COMMENT ON

"PATH INTEGRATION OF A QUADRATIC ACTION WITH A GENERALIZED MEMORY"

E. BOSCO

*Department of Inorganic and Physical Chemistry, Indian Institute of Science,
Bangalore - 560 012, India*

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Using the promeasure technique, we give an alternative evaluation of a path integral corresponding to a quadratic action with a generalized memory.

Recently, Khandekar et al. [1] considered the path integration of a quadratic action with a generalized memory. Using Feynman's polygonal method and the action functional^{*1}

$$S[x(t)] = \int_T \left(\frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2 \right) dt + \mu F(\sigma^{-1} \int_T x(t) dt), \quad (1)$$

where $F(u)$ is an arbitrary function of u , the mass m and $\hbar$ are normalised to unity, the path integral was evaluated exactly. Here we rederive the same using the powerful technique of global integration on function spaces, originally developed by DeWitt-Morette [2]. This method is elegant, shorter and without any limiting procedure.

The path integral in question is:

$$K(x, t_b; x_0, t_a) = \int_{x(t_a)=x_0}^{x(t_b)=x} D[x(t)] \exp\{iS[x(t)]\}, \quad (2)$$

with $S[x(t)]$ given by eq. (1).

After absorbing the harmonic part of the action into the promeasure ω' , the propagator in eq. (2) can be written as [3]

$$K = K_0(x, t_b; x_0, t_a) \int_{\Phi_{xx_0}} \exp\{i\mu F(\sigma^{-1} \int_T x(t) dt)\} d\omega'(x). \quad (3)$$

The function-space integral in eq. (3) is over Φ_{xx_0} , the space of continuous functions $x \in \Phi_{xx_0}$ such that $x(t_a) = x_0$ and $x(t_b) = x$ defined over the interval $T = [t_a, t_b]$. $K_0(x, t_b; x_0, t_a)$ is the propagator for the harmonic oscillator.

Under the function-space translation $\Phi_{xx_0} \rightarrow \Phi_0$ by $x(t) \mapsto y(t) = x(t) + \bar{x}(t)$, where Φ_0 is the space of continuous functions y vanishing at t_a and t_b and $\bar{x}(t)$ is the fixed path in Φ_{xx_0} given by

$$\bar{x}(t) = \frac{x_0 \sin \omega(t_b - t) + x \sin \omega(t - t_a)}{\sin \omega T}. \quad (4)$$

^{*1} Even if the memory function $F(\sigma^{-1} \int_T x(t) dt)$ is replaced by a more general one of the form $F(\sigma^{-1} \int_T \chi(t) x(t) dt)$, the analysis will essentially be the same.

With the image ω on Φ_0 of the promeasure ω' , the integral in (3) can be written

$$\int_{\Phi_0} \exp\{i\mu F(\sigma^{-1} \int_T [x(t) + \bar{x}(t)] dt)\} d\omega(x). \quad (5)$$

The promeasure ω is defined in terms of its Fourier transform $\mathcal{F}\omega(\mu)$ on the space $\mathcal{M}$ dual of Φ_0 , which is the space of bounded measures μ on T , and the covariance $G(r, s)$:

$$\mathcal{F}\omega(\mu) = \exp[-\frac{1}{2}iW(\mu)], \quad W(\mu) = \int_T d\mu(r) \int_T d\mu(s) G(r, s),$$

$$G(r, s) = [Y^+(r-s) \sin \omega(s-t_a) \sin \omega(t_b-r) + Y^-(r-s) \sin \omega(r-t_a) \sin \omega(t_b-s)]/\omega \sin \omega T.$$

The integrand in the function-space integral on Φ_0 is a cylindrical function and hence the integral in eq. (5) can be readily evaluated by a linear mapping $\tau: \Phi_0 \rightarrow R$ by $x(t) \mapsto u = \langle \lambda, x \rangle$ where λ is the Lebesgue measure on the interval T . The image of $d\omega$ under τ is

$$[2\pi i W(\lambda)]^{-1/2} \exp[iu^2/2W(\lambda)] du,$$

where

$$W(\lambda) = \int_T dr \int_T ds G(r, s) = d/\omega^3 \sin \omega T,$$

and

$$d = 2(1 - \cos \omega T) - \omega T \sin \omega T.$$

Thus

$$\int_{\Phi_0} \exp\{i\mu F(\sigma^{-1} \int_T [x(t) + \bar{x}(t)] dt)\} d\omega(x) = [2\pi i W(\lambda)]^{-1/2} \int du \exp[i\mu F(u/\sigma + \langle \lambda, \bar{x} \rangle/\sigma)] \exp[\frac{1}{2}iu^2/W(\lambda)]. \quad (6)$$

On combining eqs. (3)–(6) and simplifying, we get

$$K(x, t_b; x_0, t_a) = (\omega^2/2\pi i \sqrt{d}) \int du \exp\{(i\omega/2d) [(x^2 + x_0^2) (\sin \omega T - \omega T \cos \omega T) + 2xx_0(\omega T - \sin \omega T) + 2(x + x_0)u\omega(\cos \omega T - 1) + u^2\omega^2 \sin \omega T] + i\mu F(u/\sigma)\},$$

the expression obtained by Khandekar et al.

References

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