Performance Analysis of Coded Communication Systems on Nakagami Fading Channels With Selection Combining Diversity

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***Abstract*—In this paper, we develop analytical tools for the performance analysis of coded, coherent communication systems on independent and identically distributed Nakagami-m fading channels with selection combining (SC) diversity. First, we derive an *exact* expression for the moment generation function (MGF) of the signal-to-noise ratio (SNR) of a code symbol at the output of the selection combiner. Next, based on Gauss–Chebyshev quadrature and Gauss–Laguerre quadrature rules, we propose a simple to compute, yet accurate, numerical solution for the pairwise error probability (PEP) of coded M-phase-shift keying (PSK) signals. Using the PEP expressions, we present the union bound-based bit-error performance of trellis-coded modulation schemes and turbo codes. Finally, we derive an exact expression for the computational cutoff rate of a coded system with M-PSK signaling and SC diversity, and show that the cutoff rate expression is a simple function of the MGF of the SNR at the output of the diversity combiner.**

***Index Terms*—Coded systems, cutoff rate, Nakagami fading, selection combining (SC), trellis-coded modulation (TCM), turbo codes.**

I. Introduction

DIVERSITY reception, together with channel coding, is a popular technique to mitigate the deleterious effects of multipath fading in a mobile radio environment [1], [2]. Trellis-coded modulation (TCM) schemes are known to provide good coding gains without incurring bandwidth expansion, and are considered for bandwidth-limited wireline communication systems and bandwidth- and power-limited wireless communication systems [3], [4], whereas turbo codes are popular in coding theory literature for their near-Shannon-limit error performance [5]. Indeed, turbo codes have been proposed for third-generation wireless communication standards [6]. Union bounding techniques are usually employed to obtain upper bounds on the error probability of coded systems, but one of the difficulties with this approach lies in obtaining a closed-form expression for the pairwise error probability (PEP) between two codewords. Often, for simplicity, a Chernoff-bound-type PEP expression is used, resulting in a loss of tightness by about 4 dB in the union bound on the error probability on fading channels [4], [7]–[9]. For simple channel models, like the additive white Gaussian noise (AWGN) and the Rayleigh fading channels, one can obtain the PEP in closed form [10], [11]. Recently, for Ricean fading and log-normal shadowing channels, Tellambura [12] has obtained a numerical approximation for the PEP using the Gauss–Chebyshev quadrature (GCQ) rule.

Knowledge of the channel cutoff rate [13] is very valuable in assessing the performance of practical coding schemes. Al-Semari and Fuja [14] derived both the cutoff rate and the union bound on the error probability using the Chernoff bound on the PEP for various diversity schemes on Rayleigh fading channels. In this paper, we are concerned with the performance, as measured by the channel cutoff rate, as well as the error performance of practical coding schemes, like TCM and turbo codes, on Nakagami fading channels with selection combining (SC) diversity. In particular, we obtain an exact expression for the moment generating function (MGF) of the signal-to-noise ratio (SNR) for an M-phase-shift keying (PSK) code symbol at the output of the selection combiner. Next, using GCQ and Gauss–Laguerre quadrature (GLQ) rules, we provide a simple to compute, yet accurate, numerical solution for the PEP with M-PSK modulated codewords on independent and identically distributed (i.i.d.) Nakagami-m fading channels with L-antenna SC diversity. Using the union bounding technique, we present the bit-error probability (BEP) performance of TCM and turbo codes. We also derive an exact expression for the cutoff rate of coded communication systems on i.i.d. Nakagami-m fading with L-antenna SC diversity, and show that it is a simple function of the MGF of the diversity combiner's output SNR.

The rest of this paper is organized as follows. In Section II, we present the system model. An exact expression for the MGF of the code symbol SNR, as well as a simple GCQ/GLQ-based expression for the PEP with SC diversity, is presented in Section III. An exact expression for the cutoff rate with L-path SC diversity is derived in Section IV. In Section V, we present the union-bound-based error-probability performance of both TCM and turbo codes. Finally, we conclude this paper in Section VI.

II. System Model

We assume that the transmitted code symbols are M-PSK modulated, and are coherently demodulated at the receiver. The receiver employs L antennas for diversity combining to miti-

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gate the effect of multipath fading. Let $\mathbf{X} = (X_1, \ldots, X_N)$ denote the transmitted code symbol sequence. Then the received symbol sequence on the lth antenna, $r_k^{(l)}$, is given by

$$r_k^{(l)} = \sqrt{E_s}\alpha_k^{(l)} e^{j\theta_k^{(l)}} X_k + \eta_k^{(l)}, \quad k = 1, 2, \ldots, N; \quad l = 1, 2, \ldots, L \tag{1}$$

where $\alpha_k^{(l)}$ is the fade random variable, and $\eta_k^{(l)}$ is the AWGN component associated with the lth antenna path when the transmitted data symbol $X_k \in \{e^{j2\pi p/M}\}_{p=0}^{M-1}$. The code symbols X_k are assumed to have unit energy, the $\eta_k^{(l)}$ are assumed to be i.i.d. complex Gaussian with zero mean and variance $\sigma^2 = N_0$, where $N_0/2$ is the two-sided power spectral density of the underlying noise random process $\eta(t)$, and E_s is the energy-per-code symbol-per-branch. Further, the fade random variables, $\alpha_k^{(l)}$, are assumed to be i.i.d. and are Nakagami-m distributed with the probability density function (pdf) and the cumulative distribution function (cdf), respectively, given by

$$f_\alpha(x) = \frac{2m^m}{\Gamma(m)} e^{-mx^2} x^{2m-1}, \quad x \geq 0 \tag{2}$$

$$F_\alpha(x) = \gamma(m, mx^2), \quad x \geq 0 \tag{3}$$

where $\Gamma(m)$ is the standard Gamma integral and $\gamma(a, x) = \int_{u=0}^{x} e^{-u} u^{a-1} du$ is the incomplete Gamma function [15]. Here, we normalized the second moment of the fade, $E(\alpha^2)$, to unity, and $m \geq 1/2$. The random phase, $\theta_k^{(l)}$, associated with the kth code symbol on the lth antenna, is uniformly distributed in $(-\pi, \pi)$. Both $\alpha_k^{(l)}$ and $\theta_k^{(l)}$ are assumed to remain constant over one code symbol duration. In the following section, we obtain expressions for the MGF and the PEP with L-path SC diversity.

III. MGF and PEP With SC Diversity

The PEP with SC diversity, $P^{\text{SC}}(\mathbf{X} \to \mathbf{Y})$, is the probability that the transmitted codeword $\mathbf{X} = (X_1, \ldots, X_N)$ is incorrectly decoded as $\mathbf{Y} = (Y_1, \ldots, Y_N)$. If the two codewords $\mathbf{X}$ and $\mathbf{Y}$ differ in d positions,[1] then the PEP is denoted by $P^{\text{SC}}{}_2(d)$, which we derive as follows.

The selection combiner picks the received signal for coherent combining from the path on which the instantaneous fade amplitude $\alpha_k^{(n)}$ is maximum. Accordingly, from (1), the decision statistic r_k, at the output of the selection combiner, is given by $r_k = r_k^{(n)} \alpha_k^{(n)} e^{-j\theta_k^{(n)}}$, where $\alpha_k^{(n)} = \max(\alpha_k^{(1)}, \ldots, \alpha_k^{(L)})$. By defining

$$\beta_k = [\tilde{\alpha}_k]^2 = \max\left(\left[\alpha_k^{(1)}\right]^2, \ldots, \left[\alpha_k^{(L)}\right]^2\right) \tag{4}$$

we can simplify r_k as

$$r_k = \beta_k \sqrt{E_s} X_k + \eta_k \tag{5}$$

where η_k is easily shown to be a conditional zero-mean complex Gaussian random variable with conditional variance $\beta_k N_0$. With this statistic at the output of the demodulator, the conditional PEP $P_2^{\text{SC}}(d|\beta_1, \ldots, \beta_d)$, conditioned on the fading random variables $\beta_1, \beta_2, \ldots, \beta_d$, is given by [16]

$$P_2^{\text{SC}}(d|\beta_1, \ldots, \beta_d) = Q\left(\sqrt{\sum_{n=1}^{d} a_n \beta_n}\right) \tag{6}$$

where $a_n = (E_s|X_n - Y_n|^2)/(2N_0)$.

The unconditional PEP $P^{\text{SC}}{}_2(d)$ is obtained by averaging the above expression over the distributions of $\beta_1, \beta_2, \ldots, \beta_d$, and is given by

$$\begin{aligned} P_2^{\text{SC}}(d) &= E_{\beta_1, \beta_2, \ldots, \beta_d}\left[Q\left(\sqrt{\sum_{n=1}^{d} a_n \beta_n}\right)\right] \\ &= \frac{1}{\pi} \int_{\phi=0}^{\frac{\pi}{2}} d\phi \prod_{n=1}^{d} E_{\beta_n}\left[e^{-\frac{a_n \beta_n}{2\sin^2\phi}}\right] \\ &= \frac{1}{\pi} \int_{\phi=0}^{\frac{\pi}{2}} d\phi \prod_{n=1}^{d} \text{MGF}_{\beta_n}\left(-\frac{a_n}{2\sin^2\phi}\right) \end{aligned} \tag{7}$$

where we have used the identity $Q(x) = (1/\pi)\int_{\phi=0}^{(\pi/2)} e^{-(x^2)/(2\sin^2\phi)} d\phi$ for $x \geq 0$ from [17], and the independence among the random variables $\beta_1, \ldots, \beta_d$. The last step in (7) is due to the definition of the MGF of the random variable Z, defined as $\text{MGF}_Z(s) = E_Z[e^{sZ}]$.

A. An Exact Expression for the MGF

Using (2), (3), and (4), we can directly obtain the pdf of β_k as

$$f_{\beta_k}(x) = \frac{Lm^m}{\Gamma(m)} e^{-mx} x^{m-1} [\gamma(m, mx)]^{L-1}, \quad x \geq 0 \tag{8}$$

and the MGF of β_k can be expressed as

$$\text{MGF}_{\beta_k}(s) = \frac{Lm^m}{\Gamma(m)} \int_{x=0}^{\infty} e^{-x(m-s)} x^{m-1} [\gamma(m, mx)]^{L-1} dx. \tag{9}$$

In order to simplify the above equation, we express $\gamma(m, mx)$ of (9) in terms of the confluent hypergeometric function ${}_1F_1(\cdot;\cdot;\cdot)$ using the relationship $\gamma(n, x) = (1/n)e^{-x}x^n {}_1F_1(1; 1+n; x)$ [15, eq. (8.351.2)]. With this, a closed-form solution for (9) can be obtained as

$$\begin{aligned} \text{MGF}_{\beta_k}(s) &= \frac{Lm^{L(m-1)+1}}{\Gamma(m)} \int_{x=0}^{\infty} e^{-x(mL-s)} x^{mL-1} \\ &\quad \times [{}_1F_1(1; 1+m; mx)]^{L-1} dx \\ &= \frac{Lm^{L(m-1)+1}\Gamma(mL)}{\Gamma(m)} \frac{1}{(mL-s)^{mL}} \\ &\quad \times F_A\Bigg(mL; \underbrace{1, \ldots, 1}_{L-1 \text{ times}}; \underbrace{1+m, \ldots, 1+m}_{L-1 \text{ times}}; \\ &\quad \underbrace{\frac{m}{mL-s}, \ldots, \frac{m}{mL-s}}_{L-1 \text{ times}}\Bigg) \end{aligned} \tag{10}$$

[1]Without loss of generality, we assume that codewords $\mathbf{X}$ and $\mathbf{Y}$ differ in the first d positions.

where the simplification is due to [15, eq. (9.19)] and $F_A(\cdot;\cdots;\cdots;\cdots)$ is Laurecella's hypergeometric function [15, eq. (9.19)].

B. A Simple and Accurate Numerical Solution

Direct substitution of (10) in (7) does not seem to result in a closed-form expression for $P_2^{\mathrm{SC}}(d)$. Also, for large values of L, a numerical approximation to $\mathrm{MGF}_{\beta_n}(s)$ is simpler to implement than the direct evaluation using (10). With this motivation, we develop accurate numerical approximations for both $\mathrm{MGF}_{\beta_n}(s)$ and $P_2^{\mathrm{SC}}(d)$.

According to the GLQ rule, we have [18]

$$\int_{x=0}^{\infty} e^{-x} x^Q f(x)\,dx \approx \sum_{j=1}^{J} w_Q(j) f(x_Q(j)) \tag{11}$$

where $\{w_Q(j)\}$ are the weights of the GLQ integration rule for a specific Q [18]. Using this to approximate $\mathrm{MGF}_{\beta_n}(-(a_n)/(2\sin^2\phi))$, we obtain

$$\begin{aligned}
&\mathrm{MGF}_{\beta_n}\left(-\frac{a_n}{2\sin^2\phi}\right)\\
&= \frac{Lm^m}{\Gamma(m)} \int_{x=0}^{\infty} e^{-\frac{a_n x}{2\sin^2\phi}} e^{-mx} x^{m-1} \left[\gamma(m, mx)\right]^{L-1} dx\\
&= \frac{L}{\Gamma(m)} \left(\frac{\sin^2\phi}{\sin^2\phi + \frac{a_n}{2m}}\right)^m \int_{y=0}^{\infty} e^{-y} y^{m-1}\\
&\quad \times \left[\gamma\left(m, \frac{y\sin^2\phi}{\sin^2\phi + \frac{a_n}{2m}}\right)\right]^{L-1} dy\\
&\approx \frac{L}{\Gamma(m)} \left(\frac{\sin^2\phi}{\sin^2\phi + \frac{a_n}{2m}}\right)^m \sum_{j=1}^{J} w_{m-1}(j)\\
&\quad \times \left[\gamma\left(m, \frac{x_{m-1}(j)\sin^2\phi}{\sin^2\phi + \frac{a_n}{2m}}\right)\right]^{L-1}.
\end{aligned} \tag{12}$$

In the above equation, the second step is arrived at by the change of integration variable from x to y through $y = x(m + (a_n)/(2\sin^2\phi))$, and the third step is due to the application of (11).

Using (7) and (11), we obtain the following simple and accurate numerical approximation for $P_2^{\mathrm{SC}}(d)$:

$$\begin{aligned}
&P_2^{\mathrm{SC}}(d)\\
&= \frac{1}{\pi} \int_{\phi=0}^{\frac{\pi}{2}} d\phi \prod_{n=1}^{d} \mathrm{MGF}_{\beta_n}\left(-\frac{a_n}{2\sin^2\phi}\right)\\
&\approx \frac{1}{2P} \sum_{k=1}^{P} \prod_{n=1}^{d} \mathrm{MGF}_{\beta_n}\left(-\frac{a_n}{1-\cos\frac{(2k-1)\pi}{2P}}\right)\\
&\approx \frac{1}{2P}\left[\frac{L}{\Gamma(m)}\right]^d \sum_{k=1}^{P} \prod_{n=1}^{d} \Bigg\{ \left(\frac{\sin^2\frac{(2k-1)\pi}{4P}}{\sin^2\frac{(2k-1)\pi}{4P} + \frac{a_n}{2m}}\right)^m\\
&\quad \times \sum_{j=1}^{J} w_{m-1}(j) \left[\gamma\left(m, \frac{x_{m-1}(j)\sin^2\frac{(2k-1)\pi}{4P}}{\sin^2\frac{(2k-1)\pi}{4P} + \frac{a_n}{2m}}\right)\right]^{L-1} \Bigg\}.
\end{aligned} \tag{13}$$

In the above equation, the second step is arrived at by using the GCQ rule [18], and the last step is due to the substitution of (12).

IV. Cutoff Rate With Selection Diversity

In this section, we derive the computational cutoff rate R_0 for a coded communication system with coherent detection on Nakagami fading channels with L-path SC diversity. First, we introduce some notation. Let $\mathbf{X} = (X_1, \ldots, X_N)$ denote an N-length vector of complex-valued M-PSK code symbols. The LN-length received code-symbol vector is denoted by $\mathbf{r} = (r_1^{(1)}, \ldots, r_1^{(L)}, \ldots, r_N^{(1)}, \ldots, r_N^{(L)})$, and the corresponding LN-length complex fade vector is denoted by $\mathbf{A}_\alpha = (\alpha_1^{(1)} e^{j\theta_1^{(1)}}, \ldots, \alpha_1^{(L)} e^{j\theta_1^{(L)}}, \ldots, \alpha_N^{(1)} e^{j\theta_N^{(1)}}, \ldots, \alpha_N^{(L)} e^{j\theta_N^{(L)}})$. The L-length vector of received symbols and the fade random variables, corresponding to the ith code symbol X_i, are denoted by $\underline{r}_i = (r_i^{(1)}, \ldots, r_i^{(L)})$, and $\underline{\alpha}_i = (\alpha_i^{(1)} e^{j\theta_i^{(1)}}, \ldots, \alpha_i^{(L)} e^{j\theta_i^{(L)}})$, respectively. That is, $\mathbf{r} = (\underline{r}_1, \ldots, \underline{r}_N)$ and $\mathbf{A}_\alpha = (\underline{\alpha}_1, \ldots, \underline{\alpha}_N)$. The function $q(\mathbf{X})$ denotes the probability distribution of the codeword $\mathbf{X}$. Finally, an M-PSK symbol is denoted by $s_p = e^{j2\pi p/M}, p = 0, 1, \ldots, M-1$, and $X_i \in \{e^{j2\pi p/M}\}_{p=0}^{M-1}$.

For a discrete M-ary input and continuous output channel, with perfect knowledge of the channel state information (CSI) at the receiver, the cutoff rate is defined as [13, pp. 182–183]

$$\begin{aligned}
R_0 = \lim_{N\to\infty} \max_{q(\mathbf{X})} \Bigg\{ -\frac{1}{N} \log_2 \Bigg(\int_{\mathcal{C}^{NL}} \int_{\mathcal{C}^{NL}}\\
\left[\sum_{\mathbf{X}} q(\mathbf{X}) \sqrt{p(\mathbf{r}, \mathbf{A}_\alpha|\mathbf{X})}\right]^2 d\mathbf{r}\,d\mathbf{A}_\alpha \Bigg) \Bigg\}
\end{aligned} \tag{14}$$

where $p(\mathbf{r}, \mathbf{A}_\alpha|\mathbf{X})$ is the joint pdf of the received sequence $\mathbf{r}$ and the complex fading sequence $\mathbf{A}_\alpha$, given the transmitted codeword $\mathbf{X}$, and $\mathcal{C}^{NL}$ is the LN-dimensional complex space. The conditional joint pdf, $p(\mathbf{r}, \mathbf{A}_\alpha|\mathbf{X})$, can be written as

$$p(\mathbf{r}, \mathbf{A}_\alpha|\mathbf{X}) = p(\mathbf{r}|\mathbf{X}, \mathbf{A}_\alpha) p(\mathbf{A}_\alpha). \tag{15}$$

For a symmetric channel, the expression in (14) can be maximized with the equiprobable input distribution

$$q(\mathbf{X}) = \prod_{i=1}^{N} p(X_i) = \frac{1}{M^N}. \tag{16}$$

Substituting (16) in (14), we obtain

$$\begin{aligned}
R_0 = -\lim_{N\to\infty} \frac{1}{N} \Bigg\{ \log_2 \Bigg(\int_{\mathcal{C}^{NL}} \int_{\mathcal{C}^{NL}}\\
\left[\frac{1}{M^N} \sum_{\mathbf{X}} \sqrt{p(\mathbf{r}|\mathbf{X}, \mathbf{A}_\alpha)}\right]^2 d\mathbf{r}\,p(\mathbf{A}_\alpha)\,d\mathbf{A}_\alpha \Bigg) \Bigg\}.
\end{aligned} \tag{17}$$

The expression $\sum_{\mathbf{X}} (1/(M^N)) \sqrt{p(\mathbf{r}|\mathbf{X}, \mathbf{A}_\alpha)}$ in (17) can be further simplified as follows:

$$\begin{aligned}
\sum_{\mathbf{X}} \frac{1}{M^N} \sqrt{p(\mathbf{r}|\mathbf{X}, \mathbf{A}_\alpha)} &= \sum_{\mathbf{X}} \left[\prod_{i=1}^{N} \frac{1}{M} \sqrt{p(\underline{r}_i|X_i, \underline{\alpha}_i)}\right]\\
&= \prod_{i=1}^{N} \left[\sum_{p=0}^{M-1} \frac{1}{M} \sqrt{p(\underline{r}_i|s_p, \underline{\alpha}_i)}\right]\\
&= \prod_{i=1}^{N} \mathcal{T}(i)
\end{aligned} \tag{18}$$

where $\mathcal{T}(i) = \sum_{p=0}^{M-1} (1/M) \sqrt{p(\underline{r}_i | s_p, \underline{\alpha}_i)}$. Substituting (18) in (17), we obtain

$$R_0 = -\lim_{N\to\infty} \frac{1}{N} \times \log_2\left(\int_{\mathcal{C}^{NL}} \int_{\mathcal{C}^{NL}} \prod_{i=1}^{N} \mathcal{T}^2(i) d\mathbf{r} p(\mathbf{A}_\alpha) d\mathbf{A}_\alpha\right). \quad (19)$$

Noting that $p(\mathbf{A}_\alpha) = \prod_{i=1}^{N} p(\underline{\alpha}_i)$, and $\mathcal{T}(1), \ldots, \mathcal{T}(N)$ are i.i.d., (19) can be further simplified as

$$\begin{aligned} R_0 &= -\lim_{N\to\infty} \frac{1}{N} \log_2\left(\left[\int_{\mathcal{C}^L} \int_{\mathcal{C}^L} \mathcal{T}^2(i)\, d\underline{r}_i\, p(\underline{\alpha}_i)\, d\underline{\alpha}_i\right]^N\right) \\ &= 2\log_2(M) - \log_2 \\ &\quad \times \left(\sum_{p=0}^{M-1} \sum_{n=0}^{M-1} \int_{\mathcal{C}^L} \int_{\mathcal{C}^L} \right. \\ &\quad \left. \times \sqrt{p(\underline{r}_i | s_p, \underline{\alpha}_i) p(\underline{r}_i | s_n, \underline{\alpha}_i)} d\underline{r}_i p(\underline{\alpha}_i) d\underline{\alpha}_i \right). \end{aligned} \quad (20)$$

When $X_i = s_m$ is the transmitted code symbol, the received symbol, r_i, at time i, at the output of the selection combiner is given by (5). With this, it is not difficult to show that [1, pp. 411–412]

$$\int_{\mathcal{C}^L} \sqrt{p(\underline{r}_i | s_p, \underline{\alpha}_i) p(\underline{r}_i | s_n, \underline{\alpha}_i)} d\underline{r}_i = \exp\left(-\frac{\beta_i E_s |s_p - s_n|^2}{4N_0}\right) \quad (21)$$

where $\beta_i = \max([\alpha_i^{(1)}]^2, \ldots, [\alpha_i^{(L)}]^2)$ as given in (4). Substituting (21) in (20), and noting that $\beta_1, \ldots, \beta_N$ are i.i.d., the cutoff rate R_0 can be conveniently written as

$$R_0 = 2\log_2(M) - \log_2\left(\sum_{p=0}^{M-1} \sum_{n=0}^{M-1} \mathrm{MGF}_\beta\left(-\frac{E_s |s_p - s_n|^2}{4N_0}\right)\right). \quad (22)$$

Notice that the cutoff rate expression in the above equation is a simple-to-compute function of the MGF of the code symbol SNR at the output of the selection diversity combiner. The R_0 in (22) can be evaluated efficiently either using the exact expression of (10) or an equivalent GLQ approximation of (12). As a quick check, for binary phase-shift keying (BPSK) signals (i.e., $M = 2$) on an AWGN channel (i.e., $f_\beta(x) = \delta(x-1)$), we obtain $\mathrm{MGF}_\beta(-(E_s|s_p - s_n|^2)/(4N_0)) = e^{-(E_s|s_p - s_n|^2)/(4N_0)}$ and the cutoff rate $R_0 = 1 - \log_2(1 + e^{-(E_s)/(N_0)})$, in agreement with [1, eq. (7-2-20)]. We also notice that after extracting a factor of $\log_2(M)$ from (22), the resulting expression is similar to [19, eq. (12.14)].

We evaluate the cutoff rate expression derived in (22) for Nakagami-$m, m \in \{0.5, 5\}$, fading channels for various orders of diversity, $L \in \{1, 2, 3\}$, and for different M-PSK constellations, $M \in \{2, 4, 8\}$. In this paper, in all the numerical and simulation results, the symbol energy per path, E_s, is scaled by the average value of β, $E(\beta)$, so that the average received SNR at the output of the selection combiner is the same, irrespective

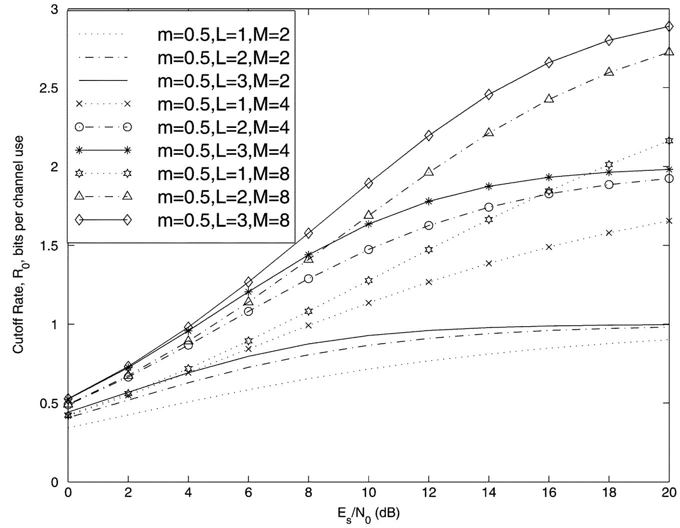

Fig. 1. Computational cutoff rate R_0, for Nakagami-m fading channels with L-antenna SC diversity. M is the constellation size of PSK signal set. $m = 0.5$.

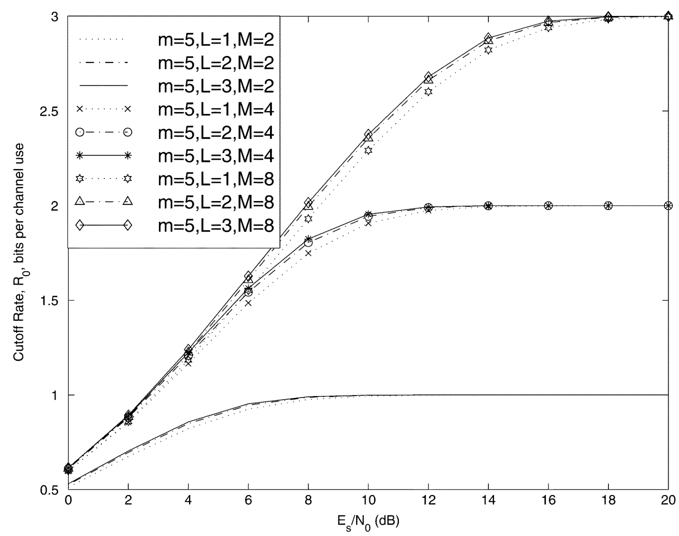

Fig. 2. Computational cutoff rate R_0, for Nakagami-m fading channels with L-antenna SC diversity. M is the constellation size of PSK signal set. $m = 5$.

of the Nakagami parameter m and the diversity order L, which allows a fair comparison of the performances with various combinations of m and L. The cutoff rate plots for various values of m, L, and M are shown in Figs. 1 and 2. From these figures, we observe that: 1) the cutoff rate is significantly affected when the fading severity index, m, is at its lowest value (i.e., for $m = 0.5$); 2) there is a diminishing-returns effect on R_0 as the number of diversity paths, L, is increased beyond two; and 3) for benign (light fading, large m) channels, diversity has a minimal effect on the cutoff rate.

V. BEP Bounds for TCM and Turbo Codes

In this section, using the exact expression for $\mathrm{MGF}_{\beta_k}(s)$ (10), and GCQ/GLQ-based approximation for the PEP (13), we present union-bound-based BEP performance for both TCM and turbo codes.

A. Union Bound on BEP: TCM

An upper bound on the BEP of a TCM scheme, using the transfer-function bounding technique [10], can be written as [2], [12]

$$P_b \leq \frac{1}{k\pi}\int_{\theta=0}^{\frac{\pi}{2}}\left\{\sum_{\mathbf{X}\neq\mathbf{Y}} a(\mathbf{X}\to\mathbf{Y})\prod_{i\in\mathcal{P}} E_{\beta_i}\left[D(\theta)^{\beta_i|X_i-Y_i|^2}\right]\right\}d\theta$$
$$= \frac{1}{k\pi}\int_{\theta=0}^{\frac{\pi}{2}}\left\{\sum_{\mathbf{X}\neq\mathbf{Y}} a(\mathbf{X}\to\mathbf{Y})\prod_{i\in\mathcal{P}} \mathrm{MGF}_{\beta_i} \times\left(-\frac{E_s|X_i-Y_i|^2}{4N_0\sin^2\theta}\right)\right\}d\theta$$
$$= \frac{1}{k\pi}\int_{\theta=0}^{\frac{\pi}{2}}\frac{d}{dI}T\left(\overline{D(\theta)},I\right)|_{I=1}d\theta \quad (23)$$

where k is the number of information bits per encoding interval, $\mathbf{X}\to\mathbf{Y}$ is the event that the transmitted codeword $\mathbf{X}$ is incorrectly decoded as the codeword $\mathbf{Y}$, $a(\mathbf{X}\to\mathbf{Y})$ is the number of information bit errors due to this event, $D(\theta) = e^{-(E_s)/(4N_0\sin^2\theta)}$, $\mathcal{P}$ is the set of all p such that $X_p \neq Y_p$, and $T(\overline{D(\theta)},I)$ is the transfer function of the underlying trellis code where each branch gain is replaced by $E_\beta[D^{\beta|X_n-Y_n|^2}(\theta)]$.

We evaluate the BEP bound for a 2-state, rate-1/2 encoded TCM scheme on Nakagami fading channels with $m=0.5$ and 5, and the number of paths $L\in\{1,2,3\}$. The encoder and the signal set mapping are same as that of [8] with the transfer function

$$T(\overline{D(\theta)},I) = \frac{IE_\beta[D^{2\beta}(\theta)]\times E_\beta[D^{4\beta}(\theta)]}{1-IE_\beta[D^{2\beta}(\theta)]} = \frac{I\mathrm{MGF}_\beta\left(-\frac{E_s}{2N_0\sin^2\theta}\right)\mathrm{MGF}_\beta\left(-\frac{E_s}{N_0\sin^2\theta}\right)}{1-I\mathrm{MGF}_\beta\left(-\frac{E_s}{2N_0\sin^2\theta}\right)}. \quad (24)$$

Figs. 3 and 4 show the union bound on the BEP for $m=0.5$ and 5, respectively. Simulation results are also plotted to evaluate the accuracy of the bound on BEP. From Figs. 3 and 4, we see that the bounds are increasingly tight for moderate-to-high SNRs. We also note that, at an error rate of 1×10^{-3}, more than 2-dB improvement in the performance can be achieved with an additional antenna (from $L=2$ to $L=3$) for $m=0.5$, whereas this improvement becomes less than 0.5 dB for $m=5$. This is due to the fact that $m=5$ corresponds to light fading, and diversity has minimal effect on this channel.

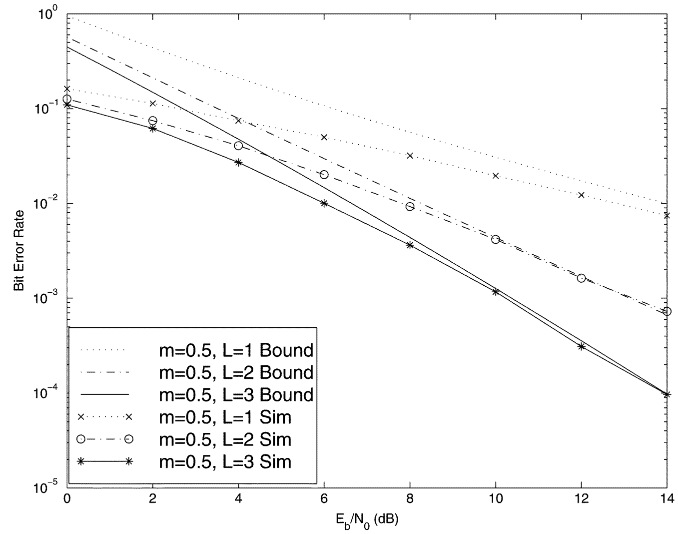

Fig. 3. Bit-error performance of TCM scheme on Nakagami-m i.i.d. fading channel with SC diversity. Number of antennas, $L=1,2,3$, and $m=0.5$. Code rate is 1/2, and the number of encoder states is two.

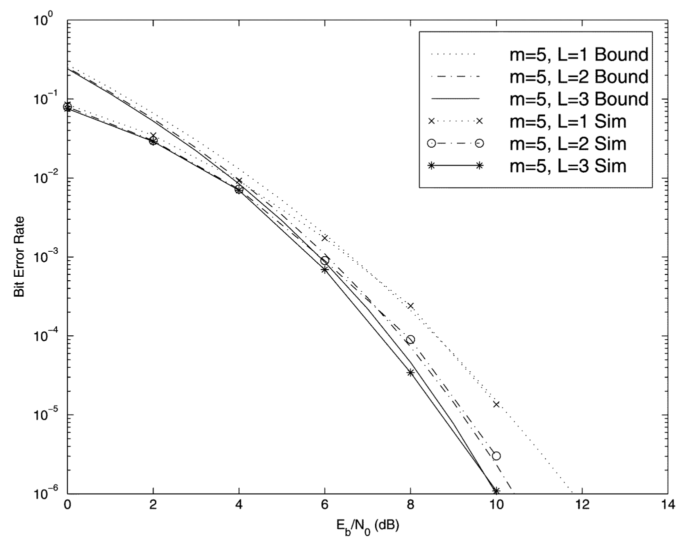

Fig. 4. Bit-error performance of TCM scheme on Nakagami-m i.i.d. fading channel with SC diversity. Number of antennas, $L=1,2,3$, and $m=5$. Code rate is 1/2, and the number of encoder states is two.

B. Union Bound on BEP: Turbo Codes

In [20], a transfer-function-based union upper bound is obtained for turbo codes on an AWGN channel. Later, in [9], the same approach is extended for Rayleigh fading channels. In this section, using the approach of [9], we provide upper bounds on the BEP of turbo codes on Nakagami fading channels with SC diversity. We assume an (N,K) parallel concatenated turbo code with a uniform interleaver, and with a minimum distance of $d_{\min}$. If we denote

$$t(d) = \sum_{i=1}^{K}(i/K)\binom{K}{i}p(d|i)$$

as the distance spectra of the turbo code, where $p(d|i)$ is the conditional probability of the codeword weight being d, given the information weight i, an upper bound on the BEP can be written as [9]

$$\overline{P_b} \leq \sum_{d=d_{\min}}^{N} t(d)P_2(d). \quad (25)$$

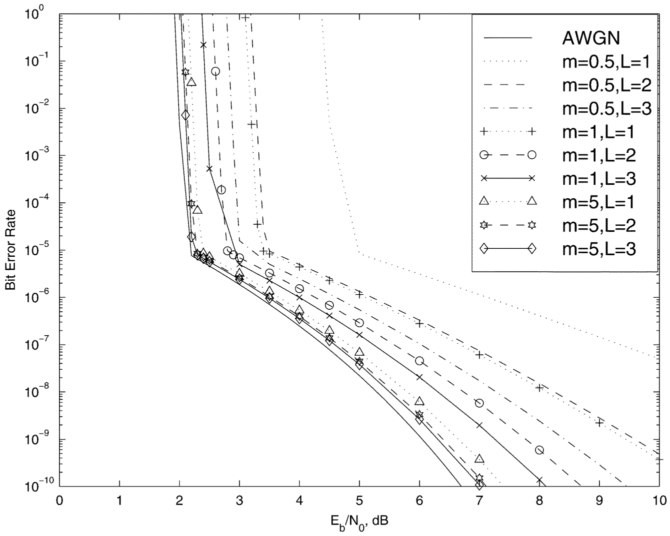

Fig. 5. BEP bounds for turbo codes on interleaved Nakagami-m fading channels with L-antenna SC diversity. Information block length is 500 bits with uniform turbo interleaver and code rate 1/3.

TABLE I
E_b/N_0 (IN dB) AT WHICH THE BEP OF RATE-1/3 TURBO CODE DIVERGES ON NAKAGAMI-m FADING CHANNEL WITH L-PATH SC DIVERSITY. THE CORRESPONDING E_b/N_0 REQUIRED TO ACHIEVE CUTOFF RATE R_0 OF 1/3 ARE ALSO GIVEN

	E_b/N_0, dB ($m = 0.5$)		E_b/N_0, dB ($m = 1$)		E_b/N_0, dB, ($m = 5$)	
L	R_0	Turbo code	R_0	Turbo code	R_0	Turbo code
1	3.7712	4.4	2.5712	3.0	1.7712	2.1
2	2.6712	3.1	2.0712	2.5	1.6712	2.0
3	2.3712	2.7	1.9712	2.3	1.5712	2.0

In Fig. 5, we plot the union bound on BEP for a rate-1/3 turbo code, whose constituent encoders' generator polynomials are $(1, 21/37)_8$, and the encoders are separated by a uniform interleaver. The bound is evaluated for Nakagami-m fading channels, $m \in \{0.5, 1, 5\}$, with L-path SC diversity, $L \in \{1, 2, 3\}$, and for an input block length of 500 information bits (i.e., 1500 code symbols). Also shown in Fig. 5 is the bound for the AWGN channel (i.e., $m \to \infty$ and $L = 1$). An important observation to make in Fig. 5 is that the union bound on BEP diverges at low E_b/N_0 values and renders the analytical results ineffective in this SNR range. In [20], it is reported that for the E_b/N_0 values less than the computational cutoff rate values, turbo code BEP bounds exhibit divergence behavior. We would like to point out that such a phenomenon is observed for Nakagami-m fading with L-path SC diversity, as well. To do this, we calculate the minimum E_b/N_0 required to achieve a cutoff rate of 1/3 with BPSK signals, and the E_b/N_0 at which our rate-1/3 turbo-code BEP bound diverges. These two SNR values are given in Table I for various values of m and L for a block length of 500 information bits. From Table I, we observe the following. First, the union bound on BEP for turbo codes with SC diversity on fading channels diverges at E_b/N_0 values below the SNR required to achieve the computational cutoff rate, thus establishing similar observations made in [20]. Second, for small orders of diversity (L) and for severe fading (small m), the difference between the E_b/N_0 at which the BEP bound diverges and that which is required to achieve $R_0 = 1/3$ is high, relative to what it is when we have large L and/or large m.

VI. CONCLUSION

We derived some analytical tools for the performance analysis of coded coherent communication systems on i.i.d. Nakagami-m fading channels with SC diversity. First, we obtained an *exact* expression for the MGF of the SNR of a code symbol at the output of the selection combiner. Next, we proposed a simple, yet accurate, numerical solution, based on GCQ and GLQ rules, for the PEP of coded M-PSK signals on Nakagami-m fading channels with L-path SC diversity. Using the PEP expressions, we presented union bounds on the bit-error performance of TCM and turbo codes. Finally, we derived an exact expression for the channel cutoff rate for the aforementioned coded system, as a function of the MGF of the SNR at the output of the combiner.

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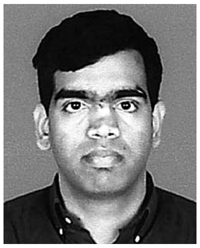

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