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Approximate solutions of Fredholm integral equations of the second kind

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ABSTRACT

This note is concerned with the problem of determining approximate solutions of Fredholm integral equations of the second kind. Approximating the solution of a given integral equation by means of a polynomial, an over-determined system of linear algebraic equations is obtained involving the unknown coefficients, which is finally solved by using the least-squares method. Several examples are examined in detail.

1. Introduction

Integral equations can be viewed as equations which are results of transformation of points in a given vector space of integrable functions by the use of certain specific integral operators to points in the same space. If, in particular, one is concerned with function spaces spanned by polynomials for which the kernel of the corresponding transforming integral operator is separable being comprised of polynomial functions only, then several approximate methods of solution of integral equations can be developed. Recently, Mandal and Bhattacharya [1] has described a special approximate method of solution of Fredholm integral equations by using Bernstein polynomials which suits the integral equations associated with function spaces spanned by polynomials only.

Varieties of integral equations have been solved numerically in recent times by several workers, utilizing various approximate methods (see Mandal and Bhattacharya [1], Chakrabarti et al. [2], Chakrabarti and Mandal [3], Golberg and Chen [4], Kanwal [5], Mandal and Bera [6] and Polyanin and Manzhirov [7]).

In the present note, we have developed a straightforward method involving expansion of the unknown function of a Fredholm integral equation of the second kind in terms of polynomials $\{x^j\}_{j=0}^n$ and obtained an approximate solution of the given integral equation by the use of the method of least-squares. Simple illustrative examples have been dealt with.

We consider here the problem of solving approximately the integral equation of the form

$$L\phi = f, \tag{1.1}$$

with L being an integral operator of the type

$$L\phi(x) = \phi(x) + \int_\alpha^\beta k(x,t)\phi(t)\,dt, \quad (\alpha < x < \beta), \tag{1.2}$$

where $\phi(t)$ is an unknown square-integrable function to be determined, $k(x,t)$ is the known kernel which is a continuous and square integrable function, and $f(x)$ is a known square-integrable function. We will assume that the integral Eq. (1.1) possesses a unique solution.

Recently, the integral equations of the above type have been solved approximately by Mandal and Bhattacharya [1], by using the expansion of the solution function $\phi(x)$, in the form

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$$\phi(x) = \sum_{i=1}^{n+1} c_{i-1} B_{i-1,n}(x), \tag{1.3}$$

where c_{i-1} $(i = 1, 2, \ldots, n+1)$'s are unknown constants and $B_{i-1,n}(x)$ $(i = 1, 2, \ldots, n+1)$ are Bernstein polynomials of degree n defined on an interval (α, β), and are given by

$$B_{i-1,n}(x) = \binom{n}{i-1} \frac{(x-\alpha)^{i-1}(\beta - x)^{n-i+1}}{(\beta-\alpha)^n}, \quad i = 1, 2, \ldots, n+1 \tag{1.4}$$

in standard notations.

Substitution of the relation (1.3) in the Eq. (1.1) gives rise to the relation

$$\sum_{i=1}^{n+1} c_{i-1} \Psi_{i-1}(x) = f(x), \quad \alpha < x < \beta, \tag{1.5}$$

where,

$$\Psi_{i-1}(x) = B_{i-1,n}(x) + \int_\alpha^\beta k(x,t) B_{i-1,n}(t)\,dt. \tag{1.6}$$

The determination of the solution (1.3) can then be completed by solving the over-determined system of linear algebraic equations (1.5) for the unknown constants c_{i-1} $(i = 1, 2, \ldots, n+1)$.

The best solution of the over-determined system of Eq. (1.5) is obtainable by the method of least-squares giving rise to the system of determinate linear algebraic equations, as given by:

$$\sum_{i=1}^{n+1} c_{i-1} D_{ij} = B_j, \quad j = 1, 2, \ldots, n+1, \tag{1.7}$$

where

$$D_{ij} = \int_\alpha^\beta \Psi_{i-1}(x) \Psi_{j-1}(x)\,dx, \tag{1.8}$$

$$B_j = \int_\alpha^\beta f(x) \Psi_{j-1}(x)\,dx. \tag{1.9}$$

The above system of Eq. (1.7) is different from the one obtained by Mandal and Bhattacharya [1], which was derived by multiplying the relation (1.5) by $B_{j-1,n}(x)$ and integrating.

We observe that the above procedure of the determination of the coefficients $c_{i-1}(i = 1, 2, \ldots, n+1)$ gives rise to computational difficulties because of the fact that a large number of integrals need to be evaluated which involve the Bernstein polynomials, even by selecting n to be as small as $n = 4$. We have avoided these difficulties by recasting the expression (1.3) as

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n, \tag{1.10}$$

where, if $\alpha = 0, \beta = 1$, we get

$$\begin{aligned}
a_0 &= c_0,\\
a_1 &= -nc_0 + nc_1,\\
a_2 &= \frac{n(n-1)}{2}\{c_0 + c_2\} - n(n-1)c_1,\\
&\ldots\\
a_{n-1} &= (-1)^{n-1} nc_0 + (-1)^{n-2} n(n-1)c_1 + (-1)^{n-3} \frac{n(n-1)(n-2)}{2} c_2 + \cdots + nc_{n-1},\\
a_n &= (-1)^n c_0 + (-1)^{n-1} nc_1 + (-1)^{n-2} \frac{n(n-1)}{2} c_2 + \cdots - nc_{n-1} + c_n.
\end{aligned}$$

We now make the following observations:

If an approximate solution of the Eq. (1.1) is expressed in the form of a polynomial, as given by

$$\phi(x) = \sum_{i=1}^{N+1} a_{i-1} x^{i-1}, \tag{1.11}$$

where $a_{i-1}(i =, 1, \ldots, N+1)$ are unknown constants to be determined then it amounts to determining the values of $\phi(x)$ at $N+1$ points in its domain of definition (see interpolation formula). This forces us to approximate the integral term (see relations(1.1) and (1.2)) of the integral equation by a suitable quadrature formula requiring the knowledge of these $(N+1)$ values of ϕ.

But, if the integral in the above Eq. (1.1) is replaced by a quadrature formula (see Fox and Goodwin [8]), we get

$$\phi(x) + \sum_{k=0}^{N} w_k \phi(t_k) k(x, t_k) = f(x), \ \alpha < x < \beta, \tag{1.12}$$

where w_k are the weights and t_k's are appropriately chosen interpolation points.

The Eq. (1.12) represents an over-determined system of linear algebraic equations for the determination of $N+1$ unknowns $\phi(t_k)$ $(k = 0, \ldots, N)$.

So, if from theoretical considerations it is already known that the given integral Eq. (1.1) possesses a unique solution, then varieties of methods can be used to cast the over-determined system of Eq. (1.12) into a system of $(N+1)$ equations and the method of least-squares provides the most appropriate procedure to handle the situation completely.

Note that one can obtain exactly $(N+1)$ equations for the $(N+1)$ unknowns $\phi_0, \ldots, \phi_N$ from the over-determined system of Eq. (1.12) by selecting $(N+1)$ interpolating points $x = t_k, k = 0, 1, 2, \ldots, N, (0 < x < 1)$.

Substituting the approximate solution (1.11) into the integral Eq. (1.1) we obtain the relation

$$\sum_{i=1}^{N+1} a_{i-1} \psi_{i-1}(x) = f(x), \quad \alpha < x < \beta, \tag{1.13}$$

giving rise to an over-determined system of linear algebraic equations for the determination of the unknown constants $a_{i-1} (i = 1, 2, \ldots, N+1)$ where

$$\psi_{i-1}(x) = x^{i-1} + \int_{\alpha}^{\beta} k(x,t) t^{i-1}\, dt, \quad i = 1, 2, \ldots, N+1. \tag{1.14}$$

On using the least-squares method, we obtain the normal equations

$$\sum_{i=1}^{N+1} a_{i-1} c_{ij} = b_j, \quad j = 1, 2, \ldots, N+1. \tag{1.15}$$

where

$$c_{ij} = \int_{\alpha}^{\beta} \psi_{i-1}(x) \psi_{j-1}(x)\, dx, \quad i = 1, 2, \ldots, N+1, \ j = 1, 2, \ldots, N+1, \tag{1.16}$$

and

$$b_j = \int_{\alpha}^{\beta} f(x) \psi_{j-1}(x)\, dx, \quad j = 1, 2, \ldots, N+1. \tag{1.17}$$

The solution of the system of Eq. (1.15) along with the relation (1.11), finally determines an approximate solution $\phi(x)$.

2. Illustrative examples

We illustrate the above procedure through the following examples.

Examples

(i) $k(x,t) = -(xt + x^2t^2), f(x) = 1, \alpha = -1, \beta = 1.$
(ii) $k(x,t) = -(x^2 + t^2), f(x) = x^2, \alpha = 0, \beta = 1.$
(iii) $k(x,t) = -(\sqrt{x} + \sqrt{t}), f(x) = 1 + x, \alpha = 0, \beta = 1.$
(iv) $k(x,t) = -(\cos x + \cos t), f(x) = \sin x, \alpha = 0, \beta = \pi.$

It can be verified that all the above integral equations possess a unique solution, by examining the eigenvalues of the associated operators.

Solution:

Using the method described in Section 1, if $\phi(x)$ is approximated by the relation (1.11), then we find that the constants $a_{i-1} (i = 1, 2, \ldots, N+1)$ satisfy the system of Eq. (1.15) where

in example 2(i),

$$c_{ij} = \frac{1 - (-1)^{i+j-1}}{i+j-1} - \frac{4}{3} \frac{\{1 - (-1)^{i+1}\}\{1 - (-1)^{j+1}\}}{(i+1)(j+1)} - \frac{8}{5} \frac{\{1 - (-1)^{i+2}\}\{1 - (-1)^{j+2}\}}{(i+2)(j+2)}, \tag{2.18}$$

$$b_j = \frac{1 - (-1)^j}{j} - \frac{2}{3} \frac{\{1 - (-1)^{j+2}\}}{j+2}. \tag{2.19}$$

By choosing $N=2$, we obtain the approximate solution as given by

$$\phi(x)=1+1.1111x^2, \tag{2.20}$$

which satisfies the integral Eq. (1.1).

Also, if we choose $N\geqslant 3$, we get the same approximate solution as obtained in Eq. (2.20).

This example was also considered in [1] where an approximate solution of the integral equation was determined by choosing $n=4$.

Hence, we find that approximating the solution by means of a polynomial is better than the approximation by using Bernstein polynomials for $n=2$.

In example- 2(ii),

$$c_{ij}=\frac{1}{i+j-1}-\frac{5}{3i(j+2)}-\frac{5}{3j(i+2)}+\frac{1}{5ij}+\frac{1}{(i+2)(j+2)}, \tag{2.21}$$

$$b_j=\frac{1}{j+2}-\frac{1}{5j}-\frac{1}{3(j+2)}. \tag{2.22}$$

By choosing $N=2$, we obtain the approximate solution as

$$\phi(x)=0.8182+2.7273x^2, \tag{2.23}$$

which satisfies the integral Eq. (1.1).

Now if we choose $N\geqslant 3$, we get the same solution.

In example – 2(iii),

the exact solution of the corresponding integral equation is given by

$$\phi(x)=\frac{-129}{70}-\frac{141}{35}\sqrt{x}+x. \tag{2.24}$$

For an approximate solution by least-squares method, we follow the procedure described in Section 1 and obtain

$$c_{ij}=\frac{-3+13j-10j^2+i(13+14j-16j^2)+2i^2(-5-8j+12j^2)}{6i(1+2i)j(i+j-1)(1+2j)}, \tag{2.25}$$

$$b_j=\frac{-1}{15j}-\frac{3}{2j+1}+\frac{1}{j+1}. \tag{2.26}$$

By choosing $N=2$, we obtain the approximate solution as given by

$$\phi(x)=\phi_1(x)=-2.5277-4.5208x+2.3003x^2. \tag{2.27}$$

Now, if we assume that $\phi(x)=a_0+a_1x+a_2x^2$ is an approximate solution to the integral Eq. (1.1), with the kernel of example 2(iii), we get

$$a_0-a_0\left(\frac{2}{3}+\sqrt{x}\right)+a_1x-\frac{1}{10}a_1\left(4+5\sqrt{x}\right)-\frac{1}{21}a_2\left(6+7\sqrt{x}\right)+a_2x^2=1+x. \tag{2.28}$$

Then, instead of using the least-squares method, we may derive a system of linear algebraic equations in an artificial manner, as derived below:

Multiplying the Eq. (2.28) by 1, x and x^2, and integrating from $x=0$ to $x=1$, we obtain

$$\frac{-1}{3}a_0-\frac{7}{30}a_1-\frac{11}{63}a_2=\frac{3}{2}, \tag{2.29}$$

$$\frac{1}{420}(-98a_0-28a_1-11a_2)=\frac{5}{6}, \tag{2.30}$$

$$\frac{1}{1260}(-220a_0-33a_1+12a_2)=\frac{7}{12}. \tag{2.31}$$

Solving the Eqs. (2.29)–(2.31), we get an approximation solution of the integral Eq. (1.1), in this case, as given by

$$\phi(x)=\phi_2(x)=-2.5356-4.5307x+2.2898x^2. \tag{2.32}$$

To compare the least-squares solution ϕ_1 and the solution ϕ_2 obtained above, in an artificial way, with the exact solution ϕ, we have then calculated $||\phi-\phi_1||=0.0817197$ and $||\phi-\phi_2||=0.0820618$, where the norm, $||.||$ is defined as

$$||\phi||^2=\int_0^1|\phi|^2\,dx. \tag{2.33}$$

Values of $\phi(x)$ and then the errors are calculated by using the exact expression (2.24) and the approximate expressions (2.27) and (2.31) at the points $x=0,0.25,0.5,0.75,1$ and are presented in the Table 1.

From the values of the norms and the results in Table 1, it is observed that the least-squares solution is better than the one obtained artificially.

Table 1
Exact and approximate solutions of integral equation given in example-2(iii).

x	0	0.25	0.50	0.75	1.00
$\phi(x)$ (exact sol.)	−1.8429	−3.6071	−4.1915	−4.5817	−4.8714
$\phi_1 = \phi(x)$ (least-squares sol.)	−2.5277	−3.5141	−4.2130	−4.6244	−4.7482
$\phi_2 = \phi(x)$ (sol. by artificial way)	−2.5356	−3.5252	−4.2285	−4.6456	−4.7765
$\lvert\phi - \phi_1\rvert$	0.6848	0.0931	0.0215	0.0426	0.1233
$\lvert\phi - \phi_2\rvert$	0.6927	0.0820	0.0369	0.0638	0.0950

In example - 2(iv),
the exact solution of the corresponding integral equation is given by

$$\phi(x) = \sin x + \frac{4}{2-\pi^2}\cos x + \frac{2\pi}{2-\pi^2}. \tag{2.34}$$

Assuming the solution as $\phi(x) = a_0 + a_1 x$, we get

$$(1 - \pi\cos x)a_0 + \left(x + 2 - \frac{\pi^2}{2}\cos x\right)a_1 = \sin x, \quad 0 \leqslant x \leqslant \pi. \tag{2.35}$$

Applying least-squares method to the system (2.35), we obtain the normal equations as given by

$$\frac{1}{2}\pi(2+\pi^2)a_0 + \left\{2\pi + \frac{\pi}{4}(8 + 2\pi + \pi^3)\right\}a_1 = 2, \tag{2.36}$$

$$\left\{2\pi + \frac{\pi}{4}(8 + 2\pi + \pi^3)\right\}a_0 + \left(4\pi + 4\pi^2 + \frac{\pi^3}{3} + \frac{\pi^5}{8}\right)a_1 = 4 + \pi. \tag{2.37}$$

Solving the system of Eqs. (2.36) and (2.37), we find an approximate solution as given by

$$\phi(x) = \phi_1(x) = \frac{-4(48 - 12\pi - \pi^2 + 6\pi^3)}{\pi(-288 + 50\pi^2 + \pi^4)} + \frac{48(-2+\pi^2)}{\pi(-288 + 50\pi^2 + \pi^4)}x. \tag{2.38}$$

Now, multiplying the Eq. (2.35) by 1 and x, and integrating from $x = 0$ to $x = \pi$, we obtain

$$\pi a_0 + \frac{1}{2}\pi(4+\pi)a_1 = 2, \tag{2.39}$$

$$\left(2\pi + \frac{\pi^2}{2}\right)a_0 + \left(2\pi^2 + \frac{\pi^3}{3}\right)a_1 = \pi. \tag{2.40}$$

Solving the Eqs. (2.39) and (2.40), we get an approximation solution as given by

$$\phi(x) = \phi_2(x) = \frac{2(12+\pi)}{-48+\pi^2} + \frac{-48}{\pi(-48+\pi^2)}x. \tag{2.41}$$

As in the example-(iii), here also we find that $||\phi - \phi_1|| = 0.5509$ and $||\phi - \phi_2|| = 0.5510$, where the norm, $||.||$ is same as given by the relation (2.33) with the interval $[0, \pi]$.

Here also, the values of $\phi(x)$ and then the errors are calculated by using the exact expression (2.24) and the approximate expressions (2.27) and (2.31) at the points $x = 0, 0.25, 0.5, 0.75, 1$ and are presented in the Table 2.

A better approximate solution of the form $\phi(x) = a_0 + a_1 x + a_2 x^2$, can also be derived easily. It is seen that the least-squares solution is better than the solution obtained artificially.

3. Remark

(A) If the integral Eq. (1.1) has a unique solution then multiplying both sides of the relation (1.5) by any arbitrary function and integrating with respect to x from $x = \alpha$ to $x = \beta$, gives rise to a linear system of equations of the form $\sum_{i=1}^{n+1} c_{i-1} d_{ij} = b_j, j = 1, 2, \ldots, n+1$ which may be solvable only in certain special circumstances, depending heavily on the kernel $k(x,t)$ as well as the forcing term $f(x)$.

(B) We emphasize that if we multiply the Eq. (1.5) by any arbitrary function and integrate, we may obtain a system of linear algebraic equations giving rise to a matrix which may be singular for non-eigenvalues of the corresponding integral equation. The following examples clarify this:

Example-I:

$$\phi(x) - \lambda\int_0^1 \left(\alpha\sqrt{x} + \sqrt{t}\right)\phi(t)\,dt = f(x), \quad 0 \leqslant x \leqslant 1. \tag{3.42}$$

Table 2
Exact and approximate solutions of integral equation given in example-2(iv).

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$\phi(x)$ (exact sol.)	−1.3067	−0.4507	0.2016	0.2681	−0.2901
$\phi_1(x)$ (least-squares sol.)	−0.7838	−0.4721	−0.1603	0.1515	0.4633
$\phi_2(x)$ (sol. by artificial way)	−0.7942	−0.4795	−0.1648	0.1499	0.4646
$\lvert\phi - \phi_1\rvert$	0.5228	0.0214	0.3619	0.1166	0.7534
$\lvert\phi - \phi_2\rvert$	0.5125	0.0288	0.3664	0.1182	0.7548

The eigenvalues of the integral equation are

$$\lambda = \frac{0.5\{-12(\alpha+1) \pm 12\sqrt{\alpha+2}\sqrt{\alpha+0.5}\}}{\alpha}, \quad (\alpha \neq 0), \tag{3.43}$$

and hence, for any non-eigenvalue $\mu \neq \lambda$, the integral Eq. (3.42) has a unique solution.

Now, let $\phi(x) = a_0 + a_1 x$ be an approximate solution to the Eq. (3.42). Substituting this approximate solution in the Eq. (3.42), we obtain

$$a_0 - \mu\left(\frac{2}{3} + \alpha\sqrt{x}\right)a_0 - \frac{\mu}{10}(4 + 5\alpha\sqrt{x})a_1 + xa_1 = f(x), \quad 0 \leqslant x \leqslant 1. \tag{3.44}$$

Multiplying the Eq. (3.44) by 1 and x and then integrating from $x = 0$ to $x = 1$, we get

$$-\frac{1}{3}\{-3 + 2(1+\alpha)\mu\}a_0 - \frac{1}{30}\{-15 + 2(6+5\alpha)\mu\}a_1 = f_1 \tag{3.45}$$

and

$$\frac{1}{30}\{15 - 2(5+6\alpha)\mu\}a_0 + \frac{1}{15}\{5 - 3(1+\alpha)\mu\}a_1 = f_2, \tag{3.46}$$

where

$$f_1 = \int_0^1 f(x)\,dx, \quad f_2 = \int_0^1 xf(x)\,dx. \tag{3.47}$$

Eqs. (3.45) and (3.46) are solvable if and only if the determinant of their coefficients is non-zero which leads to

$$\mu \neq \frac{0.125\{-50(1+\alpha) \pm 50\sqrt{\alpha+0.5068}\sqrt{\alpha+1.9732}\}}{\alpha}, \quad (\alpha \neq 0), \tag{3.48}$$

showing that there exists a value of $\mu \neq \lambda$ for which the matrix of the system of Eqs. (3.45) and (3.46) becomes singular.

Example-II:

$$\phi(x) - \lambda\int_0^\pi (\alpha\sin x + \cos t)\phi(t)\,dt = f(x), \quad 0 \leqslant x \leqslant \pi. \tag{3.49}$$

The eigenvalues of the integral Eq. (3.49) are

$$\lambda = \frac{1}{2\alpha}, (\alpha \neq 0). \tag{3.50}$$

If $\phi(x) = a_0 + a_1 x$ is an approximate solution of the Eq. (3.49), we obtain

$$(1 - \mu\alpha\pi\sin x)a_0 + \left(x + 2\mu - \frac{\pi^2}{2}\mu\alpha\sin x\right)a_1 = f(x), \quad 0 \leqslant x \leqslant 1. \tag{3.51}$$

Then, multiplying the Eq. (3.51) by x and x^2 and integrating from $x = 0$ to $x = \pi$, we get

$$\left(\frac{\pi^2}{2} - \alpha\pi^2\mu\right)a_0 + \left(-\frac{\pi^3}{2}\alpha\mu + \frac{1}{3}\pi^2(3\mu + \pi)\right)a_1 = f_3 \tag{3.52}$$

and

$$\left(\frac{\pi^3}{3} + 4\alpha\pi\mu - \alpha\pi^3\mu\right)a_0 + \left(\frac{\pi^2}{4} + 2\alpha\pi^2\mu + \frac{2}{3}\pi^3\mu - \frac{1}{2}\alpha\pi^4\mu\right)a_1 = f_4, \tag{3.53}$$

where

$$f_3 = \int_0^\pi xf(x)\,dx, \quad f_4 = \int_0^\pi x^2 f(x)\,dx. \tag{3.54}$$

For the solvability of the system of Eqs. (3.52) and (3.53), we must have

$$\mu \neq \frac{0.00977909\left(-75.3982\alpha \pm 75.3982\sqrt{\alpha}\sqrt{\alpha+1.11547}\right)}{\alpha}, \quad (\alpha \neq 0), \tag{3.55}$$

showing the existence of a value of μ, different from $\lambda = \frac{1}{2\alpha} \neq \mu$, giving rise to difficulties.

(C) Since the relation (1.5) (see also (1.13)) represents an over-determined system of linear equations, if we apply the least-squares method then a solvable determinate system of linear equations (see the Eqs. (1.7)–(1.9) and (1.15)–(1.17)) can be obtained.

(D) Though the above method of least-squares solution is expected to work well enough for Fredholm integral equations of the second kind, it may give rise to non-unique solutions of integral equations of the first kind involving varieties of kernels, as illustrated by the following examples:

(a) $\int_0^1 (1+xt)\phi(t)dt = x,\ 0 \leqslant x \leqslant 1.$
(b) $\int_0^1 (x+t)\phi(t)dt = 1,\ 0 \leqslant x \leqslant 1.$

Solution:

Using the method described in Section 1, if $\phi(x)$ is approximated by the relation (1.11), then we find that $a_{i-1}(i = 1, 2, \ldots, N+1)$ satisfy the system of normal Eqs. (1.15) where

For example (a):

$$c_{ij} = \frac{1}{ij} + \frac{1}{2j(i+1)} + \frac{1}{2i(j+1)} + \frac{1}{3(i+1)(j+1)}, \tag{3.56}$$

$$b_j = \frac{1}{2j} + \frac{1}{3(j+1)}. \tag{3.57}$$

By choosing $N = 1$, we find that an approximate solution is given by

$$\phi(t) = -6 + 12t. \tag{3.58}$$

It is verified that this $\phi(t)$ satisfies the integral equation exactly.

Again, by choosing $N = 2$, we observe the above matrix $[c_{ij}]$ is singular.

For example (b):

$$c_{ij} = \frac{1}{3ij} + \frac{1}{2j(i+1)} + \frac{1}{2i(j+1)} + \frac{1}{(i+1)(j+1)}, \tag{3.59}$$

$$b_j = \frac{1}{2j} + \frac{1}{(j+1)}. \tag{3.60}$$

By choosing $N = 1$, we find that an approximate solution is given by

$$\phi(t) = -6 + 12t. \tag{3.61}$$

It is verified that this $\phi(t)$ satisfies the integral equation exactly.

Again, if we choose $N = 2$, we encounter the same situation giving rise to a singular matrix as obtained in example (a).

The reason for encountering such singular matrices in these examples is to be attributed to the fact that the integral equations here are of the first kind, which generally produce non-unique solutions. In fact $\phi(x) = -24x + 36x^2$ is another solution of both the integral equations considered in the above examples. Though we have found in the above examples that singular systems occur for integral equations of the first kind for the special choices of the order N of the polynomial solutions where exact solutions become available for $N-1$, it is not straight forward to establish the opposite fact in general.

4. Conclusion

In order to solve a special class of Fredholm integral equations of the second kind the unknown function is approximated by a polynomial and the least-squares method is used to solve the resulting over-determined system of equations. Several illustrative examples are examined in detail.

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