Harmonic Mean of Squared Product Distance: Criterion to Design Codes for Independent Fading Channels

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***Abstract*—**
A new performance criterion called harmonic mean of squared product distance (HMSPD) to design codes for an independent fading channel is proposed. HMSPD criterion is derived based on probability of error analysis using union bounding technique. It is applied to obtain the optimum pre-rotating matrices for a BPSK modulated transmission over a fast fading channel. The optimum 2×2 pre-rotating matrix is derived and its BER performance is compared with that of the one designed based on the popular criterion of maximizing the minimum product distance of the constellation. The new criterion gives a slightly better performance than the earlier one and it is also analytically tractable for extending to higher dimensions. Simulations are carried out and they indicate improvement in performance.

***Index Terms*—Harmonic mean of squared product distance, Performance criterion, Fading Channel, Coding Techniques, Pre-coding, Pre-rotating, Minimum Product distance.**

I. Introduction

Communication over a fading channel requires that the transmitter signal constellation be designed to provide maximum diversity and coding gains. Boutros *et al.* [1] and Tarokh *et al.* [2] have shown that the criterion to ensure maximum diversity and coding gains over an independently fading Rayleigh channel is maximizing the *minimum product distance* between any two points of the signal constellation. This is based on the analysis of worst-case pairwise error probability (PEP). This need not be the optimum criterion to minimize the average probability of error. In this paper, we propose a new criterion based on the analysis of the union bound on probability of error.

It is known that a linear pre-coding applied to typical constellations can improve the performance over fading channels [3] [4]. Optimum pre-coding matrices are designed based on the minimum product distance criterion. But the pre-coding matrix that maximizes the minimum product distance for an arbitrary dimension (N) is not yet reported. Constructions are available when N is a power of 2 or if N is an Euler number [4]. A new criterion is proposed in this paper which also has the advantage of giving an analytically tractable cost function to £nd the optimum pre-coding matrices for arbitrary dimensions.

Section II describes the system model, and gives a review of the existing criterion. In section III we introduce the new criterion. Section IV shows an example of code design by applying the criterion to design optimum pre-rotating matrices for a BPSK modulated transmission over a fast fading channel. Section V summarizes the results of numerical simulations carried out to compare the new criterion with the existing one.

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II. System Model and Background

The signal constellation is denoted by $\mathfrak{C} \subset \mathbb{R}^N$, where N is its dimension (over the real £eld). Signal points are denoted in bold lowercase ($\mathbf{x} = [x_1 \ x_2 \ \ldots \ x_N]^T$). The channel fading coef£cients $\alpha_i, i = 1, \ldots, N$ are assumed to be independent Rayleigh variables with unit second moment. The additive white Gaussian noise $\mathbf{n}$ is assumed independent and with variance N_0 per dimension. The received signal $\mathbf{r}$ is given by

$$\mathbf{r} = \boldsymbol{\alpha} \odot \mathbf{x} + \mathbf{n}, \tag{1}$$

where $\boldsymbol{\alpha} = [\alpha_1 \ \alpha_2 \ \ldots \ \alpha_N]^T$ and $\odot$ denotes component-wise product.

The maximum likelihood receiver for the signal model in (1) is

$$\hat{m} = arg \min_{\mathbf{x}_m \in \mathfrak{C}} ||\mathbf{r} - \boldsymbol{\alpha} \odot \mathbf{x}_m||^2. \tag{2}$$

The upper bound on PEP for a Rayleigh fading channel is obtained using the Chernoff bound technique [1] as

$$P(\mathbf{x} \to \mathbf{y}) \leq \frac{1}{2} \prod_{i=1}^{N} \frac{1}{1 + \frac{(x_i - y_i)^2}{8N_0}}, \tag{3}$$

where L is the number of unequal components of $\mathbf{x}$ and $\mathbf{y}$, i.e. the diversity. At high signal to noise ratio (SNR), where the second term in the denominator dominates, this may be approximated as

$$P(\mathbf{x} \to \mathbf{y}) \leq \frac{1}{2} \prod_{x_i \neq y_i}^{(L)} \frac{8N_0}{(x_i - y_i)^2}. \tag{4}$$

Hence the design criterion for code design for the channel may be rewritten as [1]:

1. Maximize the diversity (L)
2. Maximize the minimum product distance (dp_{min}),

$$dp_{min} = \min_{\mathbf{x},\mathbf{y} \in \mathfrak{C}} \prod_{x_i \neq y_i}^{(L)} |x_i - y_i|$$

between any two constellation points $\mathbf{x}$ and $\mathbf{y}$

In cases where maximum diversity ($L = N$) can be achieved, the second criterion may be made to include the £rst by taking the product for all N dimensions instead of L. (N is the dimension of the constellation). So the criterion reduces to maximizing

$$dp_{min} = \min_{\mathbf{x},\mathbf{y}\in\mathfrak{C}} \prod_{i=1}^{N} |x_i - y_i|. \tag{5}$$

This is the popular criterion used in literature to design codes for fading channels [2] [4] [5].

III. Harmonic Mean of Squared Product Distance (HMSPD) Criterion

In this section we propose a new criterion called the Harmonic Mean of Squared Product Distance (HMSPD) criterion to design codes for independently fading channels.

Maximizing dp_{min} minimizes the PEP of the constellation (at high SNR), but not necessarily the actual probability of error (P_e). The union bound on error probability given that $\mathbf{x}$ was transmitted is

$$P(e|\mathbf{x}) \leq \sum_{\mathbf{y}\in\mathfrak{C},\mathbf{y}\neq\mathbf{x}} P(\mathbf{x}\rightarrow\mathbf{y}),$$

where e denotes an error event.

The average probability of error of the whole constellation (assuming equal *a priori* probabilities) is[1]

$$\begin{aligned} P(e) &= \frac{1}{|\mathfrak{C}|}\sum_{\mathbf{x}\in\mathfrak{C}} P(e|\mathbf{x}) \qquad (6)\\ &\leq \frac{1}{|\mathfrak{C}|}\sum_{\mathbf{x},\mathbf{y}\in\mathfrak{C},\mathbf{x}\neq\mathbf{y}} P(\mathbf{x}\rightarrow\mathbf{y}) \\ &\leq \frac{1}{|\mathfrak{C}|}\sum_{\mathbf{x},\mathbf{y}\in\mathfrak{C},\mathbf{x}\neq\mathbf{y}} \left(\frac{1}{2}\prod_{i=1}^{N}\frac{1}{1+\frac{(x_i-y_i)^2}{8N_0}}\right) \\ &\approx \frac{1}{|\mathfrak{C}|}\sum_{\mathbf{x},\mathbf{y}\in\mathfrak{C},\mathbf{x}\neq\mathbf{y}} \left(\frac{1}{2}\prod_{x_i\neq y_i}^{(L)}\frac{8N_0}{(x_i-y_i)^2}\right), \end{aligned}$$

where the £nal approximation is assuming that the second term in the denominator dominates.

So we might do a better job if we minimize dp_{rec_sum} de£ned by

$$dp_{rec\_sum} = \sum_{\mathbf{x},\mathbf{y}\in\mathfrak{C},\mathbf{x}\neq\mathbf{y}} \frac{1}{dp^2_{\mathbf{x}-\mathbf{y}}}, \tag{7}$$

where $dp_{\mathbf{x}-\mathbf{y}} = \prod_{i=1}^{N}|x_i - y_i|$.

We de£ne the reciprocal of dp_{rec_sum} to be the *harmonic mean of the square product distance*: dp_{hms}. We propose the new criterion for code design as:

[1] $|\mathfrak{C}|$ denotes the cardinality of $\mathfrak{C}$

Maximize dp_{hms} de£ned as:

$$dp_{hms} = \left[\sum_{\mathbf{x},\mathbf{y}\in\mathfrak{C},\mathbf{x}\neq\mathbf{y}} \frac{1}{dp^2_{\mathbf{x}-\mathbf{y}}}\right]^{-1}, \tag{8}$$

where $dp_{\mathbf{x}-\mathbf{y}} = \prod_{i=1}^{N}|x_i - y_i|$.

Remark:

Being a differentiable function of each of the pairwise product distances, dp_{hms} is easier to optimize analytically than dp_{min}.

IV. Code Design Example

The HMS product distance criterion is applied to £nd the optimum pre-rotating matrices for a BPSK modulated transmission over fast fading channels. The idea is similar to the *modulation diversity* (MD) scheme proposed in [3]. In an MD scheme, multidimensional signal constellations are used and the components of each signal point are transmitted over different, ideally independent fading channels. The *diversity order* of a multidimensional constellation is the minimum number of distinct components between any two points of the constellation.

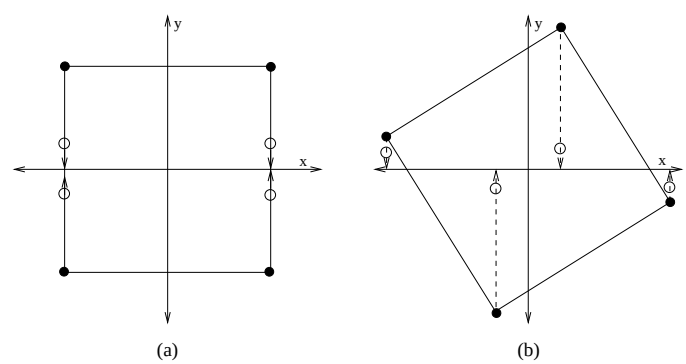

Fig. 1. Rotation for 2 dimensions

The modulation diversity of some classical constellations can be increased by applying certain rotations to the constellation. Fig 1 illustrates the idea for a 2-dimensional case. The x and y dimensions are assumed to fade independently. They can be two time slots separated more than the coherence time, two frequency bands separated more than the coherence bandwidth or two independent spatial channels. Fig 1(a) shows the case where each dimension is antipodal-modulated independently by two (independent) bits. If a deep fade hits one (only one) of the dimensions (say y), then we cannot decode the bit on that dimension even if dimension x is received strongly. Fig 1(b) shows the rotated constellation. In this case, even if one (not both) of the dimensions is in deep fade we can decode both the bits from the other dimension which we assume is received strongly. Since the probability that both the dimensions would

go into deep fade simultaneously is lower than the probability that either one of them would go into deep fade, we expect the bit error rate (BER) to go down for the rotated constellation. The diversity (L) of the constellation of Fig 1(b) is 2 while it is only 1 for that of Fig 1(a).

An interesting advantage of the *Rotation method* to increase diversity is that we are not paying anything more in terms of resources except complexity. Another advantage is that the performance over an AWGN channel does not change due to rotation since the Euclidean distance distribution does not change. So we do not loose performance if the scheme is used over an AWGN channel. Also, these constellations can be easily labeled using Gray mapping.

The question that can be posed is: *What is the optimum angle of rotation so that the probability of error is minimized?*

A. 2 Dimensions

Optimizing for two dimensions is easy since a 2-D real rotation matrix can be parametrized in one variable.

$$R = \begin{bmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{bmatrix}. \tag{9}$$

The two product distance values are

$$dp_1 = |cos\theta sin\theta|,$$
$$dp_2 = |(cos\theta - sin\theta)(cos\theta + sin\theta)|.$$

Maximizing $dp_{hms} = \left[\frac{4}{dp_1^2} + \frac{2}{dp_2^2}\right]^{-1}$ gives $tan^4 2\theta_{opt} = 8$ or $\theta_{opt} = 0.5172$ radians (29.6321 degrees). Hence,

$$R_{opt}(2) = \begin{bmatrix} 0.8692 & 0.4944 \\ -0.4944 & 0.8692 \end{bmatrix}.$$

B. Higher Dimensions

In general, a rotation in N-dimensions has $\binom{N}{2}$ degrees of freedom. An *elementary rotation* can be thought of as a rotation parallel to a plane. And any arbitrary rotation can be factored into rotations parallel to a set of mutually orthogonal planes. We can form $\binom{N}{2}$ mutually orthogonal planes in an N-dimensional space (planes formed by all possible pairs of vectors from the N-size orthogonal basis of the space). Hence an arbitrary rotation matrix R in N dimension can be factored as $\binom{N}{2}$ elementary rotation matrices $R_{i,j}$ which has the form,

$$R_{i,j}[k,l] = \begin{cases} cos(\theta_{i,j}), & k = l = j \text{ or } k = l = i \\ sin(\theta_{i,j}), & k = j \text{ and } l = i \\ -sin(\theta_{i,j}), & k = i \text{ and } l = j \\ I_N[k,l], & otherwise \end{cases} \tag{10}$$
$$i = 2, 3, \ldots, N, j = 1, 2, \ldots i-1$$
$$k = 1, 2, \ldots, N, l = 1, 2, \ldots, N,$$

where I_N is the $N \times N$ identity matrix.

We may re-index $R_{i,j}$ to R_l where l varies from 1 to $\binom{N}{2}$. And hence

$$R = \prod_{i=2}^{N} \prod_{j=1}^{i-1} R_{i,j} \tag{11}$$
$$= R_{\binom{N}{2}} R_{\binom{N}{2}-1} \cdots R_1.$$

For example, the general rotation matrix in 3-dimensions can be written as

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta_3) & sin(\theta_3) \\ 0 & -sin(\theta_3) & cos(\theta_3) \end{bmatrix}$$
$$\times \begin{bmatrix} cos(\theta_2) & 0 & sin(\theta_2) \\ 0 & 1 & 0 \\ -sin(\theta_2) & 0 & cos(\theta_2) \end{bmatrix}$$
$$\times \begin{bmatrix} cos(\theta_1) & sin(\theta_1) & 0 \\ -sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using this parametrization, the optimum rotation angles are numerically computed to maximize dp_{hms} for dimensions 3 and 4. This gives

$$R_{opt}(3) = \begin{bmatrix} 0.7453 & 0.3266 & 0.5813 \\ -0.5813 & 0.7453 & 0.3266 \\ -0.3266 & -0.5813 & 0.7453 \end{bmatrix},$$

$$R_{opt}(4) = \begin{bmatrix} 0.6732 & -0.4082 & 0.3160 & 0.5294 \\ -0.5272 & 0.3197 & 0.4104 & 0.6719 \\ -0.1523 & -0.2603 & 0.8156 & -0.4938 \\ -0.4956 & -0.8145 & -0.2580 & 0.1562 \end{bmatrix}.$$

Simulation Example 1:

We consider a BPSK modulated communication over a fast fading Rayleigh channel. N consecutive symbols are grouped together as a vector,

$$\mathbf{x}(k) = [x(kN)x(kN+1)\ldots x(kN+N-1)]^T, k = 0, 1, 2, \ldots$$

$\mathbf{x}(k)$ is pre-multiplied with the optimum pre-rotation matrix of dimension N and $R_{opt}x(k)$ is transmitted after parallel to serial conversion. We assume that the channel transfer function is constant during a symbol duration, and that it is independent between different symbols (fast fading). We also assume the channel to be frequency non-selective and Rayleigh fading.

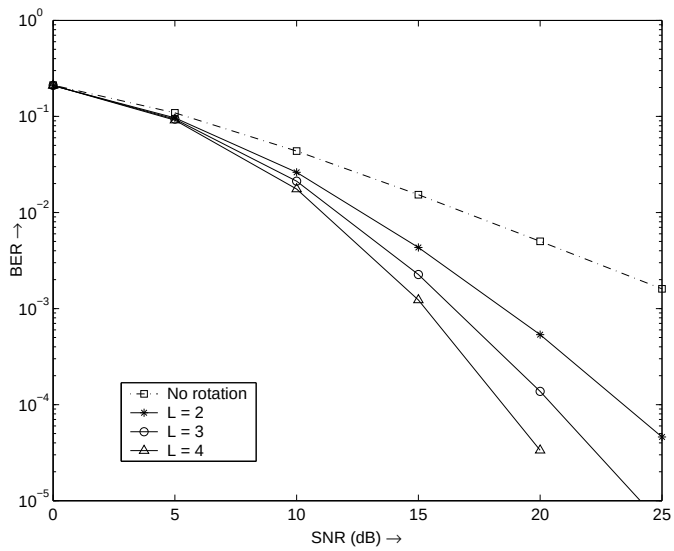

Fig. 2. BER vs SNR curves for the 2, 3 and 4 dimensional pre-rotating schemes using the pre-rotating matrix (R) that gives maximum dp_{hms}

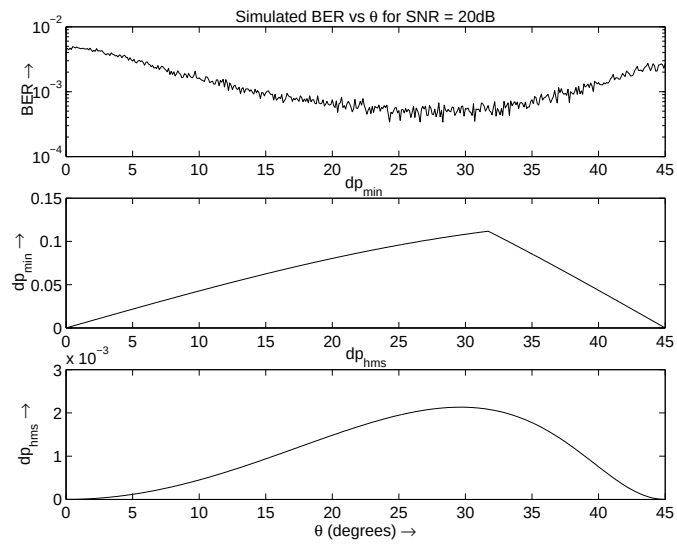

Fig. 3. Variation of (i) Simulated BER for 20 dB SNR, (ii) dp_{min} and (iii) dp_{hms} with respect to the pre-rotation angle θ

Simulations are carried out for $10e6$ bits with $N = 2$, 3 and 4 using the optimum pre-rotation matrices given above and the bit error rate (BER) vs SNR curve is plotted in Figure 2.

The pre-rotating improves the BER performance signi£cantly. For a BER of 10^{-3}, a scheme employing 2-dimensional rotation requires 8.6 dB less power than a non-rotated scheme. Going to 3 dimensions reduces the power requirement by a further 2.5 dB and 4 dimensions by a further 1.2 dB.

V. Comparison of the Two Criteria

This section compares the HMS product distance criterion with the popular minimum product distance criterion.

Simulation Example 2:

We investigate the BER variation with respect to the angle of rotation (θ). This would give an indication of the type of criterion to be used to minimize BER. In this example, the two criteria are compared with respect to their variation with θ.

With the same assumptions as in the £rst example, we have $N = 2$ and SNR = $20dB$. The pre-rotation matrices for θ ranging from 0 to 45 degrees in steps of 0.1 degrees are generated using equation (9) and applied to $\mathbf{x}(k)$. Simulations are carried out for all the cases with 10^5 bits. The values of dp_{min} and dp_{hms} are also calculated for each value of θ.

Figure 3 shows the BER, dp_{min} and dp_{hms} as a function of the angle of rotation (θ). We can observe that the BER curve varies more like the dp_{hms} curve than the dp_{min} curve which is peaky.

Simulation Example 3:

We study the BER performance of the two criteria. Consider the 2-dimensional case discussed in section IV. Maximizing dp_{min} gives $|tan2\theta_{opt}| = 2$ or $\theta_{opt} = 0.5536$ radians (31.7175 degrees) and maximizing dp_{hms} gives $\theta_{opt} = 0.5172$ radians (29.6321 degrees).

The 2-D pre-rotating scheme is simulated for 10^6 bits with the pre-rotation matrices for both these angles. Figure 4 shows the simulated BER values of the two schemes; one which maximizes dp_{min} and the second which maximizes dp_{hms} for different SNR values ranging from 0 to 25 dB. The dp_{hms} criterion shows a slight improvement in performance over dp_{min} criterion. For a BER of 10^{-3} the dp_{hms} scheme requires only 0.25 dB lesser power than the dp_{min} scheme.

Simulation Example 4:

Consider the higher dimensional cases. For dimensions 3 and 4, the optimum angles are computed numerically based on both the criteria. The two schemes are simulated for 10^6 bits with the same assumptions as in the previous examples. Figure 5 presents the result and shows that the new criterion improves performance.

Recall, that an important advantage of the HMSPD criterion is that it is analytically tractable than the dp_{min} criterion for extending to arbitrary dimensions. The pre-coding matrix that maximizes dp_{min} for an arbitrary dimension (N) is not yet reported. Constructions are available when N is a power of 2 or if N is an Euler number [4].

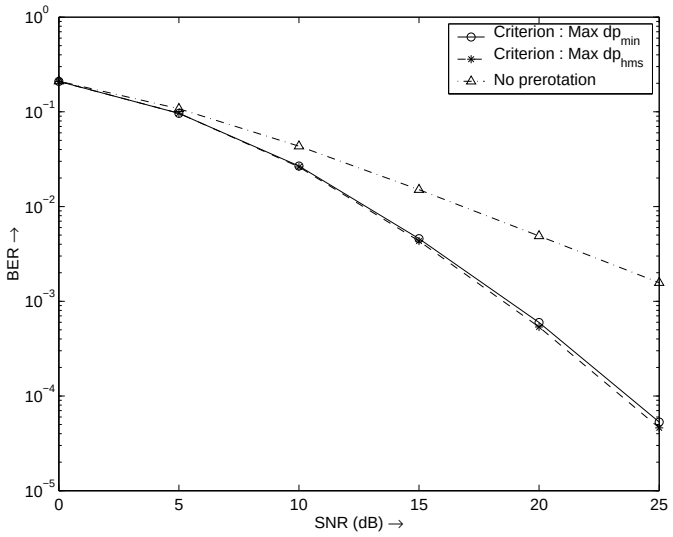

Fig. 4. BER vs SNR curves for 2D rotation schemes optimized for (i) dp_{min} , (ii) dp_{rec_sum} and (iii) no pre-rotation

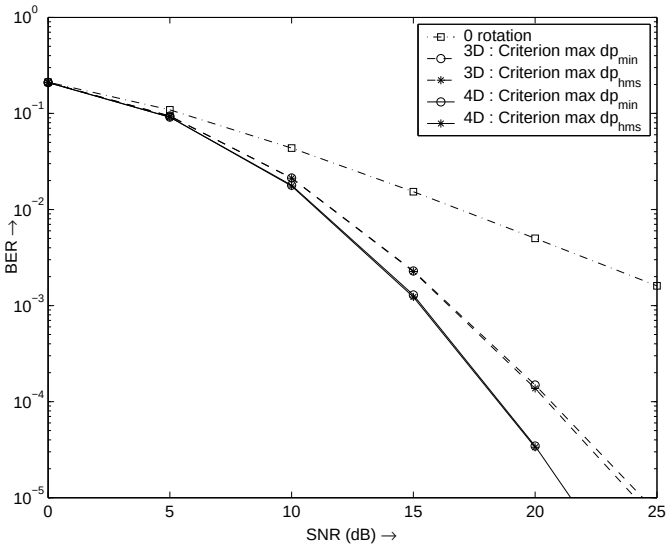

Fig. 5. BER vs SNR for higher dimensions (3 and 4) : R optimized for maximum dp_{min} and minimum dp_{rec_sum}

VI. Conclusion

A new performance criterion, harmonic mean of squared product distance (HMSPD), to design codes for an independent fading channel is proposed. It is analytically tractable than the minimum product distance criterion. The HMSPD criterion is applied to £nd the optimum pre-rotating matrices for a BPSK modulated transmission over a fast fading channel. It is shown to give a slightly better performance than the present criterion for the two dimensional case. For higher dimensions, optimum pre-rotating matrices are found by numerical optimization of both the criteria and the optimal rotation using the HMSPD criterion shows slightly better performance.

The pre-rotation scheme can be used to exploit other forms of diversity as well. For example, in a multi-carrier communication scheme, we can use it to exploit the frequency diversity offered by a frequency selective channel by applying the pre-rotation across different sub-channels that fade independently. Similarly, in a MIMO communication system, we can exploit the spatial diversity by applying the pre-rotation across independent spatial channels.

Optimizing the higher dimensional rotation matrices analytically is being pursued.

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