A Generalized Reverse Jacket Transform

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Abstract—**Generalization of the well-known Walsh–Hadamard transform (WHT), namely center-weighted Hadamard transform (CWHT) and complex reverse-jacket transform (CRJT) have been proposed and their fast implementation and simple index generation algorithms have recently been reported. These transforms are of size $2^r \times 2^r$ for integral values or r, and defined in terms of binary radix representation of integers. In this paper, using appropriate mixed-radix representation of integers, we present a generalized transform called general reverse jacket transform (GRJT) that unifies all the three classes of transforms, WHT, CWHT, and CRJT, and also applicable for any even length vectors, that is of size $2r \times 2r$. A subclass of GRJT which includes CRJT (but not CWHT) is applicable for finite fields and useful for constructing error control codes.**

Index Terms—**Error-control coding, image-coding, reverse-jacket transform, Walsh–Hadamard transform.**

I. INTRODUCTION

THE WALSH–HADAMARD transform (WHT) and discrete Fourier transform (DFT) are used widely in signal processing [1], [3], in particular image coding and processing [2] and error-control coding [13], [4], [10]. Recently, variations of these two transforms called center weighted Hadamard transform (CWHT) and complex reverse jacket transform (CRJT) [5]–[7], [9] have been reported and their applications in image processing and communications have been pointed out [15].

Both the CWHT and the CRJT include the WHT as a special case. The CRJT uses the complex number $j = \sqrt{-1}$ which is a fourth root of unity. In this paper, we first generalize the CRJT by extending it to any $2n$th root of unity, for any integer value of n. This extended CRJT is well defined for any finite field apart from the complex field and is useful in constructing new classes of error correcting codes. Then we obtain a further generalized transform called generalized reverse jacket transform (GRJT) that includes the CWHT as well as the extended CRJT and hence the WHT as special cases. Fig. 1 illustrates the interrelationship among these transforms.

CRJT has been used to construct optimal bipolar sequence [14] and it is hoped that the extended CRJT presented in this paper will lead to good multilevel sequences. Moreover, WHT is used in practical design of substitution boxes (S-box) in data encryption standard (DES) [16] and the generalized transforms presented here may be used in this design.

In this introductory section, we briefly discuss the WHT, CWHT, and CRJT, mainly to give their definitions and fix the terminology and notations.

A. The WHT

For a $n = 2^r$ length real vector $(a_o, a_1, \ldots, a_{n-1})$ the transform vector is a n-length real vector $(A_0, A_1, \ldots, A_{n-1})$ given by

$$A_j = \sum_{i=0}^{n-1} (-1)^{\langle j, i \rangle} a_i, \qquad j = 0, 1, \ldots, n-1 \qquad (1)$$

where the modulo-2 inner product $\langle j, i \rangle$ is given by

$$\langle j, i \rangle = j_{r-1} i_{r-1} \oplus j_{r-2} i_{r-2} \oplus \cdots \oplus j_0 i_0$$

where $\oplus$ denotes modulo two addition, in terms of radix-2 representation of integers $i \in \{0, 1, \ldots, n-1\}$, denoted by

$$\langle i \rangle = \langle i_{r-1}, i_{r-2}, \ldots, i_0 \rangle$$

where

$$i = i_{r-1} 2^{r-1} + i_{r-2} 2^{r-2} + \cdots + i_1 2 + i_0$$

and $i_\lambda = 0, 1$ for $\lambda = 0, 1, \ldots, r-1$. The inverse transform is given by

$$a_j = \frac{1}{n} \sum_{i=0}^{n-1} (-1)^{\langle j, i \rangle} A_i, \qquad j = 0, 1, \ldots, n-1.$$

B. The CWHT

The CWHT is obtained by weighting the center portion of the transform matrix given by (1) and is given by

$$A_j = \sum_{i=0}^{n-1} (-1)^{\langle j, i \rangle} (w)^{(i_{r-1} \ominus i_{r-2})(j_{r-1} \ominus j_{r-2})} a_i,$$
$$j = 0, 1, \ldots, n-1 \qquad (2)$$

where w is any real number is the weight. The inverse transform is given by

$$a_j = \frac{1}{n} \sum_{i=0}^{n-1} (-1)^{\langle j, i \rangle} (w^{-1})^{(i_{r-1} \ominus i_{r-2})(j_{r-1} \ominus j_{r-2})} A_i,$$
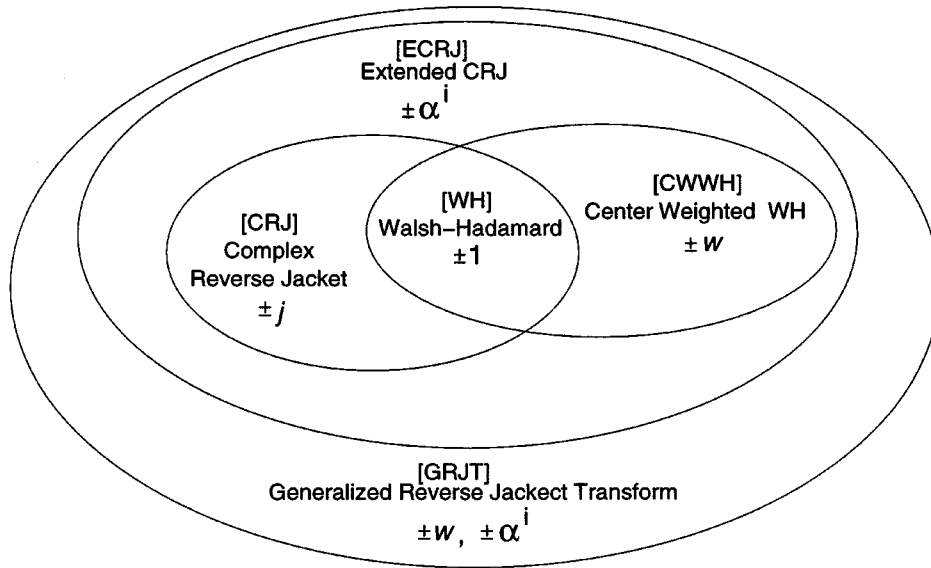$$j = 0, 1, \ldots, n-1.$$

Fig. 1. Interrelationship among the transforms.

Notice that for $w = 1$ the CWHT is same as the WHT. The 4×4 and 8×8 CWHT matrices corresponding to the weight w are, respectively

$$[W]_{4 \times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -w & w & -1 \\ 1 & w & -w & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad (3)$$

and

$$[W]_{8 \times 8} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -w & -w & w & w & -1 & -1 \\ 1 & -1 & -w & w & w & -w & -1 & 1 \\ 1 & 1 & w & w & -w & -w & -1 & -1 \\ 1 & -1 & w & -w & -w & w & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \quad (4)$$

and the inverses, respectively, are

$$[W]_{4 \times 4}^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -w^{-1} & w^{-1} & -1 \\ 1 & w^{-1} & -w^{-1} & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad (5)$$

and

$$[W]_{8 \times 8}^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -w^{-1} & -w^{-1} & w^{-1} & w^{-1} & -1 & -1 \\ 1 & -1 & -w^{-1} & w^{-1} & w^{-1} & -w^{-1} & -1 & 1 \\ 1 & 1 & w^{-1} & w^{-1} & -w^{-1} & -w^{-1} & -1 & -1 \\ 1 & -1 & w^{-1} & -w^{-1} & -w^{-1} & w^{-1} & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \quad (6)$$

C. The CRJT

In [6], a class of transform called CRJT is constructed by replacing the weight w in (2) by the complex number $j = \sqrt{-1}$, and this CRJT is orthogonal and a fast algorithm to compute CRJT is also given. In [8] a simple binary index generation algorithm is discussed for the CRJT and its applications in coded modulation pointed out. However, it is not discussed whether one can use other complex numbers, instead of $-j$ or j.

The 4×4 and the 8×8 CRJT are shown below as

$$[C]_{4 \times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & j & -1 \\ 1 & j & -j & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \quad (7)$$

and

$$[C]_{8 \times 8} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -j & -j & j & j & -1 & -1 \\ 1 & -1 & -j & j & j & -j & -1 & 1 \\ 1 & 1 & j & j & -j & -j & -1 & -1 \\ 1 & -1 & j & -j & -j & j & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}. \quad (8)$$

Observe that $-j$ is a primitive fourth root of unity on the complex unit circle and we show that by replacing j with any $2n$th (for any positive integer n) complex root of unity and using appropriate mixed-radix indexing scheme CRJT can be extended to any even length vectors. This extended CRJT is orthogonal.

The content of the paper is organized as follows: In the next section, we extend the 4×4 CRJT given by (7) to any $2n \times 2n$ by using a $2n$th complex root of unity instead of $j = \sqrt{-1}$ as an intermediate step toward a GRJT. By showing the connection with DFT, this extended CRJT transform is shown to be orthogonal. It is also observed that the extended CRJT is applicable for appropriate finite field instead of complex field and new class of error correcting codes can be obtained. In Section III, we unify the CWHT and the extended CRJT to obtain the GRJT, which is

not necessarily orthogonal. Section IV contains few concluding remarks and several directions for further research.

II. THE EXTENDED CRJT

All the transforms discussed in the previous section, WHT, CWHT, and CRJT are defined for lengths that are powers of two only, or equivalently the corresponding transform matrices are of type $2^r \times 2^r$ for some positive integer r. This is due to the fact that the CWHT is nothing but the center portion of the matrix weighted by w and in CRJT it is due to the fact that $j = \sqrt{-1}$ is an element of order 4. If, instead of j which is a fourth root of unity, α a $2n$th root of unity is used then instead of a 4×4 matrix, a $2n \times 2n$ matrix results. In this section we obtain this matrix, using a mixed radix representation of integers between zero and $2n-1$. A class of optimal bipolar sequences has been obtained in [14] using CRJT and it is hoped that the generalization presented in this section will lead to new classes of optimal sequences.

Definition 1: Let $n \geq 2$ be an arbitrary positive integer and α denote a primitive $2n$th root of unity on the complex circle. For any real or complex $2n$-length vector $(a_0, a_1, \ldots, a_{2n-1})$ the GRJT vector $(A_0, A_1, \ldots, A_{2n-1})$ is given by

$$A_j = \sum_{i=0}^{2n-1} \alpha^{[nj_1+(1-j_1)j_0+j_1(n-1-j_0)][ni_1+(1-i_1)i_0+i_1(n-1-i_0)]} a_i \tag{9}$$

where, for any integer $i \in \{0, 1, \ldots, 2n-1\}$ the mixed-radix representation of it is given by $\langle i \rangle = \langle i_1, i_0 \rangle$ where $i = ni_1 + i_0$ and $i_1 \in \{0, 1\}$ and $i_0 \in \{0, 1, \ldots, n-1\}$.

It is easy to check that the inverse GRJT is given by

$$a_j = \frac{1}{2n} \sum_{i=0}^{2n-1} \alpha^{-[nj_1+(1-j_1)j_0+j_1(n-1-j_0)][ni_1+(1-i_1)i_0+i_1(n-1-i_0)]} A_i.$$

Observe that when $n = 2$, Definition 1 gives the 4×4 CRJT given by (7).

Example 1: Let $n = 3$. Then $\alpha = e^{\sqrt{-1}(\pi/3)}$. The transform matrix and inverse matrix corresponding to Definition 1 are

$$[R]_{6 \times 6} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^5 & \alpha^4 & -1 \\ 1 & \alpha^2 & \alpha^4 & \alpha^4 & \alpha^2 & 1 \\ 1 & \alpha^5 & \alpha^4 & \alpha & \alpha^2 & -1 \\ 1 & \alpha^4 & \alpha^2 & \alpha^2 & \alpha^4 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \tag{10}$$

and

$$[R]_{6 \times 6}^{-1} = \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha^5 & \alpha^4 & \alpha & \alpha^2 & -1 \\ 1 & \alpha^4 & \alpha^2 & \alpha^2 & \alpha^4 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^5 & \alpha^4 & -1 \\ 1 & \alpha^2 & \alpha^4 & \alpha^4 & \alpha^2 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}. \tag{11}$$

The following theorem shows that the transforms given by Definition 1 is orthogonal for all values of n.

Theorem 1: The transform given by Definition 1 is unitary.

Proof: The Theorem follows from the fact that it is a permuted version of the $2n$ length DFT, given by

$$A_j = \sum_{i=0}^{2n-1} \alpha^{ij} a_i; \qquad j = 0, 1, \ldots, 2n-1$$

which is unitary. To be precise, starting from the DFT matrix, the permutation

$$j \Rightarrow nj_1 + (1-j_1)j_0 + j_1(n-1-j_0)$$

retains the first n columns unchanged and reverses the last n columns. After this column permutation, the permutation

$$i \Rightarrow ni_1 + (1-i_1)i_0 + i_1(n-1-i_0)$$

retains the first n rows unchanged and reverses the last n rows. Since column permutation and row permutation does not affect the value of inner products of columns and rows, unitariness is unaffected. $\square$

Notice that the matrix shown in (8) is not obtainable from the transform of Definition 1. This calls for the further generalization of Definition 1. Definition 2 gives such a transform which we call extended CRJT. For this definition, we need the following definition of mixed-radix representation of integers from $\{0, 1, \ldots, N-1\}$ for N of the form $N = 2^l n$, where l and n are positive integers.

Definition 2: Let $N = 2^l n$, where n and l are arbitrary positive integers. An integer $i \in \{0, 1, \ldots, N-1\}$ has a representation of the form

$$i = i_l(n2^{l-1}) + i_{l-1}2^{l-1} + i_{l-2}2^{l-2} + \cdots + i_1 2 + i_0$$

where i_l and i_λ for $\lambda = l-2, l-3, \ldots, 1, 0$ take values from $\{0, 1\}$ and $i_{l-1} \in \{0, 1, \ldots, n-1\}$, and this will be represented by

$$i = \langle i_l, i_{l-1}, i_{l-2}, \ldots, i_1, i_0 \rangle.$$

Observe that for the same value of N if l and n are different, the corresponding mixed-radix number system is different. The following example exemplifies this.

Example 2: For $N = 2^2 3$, $N = 2^3 3$ and $N = 2^2 6$, the mixed-radix representation of integers from $\{0, 1, \ldots, N-1\}$ is shown in Table I.

Definition 3 (Extended CRJT): Let $N = 2^l n$, where l and n are positive integers. For any real or complex valued N-length vector $(a_0, a_1, \ldots, a_{N-1})$ the extended CRJT vector is the N-length complex vector $(A_0, A_1, \ldots, A_{N-1})$ given by (12), shown at the bottom of the next page, and where α is a primitive nth root of unity on the complex unit circle and

$$\ll j, i \gg = j_{l-2}i_{l-2} \oplus j_{l-3}i_{l-3} \oplus \cdots \oplus j_1 i_1 \oplus j_0 i_0 \tag{13}$$

where j and i are represented by the mixed-radix representation of Definition 2.

Notice that the definition of $\ll j, i \gg$ is modulo 2 inner product of j and i without the components i_l, j_l, i_{l-1} and j_{l-1}.

It is easy to verify that the inverse of the extended CRJT is given in (12a) as shown at the bottom of the next page.

TABLE I
MIXED-RADIX REPRESENTATION CORRESPONDING TO EXAMPLE 2

$N = 2^2 3$				$N = 2^3 3$				$N = 2^2 6$			
i	$i_2 i_1 i_0$	i	$i_3 i_2 i_1 i_0$	i	$i_3 i_2 i_1 i_0$	i	$i_2 i_1 i_0$	i	$i_2 i_1 i_0$		
0	000	0	0000	12	1000	0	000	12	100		
1	001	1	0001	13	1001	1	001	13	101		
2	010	2	0010	14	1010	2	010	14	110		
3	011	3	0011	15	1011	3	011	15	111		
4	020	4	0100	16	1100	4	020	16	120		
5	021	5	0101	17	1101	5	021	17	121		
6	100	6	0110	18	1110	6	030	18	130		
7	101	7	0111	19	1111	7	031	19	131		
8	110	8	0200	20	1200	8	040	20	140		
9	111	9	0201	21	1201	9	041	21	141		
10	120	10	0210	22	1210	10	050	22	150		
11	121	11	0211	23	1211	11	051	23	151		

Now, the matrix shown in (8) is obtained from the Definition 3, for $l = 2$ and $n = 2$ by taking $\alpha = j$, which is a fourth root of unity in the complex circle. Moreover, Definition 3 gives transforms for lengths that are not powers of 2 as shown by the following example.

Example 3: For $N = 12 = 2^2 3$, the extended CRJT is same as the matrix shown in (21) with ω removed from wherever it appears or equivalently with $\omega = 1$.

Now we formally show that the transform given by Definition 3 is orthogonal.

Theorem 2: The extended CRJT is unitary.

Proof: Let H_2 denote the 2×2 Walsh–Hadamard matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$\mathbf{C}_{2n}$ the transform matrix given by Definition 1 and $\mathcal{C}_{l,n}$ denote the matrix given by Definition 3. Also, let A^{*t} denote the conjugate transposed matrix of a matrix A with complex numbers. Then we have

$$\mathcal{C}_{l,n} = H_2 \otimes H_2 \otimes \cdots (l-1) \ \times \cdots \otimes H_2 \otimes \mathbf{C}_{2n}$$

where $\otimes$ denotes the Kronecker product. (For properties of Kronecker products, see [17]). The Kronecker product has the following property:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

where A, B, C and D are matrices. Using the above property, we have

$$(\mathcal{C}_{l,n})(\mathcal{C}_{l,n}^{*t}) = [H_2 \otimes \cdots (l-1) \ \times \cdots \otimes H_2 \otimes \mathbf{C}_{2n}]$$
$$\cdot [H_2 \otimes \cdots (l-1) \ \times \cdots \otimes H_2 \otimes \mathbf{C}_{2n}^{*t}] \tag{14}$$
$$= I_2 \otimes \cdots (l-1) \ \times \cdots I_2 \otimes I_{2n} \tag{15}$$
$$= I_{2^l n} \tag{16}$$

where I_m denotes the $m \times m$ identity matrix. This completes the proof. □

Unitary transforms are useful in image enhancement using filtering in the transform domain. For a detailed discussion of transform operations and generalized linear filtering using the unitary transforms discrete cosine transform (DCT), WHT, and DFT, the readers are referred to [18]. Since extended CRJT is unitary, this can be used in such transform processing. In the following subsection we discuss an application of extended CRJT in the area of error control coding.

A. The Extended CRJT Over Finite Fields

In this section, we discuss the applicability of CRJT over finite fields and its usefulness to construct codes. For basic properties of finite fields and techniques of construction the reader is referred to [13]. Our purpose/intention is to demonstrate the applicability of CRJT over finite fields in constructing codes not actually giving all the details of the code construction.

$$A_j = \sum_{i=0}^{N-1} (-1)^{\ll j, i \gg} \alpha^{[nj_l + (1-j_l)j_{l-1} + j_l(n-1-j_{l-1})][ni_l + (1-i_l)i_{l-1} + i_l(n-1-i_{l-1})]} a_i \ , \qquad j = 0, 1, \ldots, N-1 \tag{12}$$

$$a_j = \frac{1}{N} \sum_{i=0}^{N-1} (-1)^{\ll j, i \gg} \alpha^{-[nj_1 + (1-j_l)j_{l-1} + j_l(n-1-j_{l-1})][ni_1 + (1-i_l)i_{l-1} + i_l(n-1-i_{l-1})]} A_i \ , \qquad j = 0, 1, \ldots, N-1 \tag{12a}$$

The use of DFT over finite fields to describe cyclic codes and for decoding is well known [13], [4]. Recently, WHT over finite fields have been used to describe dyadic codes and quasicyclic dyadic codes (see [10] and the references therein). The DFT is an orthogonal transform that maps cyclic convolution to point-wise multiplication-this is the property used to describe cyclic codes over finite fields. Since the extended CRJT is also orthogonal it can be used to obtain codes in the same way as DFT is used.

We first define the extended CRJT as applicable for finite fields as follows:

Definition 4: Let $N = 2^l n$, where l, n are as in Definition 3 along with the mixed-radix integer representation. Also let F_q be a finite field with q elements where N and q are relatively prime and F_{q^m} be an extension field of F_q where m is the least integer such that $2n$ divides $q^m - 1$ and $2nh = q^m - 1$. Let α be a primitive element of F_{q^m}. Then the element $\alpha^{(q^m-1)/h}$, which we denote by β has order $2n$. Then the extended CRJT equation (12) is modified as shown in (12b) at the bottom of the page, where $(a_0, a_1, \ldots, a_{N-1})$ is a vector over F_q and $(A_0, A_1, \ldots, A_{N-1})$ is its transform vector over F_{q^m}.

We construct two examples of transform matrices shown here.

Example 4: Let $N = 6$, $l = 1$, $n = 3$ and $q = 5$. Then $m = 2$, $h = 4$ and F_{25} constructed with the prime polynomial $x^2 + x + 1$ is the set

$$00, 10, 20, 30, 40, 01, 11, 21, 31, 41, 02, 12, 22, 32, 42,$$
$$03, 13, 23, 33, 43, 04, 14, 24, 34, 44$$

where ab denotes $a + bx$ where $a, b \in 0, 1, 2, 3, 4$. Multiplication of two elements is carried out modulo $x^2 + x + 1$. The element 13 is primitive and we take $\alpha = 13$. Then $\beta = \alpha^4 = 04$. For convenience, we show at the bottom of the page, all the nonzero elements in terms of the power of α. The extended CRJT in this case is

$$[F]_{6 \times 6} = \begin{bmatrix} 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 04 & 44 & 11 & 01 & 40 \\ 10 & 44 & 01 & 01 & 44 & 10 \\ 10 & 11 & 01 & 04 & 44 & 40 \\ 10 & 01 & 44 & 44 & 01 & 10 \\ 10 & 40 & 10 & 40 & 10 & 40 \end{bmatrix}. \quad (17)$$

Example 5: In Example 4, if l is changed to 2, then $N = 12$ and all other values remain same and the corresponding extended CRJT is shown

$$[F]_{12 \times 12}$$

$$= \begin{bmatrix} 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ 10 & 40 & 10 & 40 & 10 & 40 & 10 & 40 & 10 & 40 & 10 & 40 \\ 10 & 10 & 04 & 04 & 44 & 44 & 11 & 11 & 01 & 01 & 40 & 40 \\ 10 & 40 & 04 & 01 & 44 & 11 & 11 & 44 & 01 & 04 & 40 & 10 \\ 10 & 10 & 44 & 44 & 01 & 01 & 01 & 01 & 44 & 44 & 10 & 10 \\ 10 & 40 & 44 & 11 & 01 & 04 & 01 & 04 & 44 & 11 & 10 & 40 \\ 10 & 10 & 11 & 11 & 01 & 01 & 01 & 01 & 44 & 44 & 40 & 40 \\ 10 & 40 & 11 & 44 & 01 & 04 & 01 & 04 & 44 & 11 & 40 & 10 \\ 10 & 10 & 01 & 01 & 44 & 44 & 44 & 44 & 01 & 01 & 10 & 10 \\ 10 & 40 & 01 & 04 & 44 & 11 & 44 & 11 & 01 & 04 & 10 & 40 \\ 10 & 10 & 40 & 40 & 10 & 10 & 40 & 40 & 10 & 10 & 40 & 40 \\ 10 & 40 & 40 & 10 & 10 & 40 & 40 & 10 & 10 & 40 & 40 & 10 \end{bmatrix}. \quad (18)$$

III. THE GRJT

In this section, we unify the CWHT and the extended CRJT obtained in the previous section. When the CWHT is used on a $2^r \times 2^r$ size image, pixels in the first and the last 2^{r-2} rows are not weighted and among the rest 2^{r-1} rows, pixels in the first and the last 2^{r-2} columns are not weighted. The ratio of the number of pixels weighted to the total number of pixels is one-quarter, and this ratio is independent of the length of the transform. This ratio in the following GRJT is not fixed and depends on the length of the transform as we shall see in the sequel.

Definition 5 (GRJT): Let $N = 2^l n$, where l and n are positive integers and ω any real number. For any real or complex valued N-length vector $(a_0, a_1, \ldots, a_{N-1})$ the GRJT vector is the N-length complex vector $(A_0, A_1, \ldots, A_{N-1})$ given by (19) at the top of the next page, where α is a primitive $2n$th root of unity on the complex unit circle

$$\ll j, i \gg = j_{l-2} i_{l-2} \oplus j_{l-3} i_{l-3} \oplus \cdots \oplus j_1 i_1 \oplus j_0 i_0$$

and $\Delta(i) = 0$ if $i_l + i_{l-1} \ (modulo \ n) = 0$ and $\Delta(i) = 1$ otherwise.

$$A_j = \sum_{i=0}^{N-1} (-1)^{\ll j, i \gg} \beta^{[nj_l + (1-j_l)j_{l-1} + j_l(n-1-j_{l-1})][ni_l + (1-i_l)i_{l-1} + i_l(n-1-i_{l-1})]} a_i, \qquad j = 0, 1, \ldots, N-1 \quad (12b)$$

α^0 α^{15}	α^1 α^{16}	α^2 α^{17}	α^3 α^{18}	α^4 α^{19}	α^5 α^{20}	α^6 α^{21}	α^7 α^{22}	α^8 α^{23}	α^9	α^{10}	α^{11}	α^{12}	α^{13}	α^{14}
10 43	13 01	22 23	12 30	04 34	32 11	20 31	21 02	44 41	24	03	14	40	42	33

$$A_j = \sum_{i=0}^{N-1} (-1)^{\ll j, i \gg} \omega^{\Delta(i)\Delta(j)} \alpha^{[nj_l + (1-j_l)j_{l-1} + j_l(n-1-j_{l-1})][ni_l + (1-i_l)i_{l-1} + i_l(n-1-j_{l-1})]} a_i \tag{19}$$

$$a_j = \frac{1}{N} \sum_{i=0}^{N-1} (-1)^{\ll j, i \gg} (\omega^{-1})^{\Delta(i)\Delta(j)} \alpha^{-[nj_l + (1-j_l)j_{l-1} + j_l(n-1-j_{l-1})][ni_l + (1-i_l)i_{l-1} + i_1(n-1-i_{l-1})]} A_i \tag{20}$$

$$[G]_{12 \times 12} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & w\alpha & w\alpha & w\alpha^2 & w\alpha^2 & w\alpha^5 & w\alpha^5 & w\alpha^4 & w\alpha^4 & -1 & -1 \\
1 & -1 & w\alpha & -w\alpha & w\alpha^2 & -w\alpha^2 & w\alpha^5 & -w\alpha^5 & w\alpha^4 & -w\alpha^4 & -1 & 1 \\
1 & 1 & w\alpha^2 & w\alpha^2 & w\alpha^4 & w\alpha^4 & w\alpha^4 & w\alpha^4 & w\alpha^2 & w\alpha^2 & 1 & 1 \\
1 & -1 & w\alpha^2 & -w\alpha^2 & w\alpha^4 & -w\alpha^4 & w\alpha^4 & -w\alpha^4 & w\alpha^2 & -w\alpha^2 & 1 & -1 \\
1 & 1 & w\alpha^5 & w\alpha^5 & w\alpha^4 & w\alpha^4 & w\alpha & w\alpha & w\alpha^2 & w\alpha^2 & -1 & -1 \\
1 & -1 & w\alpha^5 & -w\alpha^5 & w\alpha^4 & -w\alpha^4 & w\alpha & -w\alpha & w\alpha^2 & -w\alpha^2 & -1 & 1 \\
1 & 1 & w\alpha^4 & w\alpha^4 & w\alpha^2 & w\alpha^2 & w\alpha^2 & w\alpha^2 & w\alpha^4 & w\alpha^4 & 1 & 1 \\
1 & -1 & w\alpha^4 & -w\alpha^4 & w\alpha^2 & -w\alpha^2 & w\alpha^2 & -w\alpha^2 & w\alpha^4 & -w\alpha^4 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1
\end{bmatrix} \tag{21}$$

It is easy to check that the inverse transform is given by (20) shown at the top of the page.

Example 6: The GRJT matrix for $l = 2$ and $n = 3$ is shown in (21), at the bottom of the page, where ω is the weight and $\alpha = e^{\sqrt{-1}(\pi/3)}$.

The WHT, CWHT, CRJT, and the extended CRJT are all special cases of the GRJT, obtainable from Definition 4, as follows:

- $\alpha = \omega = 1$ gives the WHT;
- $\alpha = 1$ gives the CWHT;
- $\omega = 1$ and $\alpha = -j$ gives the CRJT;
- $\omega = 1$ gives the entended CRJT.

Now, it is straight forward to see that in the GRJT the ratio of the number of pixels that are weighted to the total number of pixels is

$$\frac{2l(n-1)}{2^l n}$$

and this ratio depends on l and n. For instance, for 24 length transform $l = 2$, $n = 6$ leads to the ratio 5/6 whereas $l = 3$, $n = 3$ leads to the ratio 1/2. Recall that for the CRJT this ratio is fixed and always one-quarter.

IV. CONCLUSION

In this paper we have extended the CRJT which is defined using the so called imaginary number j to the extended CRJT that uses a complex $2n$th root of unity where n is any positive integer greater than or equal to 2. Further, the WHT, CWHT, CRJT, and extended CRJT have all been shown to be obtainable as special cases of the GRJT. The application of these transforms to image processing, error control coding and sequences have been cited. The possible directions for further research are as follows:

- Optimal bipolar sequences have been obtained using the CRJT and naturally the extended CRJT correspond to mul-

tiple valued sequences and constitute an interesting area to investigate further.
- Extension of the fast algorithm and the index generation algorithm given in [6] and [8] for CRJT to the GRJT is not straight forward because of the mixed-radix representation involved and the extent of fastness achievable is worth investigating.
- The error correcting capability of the codes obtained by using the entended CRJT for finite fields is an important direction to pursue.

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