G/H)$. Such UR's of G are typically much smaller than the regular representation.

Going beyond coset space representations, we have yet other physically interesting cases typified for example by the Schwinger oscillator representation of SU(2). Similar constructions are easily made for SU(3), etc.[24] These are not representations on spaces $L^2(G/H)$ for any choice of H; yet because of their use in various physical problems it is worthwhile to be able to set up Wigner distributions for them too.

We intend to examine some of these problems elsewhere.

APPENDIX A: THE REGULAR REPRESENTATION AND ASSOCIATED STRUCTURES

We assemble here some familiar facts concerning the regular representation of a compact Lie group, to settle notations and as preparation for setting up further operator structures. We know that the Lie group G under consideration possesses a left and right translation and inversion invariant volume element, dg say, so that the integral of a (complex valued) function $f(g)$ over G has the properties

$$\int_G \mathrm{d}g\, f(g)=\int_G \mathrm{d}g\ \ (f(g^{-1})\quad \text{or}\quad f(g'g)\quad \text{or}\quad f(gg')), \tag{A1}$$

where g' is any fixed element in G. For the compact case we normalize dg so that

$$\int_G \mathrm{d}g=1. \tag{A2}$$

With such a measure the carrier space for the unitary regular representation of G is the Hilbert space $\mathcal{H}=L^2(G)$ defined as in Eq. (4.32),

$$\mathcal{H}=\left\{\psi(g)\in\mathcal{C}\,\middle|\,\|\psi\|^2=\int_G \mathrm{d}g|\psi(g)|^2<\infty\right\}. \tag{A3}$$

A Dirac delta function can be defined with suitable invariance properties,

$$\int_G dg\, f(g)\quad \delta(g)=\int_G dg\, f(g)\quad \delta(g^{-1})=f(e),$$

$$\int_G dg\, f(g)\quad \delta(gg'^{-1}\quad \text{or}\quad g^{-1}g'\quad \text{or}\quad g'^{-1}g\quad \text{or}\quad g'g^{-1})=f(g'). \tag{A4}$$

We can introduce a convenient set of ideal basis vectors for $\mathcal{H}$ such that the wave function $\psi(g)$ is the overlap of $|\psi\rangle$ with one of these,

$$\psi(g)=\langle g|\psi\rangle,$$

$$\langle g'|g\rangle=\delta(g'g^{-1}),$$

$$\int_G dg|g\rangle\langle g|=1\ \text{ on }\ \mathcal{H}. \tag{A5}$$

The group G can be unitarily represented on $\mathcal{H}$ in two mutually commuting ways, by left or by right translations. We denote the corresponding operators by $V(g),\tilde{V}(g)$ and define them by

$$V(g')|g\rangle=|g'g\rangle,$$

$$\tilde{V}(g')|g\rangle=|gg'^{-1}\rangle. \tag{A6}$$

Both of them are unitary and obey the composition and commutation relations

$$V(g_2)V(g_1)=V(g_2g_1),$$

$$\tilde{V}(g_2)\tilde{V}(g_1)=\tilde{V}(g_2g_1), \tag{A7}$$

$$V(g_1)\tilde{V}(g_2)=\tilde{V}(g_2)V(g_1).$$

On wave functions the effects are as given in Eq. (4.33),

$$(V(g')\psi)(g)=\psi(g'^{-1}g),$$

$$(\tilde{V}(g')\psi)(g)=\psi(gg'). \tag{A8}$$

These are infinite dimensional reducible UR's of G; and in the compact case, according to the Peter–Weyl theorem, each of them contains every UIR of G as often as its dimension. Motivated by the notations in the case of SU(2), we shall use symbols $J,J',J_1,J_2,\ldots$ to label the various UIR's of G (some of which may not be faithful); so in fact J stands for several independent discrete or quantized labels, as many as the rank of G. Within the Jth UIR, in some chosen orthonormal basis, we label rows and columns by indices $M,N,M'N',\ldots$. Once again each of these stands for a collection of discrete labels: for instance the eigenvalues of as many commuting generators as the rank of G, plus further eigenvalues of chosen commuting nonlinear polynomials in the generators. In the Jth UIR, we write $\mathcal{D}^J_{MN}(g)$ for the unitary representation matrices. These obey composition, orthogonality and completeness relations:

$$\sum_{M'} \mathcal{D}^J_{MM'}(g')\mathcal{D}^J_{M'N}(g)=\mathcal{D}^J_{MN}(g'g), \tag{A9}$$

$$\int_G dg\, \mathcal{D}^{J'}_{M'N'}(g)^*\mathcal{D}^J_{MN}(g)=\delta_{J'J}\delta_{M'M}\delta_{N'N}/N_J, \tag{A10}$$

$$\sum_{JMN} N_J \mathcal{D}^J_{MN}(g)\mathcal{D}^J_{MN}(g')^* = \delta(g^{-1}g'). \tag{A11}$$

Here N_J is the dimension of the Jth UIR. With the help of these matrices we can introduce another orthonormal basis for $\mathcal{H}$ which explicitly accomplishes the simultaneous reduction of both UR's $V(\cdot), \tilde{V}(\cdot)$ into irreducibles. These basis vectors and their main properties are

$$\begin{gathered}
|JMN\rangle = N_J^{1/2}\int_G \mathrm{d}g\, \mathcal{D}^J_{MN}(g)|g\rangle, \\
\langle g|JMN\rangle = N_J^{1/2}\mathcal{D}^J_{MN}(g), \\
\langle J'M'N'|JMN\rangle = \delta_{J'J}\delta_{M'M}\delta_{N'N}, \\
\sum_{JMN} |JMN\rangle\langle JMN| = 1 \text{ on } \mathcal{H}.
\end{gathered} \tag{A12}$$

Under action by $V(\cdot), \tilde{V}(\cdot)$ they transform among themselves conserving J,

$$\begin{gathered}
V(g)|JMN\rangle = \sum_{M'} \mathcal{D}^J_{MM'}(g^{-1})|JM'N\rangle, \\
\tilde{V}(g)|JMN\rangle = \sum_{N'} \mathcal{D}^J_{N'N}(g)|JMN'\rangle.
\end{gathered} \tag{A13}$$

Therefore in $|JMN\rangle$ the index N counts the multiplicity of occurrence of the Jth UIR in the reduction of $V(\cdot)$, and the index M performs a similar function in the reduction of $\tilde{V}(\cdot)$. A general $|\psi\rangle$ can be expanded in either basis and we have

$$\begin{gathered}
|\psi\rangle = \int_G \mathrm{d}g\, \psi(g)|g\rangle = \sum_{JMN} \psi_{JMN}|JMN\rangle, \\
\psi_{JMN} = \langle JMN|\psi\rangle = N_J^{1/2}\int_G \mathrm{d}g\, \mathcal{D}^J_{MN}(g)^*\psi(g), \\
\|\psi\|^2 = \int_G \mathrm{d}g\, |\psi(g)|^2 = \sum_{JMN} |\psi_{JMN}|^2.
\end{gathered} \tag{A14}$$

Towards getting projections onto individual vectors $|JMN\rangle$ we set up the Fourier components of $V(\cdot)$ and $\tilde{V}(\cdot)$ as follows:

$$\begin{gathered}
P_{JMN} = N_J\int_G \mathrm{d}g\, \mathcal{D}^J_{MN}(g)V(g), \\
\tilde{P}_{JMN} = N_J\int_G \mathrm{d}g\, \mathcal{D}^J_{MN}(g^{-1})\tilde{V}(g).
\end{gathered} \tag{A15}$$

With respect to Hermitian conjugation the indices get interchanged,

$$P^\dagger_{JMN} = P_{JNM}, \quad \tilde{P}^\dagger_{JMN} = \tilde{P}_{JNM}, \tag{A16}$$

and their composition and multiplication laws are

$$P_{J'M'N'}P_{JMN}=\delta_{J'J}\delta_{N'M}P_{JM'N}\,,$$
$$\tilde{P}_{J'M'N'}\tilde{P}_{JMN}=\delta_{J'J}\delta_{M'N}\tilde{P}_{JMN'}\,, \tag{A17}$$
$$P_{JMN}\tilde{P}_{J'M'N'}=\tilde{P}_{J'M'N'}P_{JMN}\,.$$

Their actions on the two complementary bases for $\mathcal{H}$ are immediate,

$$P_{J'M'N'}|JMN\rangle=\delta_{J'J}\delta_{N'M}|JM'N\rangle,$$
$$\tilde{P}_{J'M'N'}|JMN\rangle=\delta_{J'J}\delta_{M'N}|JMN'\rangle,$$
$$P_{JMN}|g\rangle=N_J^{1/2}\sum_{N'}\mathcal{D}^J_{N'N}(g^{-1})|JMN'\rangle, \tag{A18}$$
$$\tilde{P}_{JMN}|g\rangle=N_J^{1/2}\sum_{M'}\mathcal{D}^J_{MM'}(g^{-1})|JM'N\rangle.$$

Therefore the projections onto $|JMN\rangle$ are

$$|JMN\rangle\langle JMN|=P_{JMM}\tilde{P}_{JNN}\,, \tag{A19}$$

and we have the completeness identities

$$\sum_M P_{JMM}=\sum_M \tilde{P}_{JMM}\,,$$
$$\sum_{JM} P_{JMM}=\sum_{JM}\tilde{P}_{JMM}=1 \text{ on } \mathcal{H}. \tag{A20}$$

Now we proceed to a construction of certain operators directly relevant to the Wigner distribution problem. Here we will be guided by analogy to what is done for the (one degree of freedom) $\hat{q}-\hat{p}$ pair and the $\hat{\theta}-\hat{M}$ pair, as recounted in Secs. II and III. In these cases we know that the unitary Weyl exponentials $U(\sigma)=\exp(i\sigma\hat{q})$, $V(\tau)=\exp(-i\tau\hat{p})$ and $U(n)=\exp(in\hat{\theta})$, $V(\tau)=\exp(-i\tau\hat{M})$ play important roles. It is seen that it is natural here to regard $\sigma(n)$ as a typical eigenvalue of $\hat{p}(\hat{M})$ and τ[in $\mathcal{R}$ or in $(-\pi,\pi)$] as a typical eigenvalue of $\hat{q}(\hat{\theta})$. Now the operator $V(\tau)$ has been generalized in the Lie group situation to the *two* families $V(g),\tilde{V}(g)$. These are indeed exponentials of the "momentum operators:" if the group element g is expressed as the exponential of an element in $\underset{\sim}{G}$, then $V(g)$ and $\tilde{V}(g)$ are corresponding exponentials in their generators (4.34) obeying (4.29):

$$g=\exp(\tau^r\ \ e_r):V(g)=\exp(-i\ \ \tau^r\hat{J}_r),$$
$$\tilde{V}(g)=\exp(-i\ \ \tau^r\hat{\tilde{J}}_r). \tag{A21}$$

With τ^r as coordinates for g, these are precisely exponentials in momenta. To generalize $U(\sigma),U(n)$ we recall on the other hand that now a typical "momentum eigenvalue" is the collection of quantum numbers JMN associated with a subspace of $\mathcal{H}$ supporting a UIR of the $\hat{J}_r,\hat{\tilde{J}}_r$. This suggests that the generalization of $U(\sigma),U(n)$ must be an operator diagonal in the "coordinate" or $|g\rangle$ basis, and labeled by JMN: it must be a function of the coordinates alone. Based on this reasoning, we define operators U_{JMN} by

$$
\begin{aligned}
U_{JMN}|g\rangle &= \mathcal{D}^{J}_{MN}(g)|g\rangle, \\
\langle g|U_{JMN} &= \mathcal{D}^{J}_{MN}(g)\langle g|.
\end{aligned}
\tag{A22}
$$

Their adjoints are also diagonal in this basis,

$$
\begin{aligned}
U^{\dagger}_{JMN}|g\rangle &= \mathcal{D}^{J}_{MN}(g)^{*}|g\rangle, \\
\langle g|U^{\dagger}_{JMN} &= \mathcal{D}^{J}_{MN}(g)^{*}\langle g|,
\end{aligned}
\tag{A23}
$$

and unitarity is expressed in a matrix sense,

$$
\sum_{M} U^{\dagger}_{JMN}U_{JMN'} = \sum_{M} U^{\dagger}_{JNM}U_{JN'M} = \delta_{N'N}\cdot 1 \quad \text{on} \quad \mathcal{H}. \tag{A24}
$$

Being simultaneously diagonal, the commutators vanish,

$$
[U_{JMN}, U_{J'M'N'}] = [U_{JMN}, U^{\dagger}_{J'M'N'}] = 0. \tag{A25}
$$

Completeness of the $\mathcal{D}$-functions $\mathcal{D}^{J}_{MN}(g)$ as expressed in Eq. (A11) now means that the operators $\{U_{JMN}\}$ form a (linear) basis for the commutative algebra $\hat{\mathcal{A}}$. In fact the map (4.27) from the classical algebra $\mathcal{A}$ to the quantized $\hat{\mathcal{A}}$ can be made explicit,

$$
\begin{aligned}
& f\in\mathcal{A}: f(g) = \sum_{JMN} f_{JMN}\mathcal{D}^{J}_{MN}(g) \rightarrow \\
& \hat{f} = \sum_{JMN} f_{JMN}U_{JMN} \in \hat{\mathcal{A}}.
\end{aligned}
\tag{A26}
$$

The relations connecting $\{U_{JMN}\}$ to $V(\cdot), \tilde{V}(\cdot)$ are easily worked out,

$$
\begin{aligned}
V(g)U_{JMN}V(g)^{-1} &= \sum_{M'} \mathcal{D}^{J}_{MM'}(g^{-1})U_{JM'N}, \\
\tilde{V}(g)U_{JMN}\tilde{V}(g)^{-1} &= \sum_{N'} \mathcal{D}^{J}_{N'N}(g)U_{JMN'}.
\end{aligned}
\tag{A27}
$$

What remains are expressions for the product of two U's, and the action of a U on $|JMN\rangle$. For both these, the Clebsch–Gordan coefficients for G have to be brought in.

Let the reduction of the direct product of the Jth and J'th UIR's of G contain various UIR's J'' with various multiplicities. This means that we have a family of Clebsch–Gordan coefficients carrying three sets of $J-M$ labels and in addition a multiplicity index λ, say; and they obey two sets of unitarity conditions,

$$
\begin{aligned}
\sum_{M,M'} C^{JJ'J''\lambda *}_{MM'M''}C^{JJ'J'''\lambda'}_{MM'M'''} &= \delta_{J''J'''}\delta_{\lambda\lambda'}\delta_{M''M'''}, \\
\sum_{J''\lambda M''} C^{JJ'J''\lambda *}_{MM'M''}C^{JJ'J''\lambda}_{NN'M''} &= \delta_{MN}\delta_{M'N'}.
\end{aligned}
\tag{A28}
$$

Using these coefficients the product of two $\mathcal{D}$-functions decomposes into a sum

$$\mathcal{D}^{J}_{MN}(g)\mathcal{D}^{J'}_{M'N'}(g)=\sum_{J''\lambda M''N''} C^{JJ'J''\lambda *}_{MM'M''}C^{JJ'J''\lambda}_{NN'N''}\mathcal{D}^{J''}_{M''N''}(g). \tag{A29}$$

In all these relations the multiplicity index λ accompanying the ''final'' UIR J'' runs over as many values as the number of times J'' occurs in the product of J and J'; and at each stage we have manifest unitary invariance under changes in the choice of λ's. Combining Eq. (A29) in turn with Eqs. (A12) and (A22) we immediately get the results for the products of two U_{JMN}'s and the action of a U_{JMN} on a state $|J'M'N'\rangle$,

$$U_{JMN}U_{J'M'N'}=\sum_{J''\lambda M''N''} C^{JJ'J''\lambda *}_{MM'M''}C^{JJ'J''\lambda}_{NN'N''}U_{J''M''N''},$$

$$U_{JMN}|J'M'N'\rangle=\sum_{J''\lambda M''N''}\sqrt{\frac{N_{J'}}{N_{J''}}}C^{JJ'J''\lambda *}_{MM'M''}C^{JJ'J''\lambda}_{NN'N''}|J''M''N''\rangle. \tag{A30}$$

The unitary invariance with respect to λ is manifest.

Thus we have expressions (A6) and (A22) for the actions of $U,\dots,V(\cdot),\widetilde{V}(\cdot)$ on $|g\rangle$, and expressions (A30) and (A13) for their actions on $|JMN\rangle$.

Last we consider the question of setting up in a natural way a complete trace orthonormal set of operators on $\mathcal{H}=L^2(G)$, involving the U's, V's, and $\widetilde{V}$'s in a ''symmetrical'' manner. In the Cartesian case the phase space displacement operators

$$e^{i(\sigma\hat{q}-\tau\hat{p})}=e^{i\sigma\hat{q}}e^{-i\tau\hat{p}}e^{-i\sigma\tau/2}=e^{-i\tau\hat{p}}e^{i\sigma\hat{q}}e^{i\sigma\tau/2} \tag{A31}$$

give us such a system, and they are basic to the Weyl correspondence. Already in the $\hat{\theta}-\hat{M}$ case we know from Eqs. (3.5a) and (3.5b) that we have to work with the operators

$$e^{in\hat{\theta}}e^{-i\tau\hat{M}}e^{-in\tau/2}=e^{-i\tau\hat{M}}e^{in\hat{\theta}}e^{in\tau/2}, \tag{A32}$$

which are again complete and trace orthonormal, but we can no longer write these as single exponentials. In the case of general G, this latter trend continues. Generalizing from the known examples, we now define a family of operators labeled by $g\in G$ together with JMN, as follows:

$$\hat{\mathcal{D}}(g;JMN)=V(g)U_{JMN}=\sum_{M'}\mathcal{D}^{J}_{MM'}(g^{-1})U_{JM'N}V(g). \tag{A33}$$

It is easy to show trace orthogonality, using Eqs. (A6) and (A22),

$$\begin{aligned}\mathrm{Tr}(\hat{\mathcal{D}}(g';J'M'N')^{\dagger}\hat{\mathcal{D}}(g;JMN))&=\int_G \mathrm{d}g''\langle g''|U^{\dagger}_{J'M'N'}V(g')^{-1}V(g)U_{JMN}|g''\rangle\\&=\int_G \mathrm{d}g''\,\mathcal{D}^{J'}_{M'N'}(g'')^{*}\mathcal{D}^{J}_{MN}(g'')\langle g''|V(g'^{-1}g)|g''\rangle\\&=\delta(g'^{-1}g)\,\delta_{J'J}\delta_{M'M}\delta_{N'N}/N_J.\end{aligned} \tag{A34}$$

As for completeness we begin with

$$\hat{\mathcal{D}}(g;JMN)|g'\rangle=\mathcal{D}^{J}_{MN}(g')|gg'\rangle, \tag{A35}$$

multiply both sides by $N_J\mathcal{D}^{J}_{MN}(g'')^{*}$, sum on JMN and use Eq. (A11) to get

$$\sum_{JMN} N_J \mathcal{D}^J_{MN}(g'')^* \hat{\mathcal{D}}(g;JMN)|g'\rangle = \delta(g''^{-1}g')|gg'\rangle = |gg''\rangle\langle g''|g'\rangle. \tag{A36}$$

Peeling off $|g'\rangle$ and then replacing gg'' by g' we get

$$|g'\rangle\langle g''| = \sum_{JMN} N_J \mathcal{D}^J_{MN}(g'')^* \hat{\mathcal{D}}(g'g''^{-1};JMN). \tag{A37}$$

This shows, albeit in a somewhat formal manner, that any operator on $\mathcal{H}$ can be linearly expanded in the set $\hat{\mathcal{D}}(g;JMN)$. If in Eq. (A33) we use $\widetilde{V}(\cdot)$ in place of $V(\cdot)$ we get the alternative results,

$$\hat{\tilde{\mathcal{D}}}(g;JMN) = \widetilde{V}(g)U_{JMN} = \sum_{N'} \mathcal{D}^J_{N'N}(g)U_{JMN'}\widetilde{V}(g), \tag{A38}$$

$$\mathrm{Tr}(\hat{\tilde{\mathcal{D}}}(g';J'M'N')^\dagger \hat{\tilde{\mathcal{D}}}(g;JMN)) = \delta(g'^{-1}g) \quad \delta_{J'J}\delta_{M'M}\delta_{N'N}/N_J, \tag{A39}$$

$$|g'\rangle\langle g''| = \sum_{JMN} N_J \mathcal{D}^J_{MN}(g'')^* \hat{\tilde{\mathcal{D}}}(g''g'^{-1};JMN). \tag{A40}$$

One can ask whether similar completeness statements can be developed for outer products of vectors of the form $|JMN\rangle\langle J'M'N'|$. This is indeed possible, but the expressions are somewhat unwieldy and involve the Clebsch–Gordan coefficients explicity, so we omit them.

The results (A38)–(A40) prove that the representation of $\hat{\mathcal{A}}, V(\cdot)$ and $\widetilde{V}(\cdot)$ on $\mathcal{H} = L^2(G)$ is irreducible, since any operator on $\mathcal{H}$ is expressible as a linear combination of the operators $\hat{\mathcal{D}}(g;JMN)$ [or $\hat{\tilde{\mathcal{D}}}(g;JMN)$].

APPENDIX B

Here we briefly outline the proofs for Eqs. (6.22) and (6.24) and also derive some useful relations similar in form to those known in the Cartesian and angle-angular momentum cases.

To prove (6.22), consider its left-hand side (LHS):

$$\sum_{JMM'} N_J^{-1} \int \mathrm{d}g \widetilde{W}_1(g;JMM')\widetilde{W}_2(g;JM'M). \tag{B1}$$

On substituting for $\widetilde{W}$ using (6.21) and carrying out the summation over JMM' using (A11), this expression becomes

$$\int \mathrm{d}g \int \mathrm{d}g_1' \int \mathrm{d}g_1'' \int \mathrm{d}g_2' \int \mathrm{d}g_2'' \langle g_1''|\hat{\rho}_1|g_1'\rangle \\ \times \langle g_2''|\hat{\rho}_2|g_2'\rangle \delta(g^{-1}s(g_1',g_1'')) \quad \delta(g^{-1}s(g_2',g_2'')) \quad \delta(g_1''g_1'^{-1}g_2''g_2'^{-1}). \tag{B2}$$

Using the fact that $\delta(gg') = \delta(g'g)$, the third delta function in the integrand can be written as $\delta(g_1'^{-1}g_2''g_2'^{-1}g_1'')$ or as $\delta((g_2''^{-1}g_1')^{-1}g_2'^{-1}g_1'')$ which in turn implies that the integral vanishes unless $g_1'' = g_2' \cdot h; g_1' = g_2'' \cdot h$, $h \in G$. This, together with the other two delta functions implies that $h = e$. The three delta functions above are therefore equivalent to $\delta(g^{-1}s(g_1',g_1''))$ $\delta(g_1'^{-1}g_2'')$ $\delta(g_1''^{-1}g_2')$. On carrying out the integrals in (B2) with the help of these delta functions one obtains the right-hand side (RHS) of (6.22).

A similar line of argument can be used to establish the relation (6.24). Next we show that, in analogy with the Cartesian and angle-angular momentum cases, the Wigner distribution in (6.20) corresponding to a density operator $\hat{\rho}$ can be written in the following compact form:

$$W(g;JMN\ \ M'N')=\mathrm{Tr}[\hat{\rho}\hat{W}(g;JMNM'N')], \tag{B3}$$

where the Wigner operator $\hat{W}(g;JMNM'N')$ can be expressed in terms of operators related to $\hat{\mathcal{D}}(g;JMN)$ as follows:

$$\hat{W}(g;JMNM'N')=N_J\hat{\mathcal{D}}_1(g;JMN)\ \ \Delta\ \ \hat{\mathcal{D}}_1^{\dagger}(g;JM'N'). \tag{B4}$$

Here

$$\hat{\mathcal{D}}_1(g;JMN)=U_{JMN}V(g) \tag{B5}$$

$$=\sum_{M'}\ \mathcal{D}^J_{MM'}(g)\hat{\mathcal{D}}(g;JM'N), \tag{B6}$$

$$\Delta=\int\ \mathrm{d}g\sum_{JMN}\ N_J\mathcal{D}^J_{MN}(e)^*\hat{\mathcal{D}}_0(g;JMN), \tag{B7}$$

$$\hat{\mathcal{D}}_0(g;JMN)=\sum_{M'}\ \mathcal{D}^J_{MM'}(s_0(g))\hat{\mathcal{D}}(g;JM'N) \tag{B8}$$

$$=\sum_{M'}\ \mathcal{D}^J_{MM'}(s_0(g^{-1}))\hat{\mathcal{D}}_1(g;JM'N). \tag{B9}$$

Note that the operator $\hat{\mathcal{D}}_0(g;JMN)$ introduced here can be regarded as the analogue of $e^{ip\hat{q}-iq\hat{p}}\equiv e^{-iq\hat{p}}e^{ip\hat{q}}e^{ipq/2}$ or of $e^{-i\tau\hat{M}}e^{in\hat{\theta}}e^{in\tau/2}$ in the angle-angular momentum case.

To show (B3), we note that the RHS of (B3) can be written as

$$\mathrm{Tr}[\hat{\rho}\hat{W}(g;JMNM'N')]=\int\ \mathrm{d}g_1\int\ \mathrm{d}g_2\langle g_2|\hat{\rho}|g_1\rangle\langle g_1|\hat{W}(g;JMNM'N')|g_2\rangle. \tag{B10}$$

Now

$$\langle g_1|\hat{W}(g;JMNM'N')|g_2\rangle=N_J\int\ \mathrm{d}g_3\int\ \mathrm{d}g_4\langle g_1|\hat{\mathcal{D}}_1(g;JMN)|g_3\rangle\langle g_3|\Delta|g_4\rangle$$
$$\times\langle g_4|\hat{\mathcal{D}}_1(g;JMN)|g_2\rangle, \tag{B11}$$

and from the definitions (B5)–(B9) of the operators that occur here it can easily be shown that

$$\langle g_1|\hat{\mathcal{D}}_1(g;JMN)|g_2\rangle=\mathcal{D}^J_{MN}(g_1)\,\delta(g_1(gg_2)^{-1}), \tag{B12}$$

$$\langle g_1|\hat{\mathcal{D}}_0(g;JMN)|g_2\rangle=\mathcal{D}^J_{MN}(s(g_1,g_2))\,\delta(g_1(gg_2)^{-1}), \tag{B13}$$

$$\langle g_1|\Delta|g_2\rangle=\int\ \mathrm{d}g\sum_{JMN}\ N_J\mathcal{D}^J_{MN}(e)^*\langle g_1|\hat{\mathcal{D}}_0(g;JMN)|g_2\rangle$$

$$=\int\ \mathrm{d}g\sum_{JMN}\ N_J\mathcal{D}^J_{MN}(e)^*\mathcal{D}^J_{MN}(s(g_1,g_2))\,\delta(g_1(gg_2)^{-1})$$

$$=\delta(s(g_1,g_2)). \tag{B14}$$

Using these in (B11) one obtains

$$\langle g_1|\hat{W}(g;JMN,M'N')|g_2\rangle = N_J\delta(g^{-1}s(g_1,g_2))\mathcal{D}^J_{MN}(g_1)\mathcal{D}^J_{M'N'}(g_2)^*, \tag{B15}$$

which when substituted in (B10) yields (B3).

On setting $N=N'$ $(M=M')$ in (B3) and summing over N (M) we obtain the following formulas for the simpler Wigner distributions in terms of simpler Wigner operators:

$$\tilde{W}(g;JMM') = \mathrm{Tr}[\hat{\rho} \quad \hat{\tilde{\tilde{W}}}(g;JMM'), \tag{B16}$$

$$\tilde{\tilde{W}}(g;JNN') = \mathrm{Tr}[\rho \quad \hat{\tilde{\tilde{W}}}(g;JNN')], \tag{B17}$$

where

$$\hat{\tilde{W}}(g;JMM') = \sum_N \hat{W}(g;JMNM'N), \tag{B18}$$

$$\hat{\tilde{\tilde{W}}}(g;JNN') = \sum_M \hat{W}(g;JMNMN'). \tag{B19}$$

The relations (B16) and (B17) can be inverted with the help of (6.22) and (6.24), respectively, to obtain

$$\hat{\rho} = \int \mathrm{d}g \sum_{JMM'} \frac{1}{N_J}\tilde{W}(g;JMM')\hat{\tilde{W}}(g;JM'M), \tag{B20}$$

$$\hat{\rho} = \int \mathrm{d}g \sum_{JNN'} \frac{1}{N_J}\tilde{\tilde{W}}(g;JNN')\hat{\tilde{\tilde{W}}}(g;JN'N). \tag{B21}$$

This can be seen as follows. Setting $\hat{\rho}_1 \equiv \hat{\rho}$ and $\hat{\rho}_2 = |g_2\rangle\langle g_1|$ in (6.22) and using (B16) for the second Wigner distribution one obtains

$$\langle g_1|\hat{\rho}|g_2\rangle = \int \mathrm{d}g \sum_{JMM'} \frac{1}{N_J}\tilde{W}(g;JMM')\langle g_1|\hat{\tilde{W}}(g;JM'M)|g_2\rangle, \tag{B22}$$

which on peeling off $\langle g_1|$ and $|g_2\rangle$ gives (B20). Equation (B21) can be derived in a similar fashion.

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