Journal of Applied Mathematics and Physics (ZAMP)
Vol. 34, July 1983

0044-2275/83/004545-05 $ 2.50/0

Kurze Mitteilungen – Brief Reports – Communications brèves

On the instability of internal Alfvén-gravity waves in stratified shear flows

By A. Satya Narayanan, Dept of Applied Mathematics, Indian Institute of Science, Bangalore-560012, India

1. Introduction

The propagation of internal Alfvén-gravity waves from the troposphere into the ionosphere is of considerable interest from a geophysical point of view. Much attention has been given to the detection and measurement of irregular motions in the *D*, *E*, and lower *F* regions of the atmosphere and to the occurence of irregular density distributions at the same heights. Many of these irregularities may have their origin in disturbances in the atmosphere, namely, Compressive and Lorentz forces.

The stability properties of a Helmholtz velocity profile for stratified shear flows in the presence of a lower rigid boundary was studied in Ref. [1]. It was shown that new unstable modes, which are similar to internal gravity modes, are present in addition to the Kelvin-Helmholtz mode. In Ref. [2] the role played by twisted magnetic fields when fluid motions are present along the lines of force was considered. The stability of helical magnetic fields on a gravitating cylinder was studied in Ref. [3]. In this paper we study the stability properties of internal Alfvén-gravity waves in a two-layer model with a lower rigid boundary. The wavelengths of the unstable modes have resemblance to propagating internal Alfvén-gravity waves in the atmosphere.

In Section 2 we formulate the problem and derive the dispersion relation. Section 3 deals with the discussion of the results while Section 4 presents the conclusions.

2. Formulation and dispersion relation

The velocity and Brunt-Väisälä frequency distributions in our model (Fig. 1) are given below:

$$U_0(z) = U + \alpha z, \quad N_0 = N, \quad -D < z < 0 \quad \text{in layer } A,$$
$$U_0(z) = U, \quad N_0 = N, \quad 0 < z < \infty \quad \text{in layer } B,$$

where U and N are constants, α is the gradient of shear. We assume that the magnetic field is zero in layer A while layer B has a uniform magnetic field H. Under the Boussinesq approximation, the differential equation governing the internal Alfvén-gravity waves can be shown to be [4]

$$\frac{d^2 w}{dz^2} - \frac{2k\Omega^2 \, dU_0/dz}{\Omega_d(\Omega_d^2 - \Omega^2)} \frac{dw}{dz} + \left\{ \frac{k^2 N^2}{\Omega_d^2 - \Omega^2} - \frac{k \, d^2 U_0}{\Omega \, dz^2} - k^2 - \frac{2k^2 \Omega^2 (dU_0/dz)^2}{\Omega_d^2 (\Omega_d^2 - \Omega^2)} \right\} w = 0 \tag{1}$$

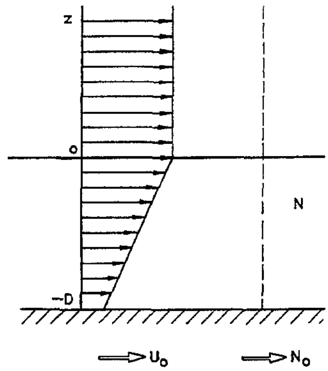

Figure 1
Two-layered model. The rigid boundary is at a distance D from the origin.

where $\Omega_d = \omega - k\,U_0$ is the Doppler-shifted frequency, $\Omega = k\,V_A$ is the Alfvén frequency and V_A is the Alfvén velocity.

The problem of the stability of the inviscid flow of a fluid in which both density and velocity vary continuously with height was first considered by Taylor and Goldstein independently as early as 1931 Refs. [5, 6]. Although they considered various layered models, the most relevant one was with a linearly increasing velocity with height and an exponentially decreasing density. Taylor speculated that if the number of layers were increased indefinitely, a critical value $R = 1/4$ might be reached for the dynamic stability of shear flows, where R is the Richardsen number ($R = N^2/(\mathrm{d}U_0/\mathrm{d}z)^2$). Subsequently the general statement that stability is assured if the local value of $R > 1/4$ everywhere was proved in Refs. [7] and [8]. The generalization of this proof to compressible flows is given in Ref. [9]. Although $R \leqq 1/4$ is a necessary condition to produce dynamic instability in a fluid undergoing shear, it is far from being a sufficient one.

The solution of Eq. (1) in layer A is obtained in terms of modified Bessel functions as

$$w = C_1\,I_\nu(x) + C_2\,K_\nu(x),$$

where $\nu = ((1/4) - N^2/\alpha^2)^{1/2}$ and $x = (\omega - k\,U)/\alpha - k\,z$. For integral values of ν, the above solution is valid. However, for non integral values of ν, the solution is written in terms of $I_\nu(x)$ and $I_{-\nu}(x)$. Since $R = N^2/\alpha^2$ is the Richardson number and $R = 1/4$ is the critical value for stability, we choose $R = 1/4$ throughout the paper.

In layer B the solution is

$$w = B_1 \exp(i\,Q_B\,z), \tag{3}$$

$$Q_B^2 = k^2\left\{\frac{N^2}{(\omega - k\,U)^2 - k^2\,V_A^2} - 1\right\}, \tag{4}$$

with $\mathrm{Imag}(Q_B) > 0$.

Applying the matching conditions at the interface ([10]) and the boundary condition on the rigid boundary, we have the following dispersion relation.

$$d_{11}(d_{22}\,d_{33} - d_{32}\,d_{23}) - d_{12}(d_{21}\,d_{33} - d_{31}\,d_{23}) = 0, \tag{5}$$

where

$$d_{11} = I_\nu[\{(\omega - k\,U)/\alpha\} + k\,D], \qquad d_{12} = K_\nu[\{(\omega - k\,U)/\alpha\} + k\,D],$$

$$d_{21} = I_\nu\{(\omega - k\,U)/\alpha\}, \qquad d_{22} = K_\nu\{(\omega - k\,U)/\alpha\}, \qquad d_{23} = 1,$$

$$d_{31} = -\frac{k}{2}(\omega - k\,U)[I_{\nu+1}\{(\omega + k\,U)/\alpha\} + I_{\nu-1}\{(\omega - k\,U)/\alpha\}]$$
$$+ k\,\alpha\,I_\nu\,\{(\omega - k\,U)/\alpha\},$$
$$d_{32} = \frac{k}{2}(\omega - k\,U)[K_{\nu+1}\{(\omega - k\,U)/\alpha\} + K_{\nu-1}\{\omega - k\,U)/\alpha\}]$$
$$+ k\,\alpha\,K_\nu\{\omega - k\,U)/\alpha\},$$
$$d_{33} = -\,i\,Q_B\{\omega - k\,U) + k^2\,V_A^2/(\omega - k\,U)\}.$$

3. Discussion of the results

The dispersion relation derived in Section 2 is solved numerically for the relevent values of the parameters $\hat{D}$ and τ. $\hat{D}$ is the non-dimensional parameter characterizing the width of the shear layer from the lower rigid boundary. τ is the non-dimensional ratio of the square of the Alfvén and the constant basic velocities, i.e. $\tau = V_A^2/U^2$ where U has already been defined in the previous section.

Fig. 2 depicts the normalized imaginary part $\hat{C}_i$ of the phase speed for $\hat{D} = 18.13$ and $\tau = 10.0$ as a function of $\hat{k}$ (equal to $k\,U/N$). Unstable modes at low wavenumbers are found which have wavelengths resembling internal Alfvén-gravity waves. The broken line corresponds to the most unstable mode for the case $\alpha = 0$. The solution of Eq. (1) in layer A for this case is similar to the solution obtained for layer B. Thus the presence of shear has a destabilizing effect on the unstable modes. Fig. 3 depicts the nature of the modes for

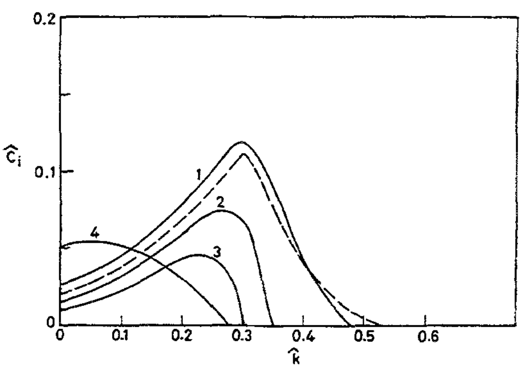

Figure 2
Properties of unstable modes for $\hat{D} = 18.13$, $\tau = 10.0$. 1–4: Alfvén-gravity modes.

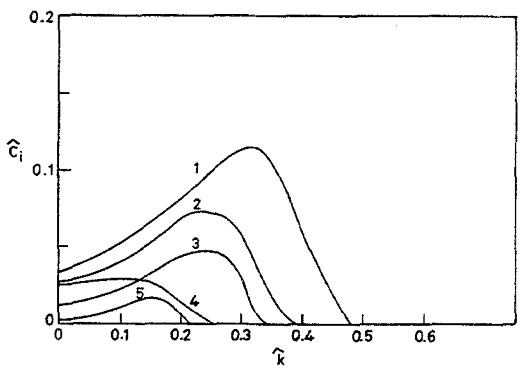

Figure 3
Same as in Fig. 2 except for $\hat{D} = 22.91$.

Table 1
Stability properties for the most unstable internal Alfvén-gravity waves for the present configuration.

Case	D km	U ms^{-1}	N s^{-1}	$\hat{D}$	wave length λ, km	C_i/U	$(kC_i)^{-1}$ s
1	80	75	0.017	18.13	93.64	0.118	1684
2	100	90	0.022	24.44	100.28	0.113	1569
3	120	110	0.021	22.91	107.34	0.115	1350

$\hat{D} = 22.91$ and $\tau = 10.0$. We see the emergence of another mode whose growth rate is smaller than all the other modes. This confirms the results obtained in Ref. [1]. As the width of the shear layer increases, new unstable modes are generated which have growth rates small compared to the other modes. As τ is increased, the maximum value of $\hat{C}_i$ for the most unstable mode decreases significantly. Thus the presence of a magnetic field has a stabilizing effect on the unstable modes at small wavenumbers.

In order to get a quantitative idea of the stability properties of internal Alfvén-gravity modes for the present model we give, in Table 1, the wavelengths of the most unstable mode for different values of the parameter $\hat{D}$. The wavelength of these unstable modes resemble internal Alfvén-gravity waves propagating in the atmosphere.

4. Conclusions

The stability results obtained for the model under consideration bring out the importance of the role played by internal Alfvén-gravity waves which propagate from the troposphere into the ionosphere. The presence of a magnetic field has a stabilizing effect on the unstable modes at low wavenumbers while shear has a destabilizing effect. More realistic distribution of winds, Brunt-Väisälä frequency, magnetic fields, electrical conductivity and the other effects need to be incorporated to enhance the relevance of the present study.

Acknowledgement

The author wishes to thank the authorities of the University Grants Commission (India) for providing financial help. He is also grateful to Dr. K. Somasundaram for useful discussions.

References

[1] R. S. Lindzen and A. J. Rosenthal, J. Geophys. Res. *81*, 1561–1571 (1976).
[2] S. K. Trehan and M. Singh, Astrophys. Space Sci. *53*, 165–172 (1978).
[3] A. M. Karnik and S. P. Talway, Astrophys. Space Sci. *79*, 379–386 (1981).
[4] N. Rudraiah and M. Venkatachalappa, J. Fluid Mech. *62*, 705–726 (1974).
[5] G. I. Taylor, Proc. Roy Soc. Lond., A132, 499–523 (1931).
[6] S. Goldstein, Proc. Roy. Soc. Lond. A132, 524–548 (1931).
[7] J. W. Miles, J. Fluid Mech. *10*, 496–508 (1961).
[8] L. N. Howard, J. Fluid Mech. *10*, 509–512 (1961).
[9] G. Chimonas., J. Fluid Mech. *43*, 833–836 (1970).
[10] S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability*. Oxford Univ. Press, Oxford 1961.

Abstract

Internal Alfvén-gravity waves and their stability characteristics for an inviscid, nondissipative, Boussinesq fluid undergoing shear in the presence of a lower rigid boundary is studied. The model consists of a layer of constant shear capped by a layer of constant velocity. The Brunt-Väisälä frequency is assumed to be uniform throughout the fluid. The unstable modes have wavelenghts which are close to those of the internal Alfvén-gravity waves which propagate from the troposphere into the ionosphere.

Zusammenfassung

Es werden in der vorliegenden Arbeit die internen Alfvén-Gravitationswellen untersucht und zwar für eine reibungslose nicht-dissipierende Boussinesq-Flüssigkeit, mit einer Scherschicht in Anwesenheit einer unteren steifen Grenze. Das Modell besteht aus einer Schicht mit konstanter Scherung, worauf eine Schicht mit konstanter Geschwindigkeit liegt. Es wird angenommen, daß die Brunt-Väisälä-Frequenz in der ganzen Flüssigkeit gleichförmig ist. Die instabilen Schwingungen haben Wellenlängen, die ganz nahe der Wellenlängen der internen Alfvén-Gravitationswellen sind, die sich sich von der Troposphäre in die Ionosphäre fortpflanzen.

(Received: March 10, 1982; revised: March 30, 1983)